NOTES ON THE UNIQUE EXTENSION PROPERTY

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Abstract. In a recent paper, Dritschel and McCullough established the existence of completely positive maps of operator algebras that have a unique extension property. In this expository note we give a more explicit rendering of that result geared to operator systems, and discuss consequences.

1. Maximality

An operator system is a self-adjoint linear subspace S of a unital C^*-algebra that contains the unit; we usually require that the C^*-algebra be generated by S, and express that by writing $S \subseteq C^*(S)$. We consider unital completely positive (UCP) maps $\phi : S \to \mathcal{B}(H)$, that is, completely positive maps that carry the unit of S to the identity operator of $\mathcal{B}(H)$. Such maps satisfy $\phi(x^*) = \phi(x)^*$, $x \in S$. A linear map $\phi : S \to \mathcal{B}(H)$ that preserves the unit is completely positive iff it is completely contractive. If S is a linear subspace of $C^*(S)$ containing $\mathbf{1}$, then every completely contractive unital map of S extends uniquely to a UCP map of $S + S^*$ (see [Arv69]).

Let $S \subseteq C^*(S)$ be an operator system. Given two UCP maps $\phi_k : S \to \mathcal{B}(H_k)$, $k = 1, 2$, we write $\phi_1 \leq \phi_2$ if $H_1 \subseteq H_2$ and $P_{H_1}\phi_2(x)\restriction_{H_1} = \phi_1(x)$, $x \in S$; ϕ_2 is called a *dilation* of ϕ_1 and ϕ_1 is called a *compression* of ϕ_2. The relation $\leq$ is transitive, and one has $\phi_1 \leq \phi_2$ and $\phi_2 \leq \phi_1$ iff $H_1 = H_2$ and $\phi_1 = \phi_2$. Thus, $\leq$ defines a *partial ordering* of UCP maps of S. Every UCP map $\phi : S \to \mathcal{B}(H)$ can be dilated in a trivial way by forming a direct sum $\phi \oplus \psi$ where $\psi : S \to \mathcal{B}(K)$ is another UCP map.

Definition 1.1. A UCP map $\phi : S \to \mathcal{B}(H)$ is said to be *maximal* if it has no nontrivial dilations: $\phi' \geq \phi \implies \phi' = \phi \oplus \psi$ for some UCP map ψ.

A dilation ϕ_2 of ϕ_1 need not satisfy $H_2 = [C^*(\phi_2(S))H_1]$, $C^*(\phi_2(S))$ denoting the C^*-algebra generated by $\phi_2(S) \subseteq \mathcal{B}(H_2)$, but it can always be replaced with a smaller dilation of ϕ_1 that has this property. Notice that this reduction imposes an upper bound on the dimension of H_2 in terms of the dimension of H_1 and the cardinality of S. ϕ_1 is maximal iff the only dilation $\phi_2 \geq \phi_1$ that satisfies $H_2 = [C^*(\phi_2(S))H_1]$ is $\phi_2 = \phi_1$ itself.

Let $\phi : S \to \mathcal{B}(H)$ be a UCP map and let F be a (possibly empty) subset of $S \times H$. We will say that ϕ is maximal on F if for every dilation ψ of ϕ acting on $K \supseteq H$, we have

$$\psi(x)\xi = \phi(x)\xi, \qquad (x, \xi) \in F.$$

A UCP map $\phi : S \to \mathcal{B}(H)$ is maximal iff it is maximal on $S \times H$. If ϕ is maximal on $F \subseteq S \times H$ and $\psi \geq \phi$, then ψ is maximal on F. We require the following result, inspired by an observation of N. Ozawa.

