INFINITE TENSOR PRODUCTS OF COMPLETELY POSITIVE SEMIGROUPS

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Abstract. We construct a new class of semigroups of completely positive maps on $\mathcal{B}(H)$ which can be decomposed into an infinite tensor product of such semigroups. Under suitable hypotheses, the minimal dilations of these semigroups to E_0-semigroups are pure, and have no normal invariant states. Concrete examples are discussed in some detail.

Dedicated to Robert T. Powers on the occasion of his sixtieth birthday

1. Introduction.

An E_0-semigroup is a semigroup of normal unital $*$-endomorphisms $\alpha = \{\alpha_t : t \geq 0\}$ acting on the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space H, which satisfies the natural continuity condition

$$\lim_{t\to 0^+} \langle \alpha_t(x)\xi, \eta\rangle = \langle x\xi, \eta\rangle, \;\; x \in \mathcal{B}(\mathcal{H}), \;\; \xi, \eta \in H.$$

E_0-semigroups are classified roughly into types I, II, III, depending on the existence of "intertwining semigroups". In general, the von Neumann algebras $N_t = \alpha_t(\mathcal{B}(H))$ are type I subfactors of $\mathcal{B}(H)$ which decrease with increasing t. An E_0-semigroup α is called pure if its "tail" algebra is trivial in the sense that

$$\bigcap_{t\geq 0} \alpha_t(\mathcal{B}(H)) = \mathbb{C}\mathbf{1}. \tag{1.1}$$

Pure E_0-semigroups emerge naturally in noncommutative dynamics. For example, Powers [P2] has developed a "standard form" for most E_0-semigroups (more

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precisely, those which are not of type III), which is analogous to the standard form for von Neumann algebras. A key element is the result that any E_0-semigroup not of type III can be perturbed by a cocycle so as to become a pure E_0-semigroup which leaves invariant a (necessarily unique) vector state. The perturbed semigroup is said to be in standard form. This standard form is unique in the sense that if two E_0-semigroups in standard form are cocycle-conjugate then they must actually be conjugate.

On the other hand, in the interaction theory developed by one of us [A7], one is presented with a pair of pure E_0-semigroups, each of which has an invariant normal state. Significantly, in the context of interactions the invariant states are typically *not* vector states. Given any pure E_0-semigroup α with an invariant normal state ω, an arbitrary normal state ρ of $\mathcal{B}(H)$ can be restricted to each of the type I subfactors $\alpha_t(\mathcal{B}(H))$, $t \geq 0$, and each of these restrictions has an eigenvalue list. The basic interaction inequality of [A2] depends upon an asymptotic formula which relates the limiting behavior of these eigenvalue lists to those of ρ and ω as $t \to \infty$.

Consideration of such issues has led us to single out pure E_0-semigroups for attention, and many significant problems remain open. For example, Powers' result above implies that every E_0-semigroup that is not of type III is a cocycle perturbation of a pure E_0-semigroup, but it is not known if *every* E_0-semigroup can be perturbed by a cocycle into a pure one. In this paper we are concerned with the construction of pure E_0-semigroups which do not have normal invariant states. The first examples of this phenomenon were given in [A3], and were based on the canonical commutation relations. In this paper we give a new and more flexible construction of such examples. As in [A3], we actually construct examples of semigroups of completely positive linear maps of $\mathcal{B}(H)$ having appropriate properties, and then obtain the examples of E_0-semigroups by a dilation procedure. However, the examples constructed here differ from from those of [A3] in that they decompose into infinite tensor products of simpler flows acting on matrix algebras.

We now describe the setting of this paper in more detail. By a CP semigroup we mean a one-parameter semigroup $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive linear maps of $\mathcal{B}(H)$ which is unital in the sense that $\phi_t(\mathbf{1}) = \mathbf{1}$ for every t, and is continuous in t as described above for E_0-semigroups. The definition of pure E_0-semigroup (1.1) must be modified appropriately for the broader category of CP semigroups. A CP semigroup ϕ is said to be pure if for every pair of normal states ρ, σ of $\mathcal{B}(H)$ we have