Lemma 1.2. *For every UCP map $\phi : S \to \mathcal{B}(H)$ and every $(x, \xi) \in S \times H$, there is a dilation of ϕ that is maximal on (x, ξ).*

Proof. Since for every dilation $\psi \geq \phi$ we have $\|\psi(x)\xi\| \leq \|x\| \cdot \|\xi\| < \infty$, we can find a dilation ϕ_1 of ϕ for which $\|\phi_1(x)\xi\|$ is as close to $\sup\{\|\psi(x)\xi\| : \psi \geq \phi\}$ as we wish. Continuing inductively, we find a sequence of UCP maps $\phi \leq \phi_1 \leq \phi_2 \leq \cdots$ such that $\phi_n : S \to \mathcal{B}(H_n)$, $H \subseteq H_1 \subseteq H_2 \subseteq \cdots$, and

$$\|\phi_{n+1}(x)\xi\| \geq \sup_{\psi \geq \phi_n} \|\psi(x)\xi\| - 1/n.$$

Let H_∞ be the closure of the union $\cup_n H_n$ and let $\phi_\infty : S \to \mathcal{B}(H_\infty)$ be the unique UCP map that compresses to ϕ_n on H_n for every n. Note that ϕ_∞ is maximal on (x, ξ). Indeed, if $\psi \geq \phi_\infty$ then $\psi \geq \phi_n$ for every $n \geq 1$, and

$$\|\phi_\infty(x)\xi\| \geq \|P_{H_{n+1}}\phi_\infty(x)\xi\| = \|\phi_{n+1}(x)\xi\| \geq \|\psi(x)\xi\| - 1/n.$$

Hence $\|\phi_\infty(x)\xi\| \geq \|\psi(x)\xi\|$. It follows that

$$\|\psi(x)\xi - \phi_\infty(x)\xi\|^2 = \|\psi(x)\xi - P_{H_\infty}\psi(x)\xi\|^2 = \|\psi(x)\xi\|^2 - \|\phi_\infty(x)\xi\|^2 \leq 0$$

so that $\psi(x)\xi = \phi_\infty(x)\xi$ as asserted. □

Theorem 1.3. *Every UCP map $\phi_0 : S \to \mathcal{B}(H_0)$ can be dilated to a maximal UCP map $\phi : S \to \mathcal{B}(H)$.*

Proof. We show first that ϕ_0 can be dilated to a UCP map $\phi_1 : S \to \mathcal{B}(H_1)$ that is maximal on $S \times H_0$. To that end, let λ be an ordinal sufficiently large that there is a surjection $\alpha \in \lambda \mapsto x_\alpha \in S \times H_0$; hence $S \times H_0 = \{x_\alpha : \alpha < \lambda\}$. We claim that there is a family of UCP maps $\phi_\alpha : S \to \mathcal{B}(H_\alpha)$, indexed by the ordinals $\alpha \leq \lambda$, which satisfy $\phi_\alpha \geq \phi_0$ together with

(i) ϕ_α is maximal on $\{x_\beta : \beta < \alpha\}$, and
(ii) $\alpha \leq \beta \implies \phi_\alpha \leq \phi_\beta$.

Once the existence of this family is established, one can set $\phi_1 = \phi_\lambda$.

Proceeding inductively, for $\alpha = 0$ we set $\phi_\alpha = \phi_0$, noting that (i) is vacuous for $\alpha = 0$. Assuming that $\alpha \leq \lambda$ is an ordinal for which $\{\phi_\beta : \beta < \alpha\}$ has been defined and satisfies (i) and (ii) on the initial segment $\{\beta < \alpha\}$, define ϕ_α as follows. If α has an immediate predecessor $\alpha - 1$, Lemma 1.2 implies that $\phi_{\alpha-1}$ can be dilated to a UCP map $\phi_\alpha : S \to \mathcal{B}(H_\alpha)$ that is maximal on $x_{\alpha-1}$. If α is a limit ordinal then the Hilbert spaces H_β, $\beta < \alpha$, are linearly ordered by inclusion; we take H_α to be the closure of their union and $\phi_\alpha : S \to \mathcal{B}(H_\alpha)$ to be the unique UCP map that compresses to ϕ_β on H_β for every $\beta < \alpha$. In either case properties (i) and (ii) persist for the augmented family $\{\phi_\beta : \beta \leq \alpha\}$. This defines $\{\phi_\alpha : \alpha \leq \lambda\}$.

Now one can use ordinary induction on the preceding result to find an increasing sequence of Hilbert spaces $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ and UCP maps $\phi_n : S \to \mathcal{B}(H_n)$ such that ϕ_{n+1} is a dilation of ϕ_n that is maximal on $S \times H_n$, $n = 0, 1, 2, \dots$. Let H_∞ be the closure of $\cup_n H_n$ and let $\phi_\infty : S \to \mathcal{B}(H_\infty)$ the unique UCP map that compresses to ϕ_n on H_n for every $n \geq 1$. Note that for every dilation $\psi : S \to \mathcal{B}(K)$ of ϕ_∞ and every $n \geq 1$, both ψ and ϕ_∞ are dilations of ϕ_{n+1}, so by maximality of ϕ_{n+1} on $S \times H_n$ we have

$$\psi(x)\xi = \phi_{n+1}(x)\xi = \phi_\infty(x)\xi, \qquad (x, \xi) \in S \times H_n.$$

It follows that ϕ_∞ is maximal on $S \times \cup_n H_n$, hence on its closure $S \times H_\infty$. □

2. The unique extension property

Dritschel and McCullough have used the term *boundary representation* for UCP maps $\phi : S \to \mathcal{B}(H)$ that have unique competely positive extensions to representations of $C^*(S)$ [DM03]. We avoid conflict with the terminology of [Arv69] and [Arv72] by making use of the following:

Definition 2.1. A UCP map $\pi : S \to \mathcal{B}(H)$ is said to have the *unique extension property* if

(i) π has a unique completely positive extension $\tilde{\pi} : C^*(S) \to \mathcal{B}(H)$, and
(ii) $\tilde{\pi}$ is a representation of $C^*(S)$ on H.

The unique extension property for $\pi : S \to \mathcal{B}(H)$ is equivalent to the assertion that every extension of π to a completely positive map $\phi : C^*(S) \to \mathcal{B}(H)$ should be *multiplicative* on $C^*(S)$. If the extension $\tilde{\pi}$ of such a map π to $C^*(S)$ is an irreducible representation then the extension is a boundary representation in the sense of [Arv69]; otherwise it is not.

In the following result we adapt an observation of Muhly and Solel [MS98] so as to relate maximality to the unique extension property.

Proposition 2.2. *A UCP map $\pi : S \to \mathcal{B}(H)$ is maximal iff it has the unique extension property.*

Proof. Assume first that π is maximal and let $\phi : C^*(S) \to \mathcal{B}(H)$ be a completely positive extension of it. We have to show that ϕ is multiplicative. By Stinespring's theorem, there is a representation $\sigma : C^*(S) \to \mathcal{B}(K)$ on a Hilbert space $K \supseteq H$ such that $\phi(x) = P_H\sigma(x) \restriction_H$, $x \in C^*(S)$. We can assume that the dilation is minimal in that $K = [\sigma(C^*(S))H] = [C^*(\sigma(S))H]$. By maximality of π, $K = H$ and $\phi = \sigma$ is multiplicative.

Conversely, suppose that π has the unique extension property and let $\phi : S \to \mathcal{B}(K)$ be a dilation of π acting on $K \supseteq H$ with $K = [C^*(\phi(S))H]$. We show that $K = H$ and $\phi = \pi$. By Theorem 1.2.9 of [Arv69] ϕ can be extended to a completely positive linear map $\tilde{\phi} : C^*(S) \to \mathcal{B}(K)$. Since the compression of $\tilde{\phi}$ to H defines a completely positive map of $C^*(S)$ to $\mathcal{B}(H)$ that restricts to π on S, the unique extension property implies that $P_H\tilde{\phi}P_H$ is multiplicative on $C^*(S)$. So for $x \in C^*(S)$,

$$P_H\tilde{\phi}(x)^*P_H\tilde{\phi}(x)P_H = P_H\tilde{\phi}(x^*x)P_H \geq P_H\tilde{\phi}(x)^*\tilde{\phi}(x)P_H,$$