$$\lim_{t\to\infty} \|\rho \circ \phi_t - \sigma \circ \phi_t\| = 0. \tag{1.2}$$

It is known that (1.1) and (1.2) are equivalent in case $\phi = \alpha$ is an E_0-semigroup [A2]. Notice that if ϕ is a pure CP semigroup and if there is a normal state ω of $\mathcal{B}(H)$ that is ϕ-invariant in the sense that $\omega \circ \phi_t = \omega$ for every $t \geq 0$, then because of (1.2) ω must be an *absorbing* state in the sense that for every normal state ρ we have

$$\lim_{t\to\infty} \|\rho \circ \phi_t - \omega\| = 0. \tag{1.3}$$

When an absorbing state exists, it is obviously the unique normal ϕ-invariant state. Conversely, if for an arbitrary CP semigroup ϕ there is a state ω of $\mathcal{B}(H)$ satisfying

(1.3) for every normal state ρ, then ω is a (normal) absorbing state and ϕ must be a pure CP semigroup.

In the physics literature the asymptotic behavior (1.3) is called *return to equilibrium*, whereas in ergodic theory it is called *mixing*. Indeed, (1.3) implies that for every pair of operators $A, B \in \mathcal{B}(H)$ and every normal state σ on $\mathcal{B}(H)$ one has

$$\lim_{t\to\infty} \sigma(\phi_t(A)B) = \omega(A)\sigma(B),$$

and in particular ω is a mixing state: $\omega(\phi_t(A)B) \to \omega(A)\omega(B)$ as $t \to \infty$.

In general, a pure CP semigroup need not have an absorbing state (equivalently, there may not exist a *normal* state that is ϕ-invariant). But in the context of CP semigroups acting on matrix algebras, (1.2) and (1.3) are equivalent. Indeed, if H is finite dimensional and ϕ is a pure CP semigroup acting on $\mathcal{B}(H)$, then a simple application of the Markov-Kakutani fixed point theorem shows that there must be a state ω of $\mathcal{B}(H)$ satisfying $\omega \circ \phi_t = \omega$ for every $t \geq 0$. Since in this case ω must be normal, the preceding observations imply that (1.3) is satisfied.

In section 2, we show how to construct examples of pure CP semigroups which decompose into an infinite tensor product of CP semigroups acting on matrix algebras (Theorem A). Under appropriate hypotheses, these CP semigroups do not have normal invariant states. In section 3 we discuss examples and show how to arrange the general hypotheses of Theorem A. These results are applied to the theory of E_0-semigroups in section 4.

2. Infinite tensor products of CP Semigroups.

In this section we discuss general problems associated with the construction of infinite tensor products of CP semigroups. We give an effective criterion for their existence, and we determine when such infinite tensor products are pure. Applications are taken up in the following section.

Starting with a sequence of CP semigroups ϕ^k acting on $\mathcal{B}(H_k)$, $k = 1, 2, \dots$, one can choose an arbitrary sequence of normal pure states ω_k of $\mathcal{B}(H_k)$, $k = 1, 2, \dots$, and with that data attempt to construct an infinite spatial tensor product $\otimes_k \phi^k$ of CP semigroups. One finds, however, that the existence of such a product semigroup depends strongly on the choice of the sequence $\omega_1, \omega_2, \dots$. In order to discuss this, we begin by introducing an intermediate C^*-algebra $\mathcal{A}$ which is in some sense an infinite tensor product of type I factors, and which carries "locally" the structure of a von Neumann algebra. On $\mathcal{A}$ there is a natural way of defining the infinite tensor product of CP semigroups, but there is no Hilbert space. We show that for appropriate sequences (ω_k), the GNS construction applied to $\otimes_k \omega_k$ gives rise to a representation of $\mathcal{A}$ on a Hilbert space H so that the tensor product of semigroups can be extended uniquely to a CP semigroup on $\mathcal{B}(H)$.

Given a pair of normal completely positive linear maps ϕ, ψ on $\mathcal{B}(H)$, $\mathcal{B}(K)$ respectively, there is a unique normal completely positive linear map $\phi \otimes \psi$ on $\mathcal{B}(H \otimes K)$, defined uniquely by its action on operators $A \otimes B$,

$$\phi \otimes \psi(A \otimes B) = \phi(A) \otimes \psi(B), \qquad A \in \mathcal{B}(H), B \in \mathcal{B}(K).$$

Similarly, there is a natural notion of finite tensor product of normal completely positive linear maps; when ϕ^k acts on $\mathcal{B}(H_k)$, $k = 1, \dots, n$, $\phi^1 \otimes \cdots \otimes \phi^n$ is a normal completely positive linear map on $\mathcal{B}(H_1 \otimes \cdots \otimes H_n)$.

Suppose now that we have a sequence of CP semigroups $\phi^k = \{\phi^k_t : t \geq 0\}$ acting on $\mathcal{B}(H_k)$, $k = 1, 2, \dots$. For each $n = 1, 2, \dots$ consider the type I factor

$$\mathcal{A}_n = \mathcal{B}(H_1 \otimes \cdots \otimes H_n),$$

and the semigroup of completely positive maps $\phi^1 \otimes \cdots \otimes \phi^n$ defined on it by

$$(\phi^1 \otimes \cdots \otimes \phi^n)_t = \phi^1_t \otimes \cdots \otimes \phi^n_t, \qquad t \geq 0.$$

One verifies easily that $\phi^1 \otimes \cdots \otimes \phi^n$ is a CP semigroup. The natural embedding of $\mathcal{A}_n$ into $\mathcal{A}_{n+1}$ is given by the unital $*$-monomorphism $A \in \mathcal{A}_n \mapsto A \otimes \mathbf{1}_{H_{n+1}} \in \mathcal{A}_{n+1}$, and since $\phi^{n+1}_t(\mathbf{1}_{H_{n+1}}) = \mathbf{1}_{H_{n+1}}$ we have the following coherence

$$(\phi^1 \otimes \cdots \otimes \phi^{n+1})_t(A \otimes \mathbf{1}_{H_{n+1}}) = (\phi^1 \otimes \cdots \otimes \phi^n)_t(A) \otimes \mathbf{1}_{H_{n+1}}, \tag{2.1}$$

for every $t \geq 0$, every $A \in \mathcal{A}_n$, and every $n = 1, 2, \dots$.

Finally, we form the inductive limit of C^*-algebras

$$\mathcal{A} = \varinjlim \mathcal{A}_n.$$

There is a natural way of identifying $\mathcal{A}_n$ with a subalgebra of $\mathcal{A}$, and after making this identification one has $\mathcal{A}_n \subseteq \mathcal{A}_{n+1} \subseteq \mathcal{A}$, and $\mathcal{A}$ is the norm closure of the unital $*$-subalgebra $\cup_n \mathcal{A}_n$. Because of (2.1) there is a unique semigroup of unital completely positive maps $\phi = \{\phi_t : t \geq 0\}$ which acts as follows on $\mathcal{A}$,

$$\phi_t \restriction_{\mathcal{A}_n} = (\phi^1 \otimes \cdots \otimes \phi^n)_t, \qquad t \geq 0, \quad n = 1, 2, \dots. \tag{2.2}$$

In order to discuss continuity in the time parameter of this semigroup, we require the notion of locally normal linear functionals on the C^*-algebra $\mathcal{A}$. A bounded linear functional ρ on $\mathcal{A}$ is said to be *locally normal* if its restriction to each subalgebra $\mathcal{A}_n$ is a normal linear functional on that type I factor. The set of all locally normal functionals is a norm-closed linear subspace of the dual space of $\mathcal{A}$. Since every CP semigroup acting on a type I factor M is the adjoint of a strongly continuous semigroup acting on its predual M_*, it follows that for every locally normal linear functional ρ on $\mathcal{A}$, we have

$$\lim_{t \to t_0} \|\rho \circ \phi_t \restriction_{\mathcal{A}_n} - \rho \circ \phi_{t_0} \restriction_{\mathcal{A}_n} \| = 0, \qquad n = 1, 2, \dots. \tag{2.3}$$

We emphasize that one cannot expect $t \mapsto \rho \circ \phi_t$ to move continuously in the norm of the dual space of $\mathcal{A}$, except in rather special circumstances. In any case, from (2.3) it follows that for every $A \in \mathcal{A}$ and every locally normal ρ, the function $t \in [0, \infty) \mapsto \rho(\phi_t(A))$ is continuous.