since $\tilde{\phi}(x^*x) \geq \tilde{\phi}(x)^*\tilde{\phi}(x)$; hence $|(\mathbf{1} - P_H)\tilde{\phi}(x)P_H|^2 \leq 0$. This implies that H is invariant under the set of operators $\tilde{\phi}(C^*(S)) \supseteq \phi(S)$, and therefore under $C^*(\phi(S))$. Thus $K = [C^*(\phi(S))H] = H$, and $\phi = \pi$ follows. □

Combining Theorem 1.3 with Proposition 2.2, we obtain the main result of [DM03]. Note that while the discussion of [DM03] is limited to operator algebras, the arguments there carry over to operator systems as well.

Corollary 2.3 (Dritschel-McCullough)**.** *Every UCP map $\phi_0 : S \to \mathcal{B}(H_0)$ can be dilated to a UCP map $\pi : S \to \mathcal{B}(H)$ with the unique extension property.*

3. Silov ideals and C^*-envelopes

We have seen that a UCP map is maximal iff it has the unique extension property. This allows one to strengthen the invariance principle of Theorem 2.1.2 of [Arv69] by a simpler argument than the original.

Proposition 3.1 (Invariance Principle)**.** *Let $S_k \subseteq C^*(S_k)$, $k = 1, 2$, be two operator systems and let $\theta : S_1 \to S_2$ be a unital completely isometric linear map of S_1 on S_2. For every UCP map $\pi_1 : S_1 \to \mathcal{B}(H)$ with the unique extension property, the UCP map $\pi_2 : S_2 \to \mathcal{B}(H)$ defined by $\pi_2 \circ \theta = \pi_1$ has the unique extension property.*

Proof. Consider the UCP map $\pi_2 : S_2 \to \mathcal{B}(H)$ defined by $\pi_2 = \pi_1 \circ \theta^{-1}$. By Proposition 2.2, it suffices to show that π_2 is maximal, given that π_1 is maximal. Let $\phi : S_2 \to \mathcal{B}(K)$ be a UCP map acting on $K \supseteq H$ that compresses to π_2 and satisfies $K = [C^*(\phi(S_2))H]$. Then $\phi\circ\theta$ is a UCP map of S_1 to $\mathcal{B}(K)$ that compresses to π_1 and satisfies $K = [C^*(\pi(S_1))H]$, so by maximality of π_1 we have $\phi \circ \theta = \pi_1$, hence $\phi = \pi_1 \circ \theta^{-1} = \pi_2$. □

We also make use of some terminology from [Arv69] and [Arv72].

Definition 3.2. Let $S \subseteq C^*(S)$ be an operator system. A *boundary ideal* for S is an ideal $J \subseteq C^*(S)$ with the property that the natural projection of $C^*(S)$ on $C^*(S)/J$ restricts to a completely isometric map on S. If there exists a boundary ideal $\bar{J}$ that contains every other boundary ideal for S, then $\bar{J}$ is called the *Silov ideal* for S.

The Silov ideal of an operator system was introduced in [Arv69] and [Arv72], where its existence was established for many examples, and applications to concrete problems in operator theory were described in detail. But the problem of the existence of the Silov ideal for general operator systems was left open. It was finally settled in the affirmative by Hamana, as a consequence of his analysis of injective operator systems [Ham79]. As pointed out by Dritschel and McCullough [DM03], Corollary 2.3 implies the existence of the Silov ideal and the C^*-envelope following an argument of [Arv69]. This has the advantage of sidestepping issues of injectivity that were essential in Hamana's proof of the existence of the C^*-envelope for operator systems. We now reiterate this point in some detail.