Suppose now that we choose a sequence ω_k of pure normal states ω_k of $\mathcal{B}(H_k)$, $k = 1, 2, \dots$, subject to the following hypothesis: there are positive constants $A_1, A_2, \dots$ satisfying $\sum_k A_k < \infty$ and

$$\|\omega_k - \omega_k \circ \phi^k_t\| \leq A_k, \qquad 0 \leq t \leq 1, \quad k = 1, 2, \dots. \tag{H}$$

For example, (H) will be satisfied when each space H_k is finite dimensional and the generators L_k of the semigroups ϕ^k, defined by $\phi^k_t = \exp tL_k$, $t \geq 0$, satisfy

$\sum_k \|\omega_k \circ L_k\| < \infty$ (see Corollary 1 below). Explicit examples will be discussed in section 3.

Since $\omega_1 \otimes \cdots \otimes \omega_n$ is a normal pure state of $\mathcal{A}_n$ for every n, the product state $\bar{\omega} = \omega_1 \otimes \omega_2 \otimes \dots$ is a locally normal pure state of $\mathcal{A}$. Applying the GNS construction to $\bar{\omega}$ we obtain a separable Hilbert space K, a faithful irreducible representation of $\mathcal{A}$ on K, and a distinguished unit vector $\Omega \in K$. We identify $\mathcal{A}$ with its image in this representation, so that it becomes a unital strongly dense $*$-subalgebra $\mathcal{A} \subseteq \mathcal{B}(K)$, and once this is done we have $\bar{\omega}(A) = \langle A\Omega, \Omega\rangle$ for $A \in \mathcal{A}$. Following is the main result of this section.

Theorem A. *Assume that hypothesis (H) is satisfied. Then for each $t \geq 0$ the CP map ϕ_t defined on $\mathcal{A}$ by (2.2) extends uniquely to a normal completely positive linear map $\tilde{\phi}_t$ on $\mathcal{B}(K)$, and $\tilde{\phi} = \{\tilde{\phi}_t : t \geq 0\}$ is a CP semigroup on $\mathcal{B}(K)$.*

$\tilde{\phi}$ is a pure CP semigroup on $\mathcal{B}(K)$ iff every ϕ_k is a pure CP semigroup on $\mathcal{B}(H_k)$, $k = 1, 2, \dots$.

Assuming that H_k is finite dimensional and ϕ^k has the normalized trace on $\mathcal{B}(H_k)$ as an absorbing state for every $k = 1, 2, \dots$, then $\tilde{\phi}$ is pure and has no normal invariant state.

Before giving the proof, we require a more concrete realization of K as the infinite tensor product of the Hilbert spaces $H_1, H_2, \dots$. Our construction is similar to that in [vN] (see also [G]). For $n = 1, 2, \dots$ let K_n be the n-fold tensor product of Hilbert spaces

$$K_n = H_1 \otimes H_2 \otimes \cdots \otimes H_n.$$

Choosing a unit vector $f_k \in H_k$ such that $\omega_k(A) = \langle Af_k, f_k\rangle$ for $A \in \mathcal{B}(H_k)$, we can map K_n isometrically into K_{n+1} as follows

$$\xi \in K_n \mapsto \xi \otimes f_{n+1} \in K_{n+1},$$

and this gives rise to an inductive system of Hilbert spaces. K is defined as the completion of the inductive limit of Hilbert spaces

$$K = \overline{\varinjlim K_n}.$$

Thus each space K_n is isometrically embedded in K and their union is dense. For a vector $\xi \in K_n$ we denote its image in K with the suggestive notation

$$\xi \otimes f_{n+1} \otimes f_{n+2} \otimes \cdots = \xi \otimes \otimes_{k=n+1}^{\infty} f_k.$$

Note that parentheses come and go with impunity; for example, the equality

$$\xi \otimes f_{n+1} \otimes f_{n+2} \otimes \cdots = (\xi \otimes f_{n+1}) \otimes f_{n+2} \otimes \dots$$

simply asserts that the image of $\xi \in K_n$ in the limit space K is the same as the image of $\xi \otimes f_{n+1} \in K_{n+1}$ in the limit space K.