Corollary 3.3 (Existence of the Silov ideal)**.** *For every operator system $S \subseteq C^*(S)$, the set of all boundary ideals for S has a largest element $\bar{J}$.*

Assuming that $\bar{J} = \{0\}$, as we may after passing to a quotient, then $C^*(S)$ is the smallest C^*-algebra that can be generated by S in the following sense: for every unital completely isometric linear map θ of S into some other unital C^*-algebra B with $B = C^*(\theta(S))$ there is a unique $*$-homomorphism σ of B onto $C^*(S)$ such that $\sigma \circ \theta$ is the identity map of S.

Proof. There is a maximal UCP map $\pi : S \to \mathcal{B}(H)$ whose restriction to S is completely isometric; indeed, starting with any completely isometric UCP map $\phi_0 : S \to \mathcal{B}(H_0)$, Theorem 1.3 implies that ϕ_0 can be dilated to such a map π. Proposition 2.2 implies that we may extend π uniquely to a representation of $C^*(S)$ on H, which we denote by the same latter π. Let $\bar{J} = \ker \pi$. $\bar{J}$ is a boundary ideal since π restricts to a completely isometric map of S.

Let J be any other boundary ideal for S, and consider the natural projection $x \in C^*(S) \mapsto \dot{x} \in C^*(S)/J$. the map $\psi : \dot{S} \to \mathcal{B}(H)$ defined by $\psi(\dot{s}) = \pi(s)$, $s \in S$ is unit preserving and completely contractive, hence completely positive, and therefore has a completely positive extension $\tilde{\psi} : C^*(S)/J \to \mathcal{B}(H)$. Since $\tilde{\psi}(\dot{s}) = \pi(s)$ for $s \in S$, $\tilde{\psi}(\dot{x}) = \pi(x)$ for $x \in C^*(S)$ by the unique extension property for π. Hence π vanishes on J and $J \subseteq \bar{J}$ follows.

The last paragraph follows from the invariance principle Proposition 3.1. Indeed, since $\bar{J} = \{0\}$ the extension of the map $\pi : S \to \mathcal{B}(H)$ of the first paragraph to $C^*(S)$ is a *faithful* representation. Consider the UCP map $\pi \circ \theta^{-1} : \theta(S) \to \mathcal{B}(H)$. Proposition 3.1 implies that $\pi \circ \theta^{-1}$ has the unique extension property. Letting $\sigma_0 : B \to \mathcal{B}(H)$ be its extension to a representation of B, we find that $\sigma_0(\theta(s)) = \pi(s)$ for $s \in S$, and therefore $\sigma_0(\theta(s_1) \cdots \theta(s_n)) = \pi(s_1 \cdots s_n)$ for $s_1, \dots, s_n \in S$, $n = 1, 2, \dots$. It follows that $\sigma_0(B) = \pi(C^*(S))$, hence the map defined on B by $\sigma = \pi^{-1} \circ \sigma_0$ is a homomorphism of B onto $C^*(S)$ that carries $\theta(s)$ to s, $s \in S$. □

Remark 3.4 (Existence of the C^*-envelope). The second paragraph of Corollary 3.3 implies that in general, the Silov ideal $\bar{J} \subseteq C^*(S)$ has the property that the natural projection $S \to \dot{S} \subseteq C^*(S)/\bar{J}$ exhibits $C^*(S)/\bar{J}$ as the C^*-envelope of S.

References

[Arv69] W. Arveson. Subalgebras of C^*-algebras. *Acta Math.*, 123:141–224, (1969).

[Arv72] W. Arveson. Subalgebras of C^*-algebras II. *Acta Math.*, 128:271–308, (1972).

[DM03] M. Dritschel and S. McCullough. Boundary representations for operator algebras. *J. Op. Th.(to appear)*, pages 1–7, (2003). arXiv:math.OA/0107217.

[Ham79] M. Hamana. Injective envelopes of operator systems. *Publ. Res. Inst. Math. Sci. (Kyoto)*, 15:773–785, (1979).

[MS98] P. Muhly and B. Solel. An algebraic characterization of boundary representations. In *Nonselfadjoint operator algebras, operator theory, and related topics*, pages 189–196, Basel, (1998). Birkhäuser.

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