The C^*-algebra $\mathcal{A}$ acts on K as follows. To define a representation of $\mathcal{A}$ it suffices to specify how an operator in $\mathcal{A}_n$ acts on K_m when $m > n$, and for an operator $A \in \mathcal{A}_m$ and a vector in $K_m \subseteq K$ of the form $\eta = \xi \otimes f_{m+1} \otimes f_{m+2} \otimes \dots$ $A\eta$ is given by

$$A\eta = [(A \otimes \mathbf{1}_{H_{n+1}} \otimes \cdots \otimes \mathbf{1}_{H_m})\xi] \otimes f_{m+1} \otimes f_{m+2} \otimes \dots.$$

It is easily checked that this defines a representation of $\mathcal{A}$ on K, and that for the vector $\Omega \in K$ given by $\Omega = f_1 \otimes f_2 \otimes \dots$, $\mathcal{A}\Omega$ is dense in K and

$$(\omega_1 \otimes \omega_2 \otimes \dots)(A) = \langle A\Omega, \Omega\rangle, \qquad A \in \mathcal{A}.$$

Thus, we settle on this Hilbert space K, this "product vector" $\Omega \in K$, and the realization of $\mathcal{A} \subseteq \mathcal{B}(K)$ described in the preceding paragraphs.

This has the following consequence for the description of normal linear functionals on $\mathcal{B}(K)$. For any vector $h \in K_n = H_1 \otimes H_2 \otimes \dots \otimes H_n$ we may consider its image $\xi = h \otimes f_{n+1} \otimes f_{n+2} \otimes \dots$ in K , and the corresponding vector state on $\mathcal{B}(K)$

$$\omega_\xi = \omega_h \otimes \omega_{n+1} \otimes \omega_{n+2} \otimes \dots .$$

Since any normal linear functional ρ on $\mathcal{A}_n$ can be expressed as a series

$$\rho(A) = \sum_{k=1}^{\infty} \theta_k \langle Au_k, v_k\rangle$$

where u_k, v_k are unit vectors in $H_1 \otimes \dots \otimes H_n$ and (θ_k) is a summable sequence of nonnegative reals, a similar argument shows that there is a unique normal state on $\mathcal{B}(K)$ of the form

$$\hat{\rho} = \rho \otimes \omega_{n+1} \otimes \omega_{n+2} \otimes \dots = \rho \otimes \bigotimes_{k=n+1}^{\infty} \omega_k, \tag{2.4}$$

The tensor product notation is intended merely to suggest the form this linear functional takes on when it is restricted to the dense C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(K)$.

Lemma 2.1. *For every normal linear functional (resp. state) σ of $\mathcal{B}(K)$ and every $\epsilon > 0$, there is an $n \in \mathbb{N}$ and a normal linear functional (resp. state) ρ on $\mathcal{A}_n$ such that*

$$\|\sigma - \rho \otimes \bigotimes_{k=n+1}^{\infty} \omega_k\| < \epsilon.$$

Proof. Let σ be a normal linear functional on $\mathcal{B}(K)$. We can approximate σ as closely as we wish with a finite linear combination of functionals of the form $\omega_{\xi,\eta}(T) = \langle T\xi, \eta\rangle$ with $\xi, \eta \in H$. Thus it suffices to show that each $\omega_{\xi,\eta}$ can be so approximated. Fixing ξ and η in H, we can find an $n \in \mathbb{N}$ and vectors $\xi_n, \eta_n \in H_n$ such that both norms

$$\|\xi - \xi_n \otimes f_{n+1} \otimes f_{n+2} \otimes \dots\|, \qquad \|\eta - \eta_n \otimes f_{n+1} \otimes f_{n+2} \otimes \dots\|$$

are less than $\epsilon/2$. It follows that

$$\|\omega_{\xi,\eta} - \omega_{\xi_n,\eta_n} \otimes \bigotimes_{k=n+1}^{\infty} \omega_k\| < \epsilon,$$

as asserted.

The argument for normal states is similar, after one notes that every normal state σ of $\mathcal{B}(K)$ can be expressed as a convex combination of vector states

$$\sigma = \theta_1 \omega_{\xi_1,\xi_1} + \theta_2 \omega_{\xi_2,\xi_2} + \dots$$

where $\theta_k \geq 0$ and $\sum_k \theta_k = 1$. $\square$

Since $\mathcal{A} \subseteq \mathcal{B}(K)$ it makes sense to speak of normal states and normal linear functionals on $\mathcal{A}$; for example, a normal state of $\mathcal{A}$ is the restriction to $\mathcal{A}$ of a normal state of $\mathcal{B}(K)$. We remark that a straightforward application of Kaplansky's density theorem shows that the restriction map $\rho \mapsto \rho\restriction_{\mathcal{A}}$ is an order-preserving isometric isomorphism of the Banach space of all normal linear functionals of $\mathcal{B}(K)$ onto the normal subspace of the dual of $\mathcal{A}$. In particular the normal subspace of $\mathcal{A}'$ is closed in the norm of $\mathcal{A}'$.

Lemma 2.2. *Let $\theta : \mathcal{A} \to \mathcal{A}$ be a unital completely positive map with the property that $\rho \circ \theta$ on $\mathcal{A}$ extends to a normal state on $\mathcal{B}(K)$ for every normal state ρ on $\mathcal{B}(K)$. Then θ extends uniquely to a normal CP map $\tilde{\theta}$ on $\mathcal{B}(K)$.*

proof. The uniqueness of the extension is clear from the fact that $\mathcal{A}$ is weak*-dense in $\mathcal{B}(K)$. Morever, if there is a normal linear map $\tilde{\theta}$ which extends θ then $\tilde{\theta}$ must be completely positive. Thus we need only show that θ can be extended to a normal map on $\mathcal{B}(K)$.

Let $\mathcal{L}^1(K)$ denote the Banach space of all trace class operators on $\mathcal{B}(K)$ and choose $T \in \mathcal{L}^1(K)$. Then $\rho(B) = \operatorname{trace}(TB)$ defines a normal linear functional on $\mathcal{B}(K)$. By hypothesis $A \in \mathcal{A} \mapsto \rho(\theta(A))$ can be extended to a normal linear functional on $\mathcal{B}(K)$, hence there is a unique trace class operator $\psi(T) \in \mathcal{L}^1(K)$ such that

$$\operatorname{trace}(\psi(T)A) = \operatorname{trace}(T\theta(A)), \qquad A \in \mathcal{A}.$$

ψ is obviously a linear map of $\mathcal{L}^1(K)$ into itself. Since the unit ball of $\mathcal{A}$ is strongly dense in the unit ball of $\mathcal{B}(K)$ and $\|\theta\| \leq 1$, the formula itself implies that

$$\operatorname{trace}|\psi(T)| = \sup_{\|A\|\leq 1} |\operatorname{trace}(\psi(T)A)| = \sup_{\|A\|\leq 1} |\operatorname{trace}(T\theta(A))| \leq \operatorname{trace}|T|$$

for every $T \in \mathcal{L}^1(K)$, hence $\|\psi\| \leq 1$. The adjoint $\psi' : \mathcal{B}(K) \to \mathcal{B}(K)$ defines a normal linear map on $\mathcal{B}(K)$ which satisfies $\operatorname{trace}(T\psi'(A)) = \operatorname{trace}(T\theta(A))$ for $A \in \mathcal{A}$ and all $T \in \mathcal{L}^1(K)$, thus ψ' extends θ. $\square$

proof of Theorem A. Fix $t \geq 0$. We show first that $\phi_t : \mathcal{A} \to \mathcal{A}$ can be extended (necessarily uniquely) to a normal completely positive map on $\mathcal{B}(K)$. Note that it suffices to prove this for $0 \leq t \leq 1$, because the semigroup property implies that $\phi_{nt} = (\phi_t)^n$ for every $n = 1, 2, \dots$ and a composition of normal maps on $\mathcal{B}(K)$ is a normal map. Thus we assume that $t \in [0, 1]$.

By Lemma 2.2, it is enough to show that for every normal linear functional ρ on $\mathcal{B}(K)$, $\rho \circ \theta_t$ is a normal functional on $\mathcal{A}$. Lemma 2.1 implies that ρ can be approximated in norm with a decomposable linear functional of the form

$$\rho_0 = \psi \otimes \bigotimes_{k=n+1}^{\infty} \omega_k$$

for some $n \in \mathbb{N}$, where ψ is a normal linear functional on $\mathcal{A}_n$ satifying $\|\psi\| = \|\rho_0\|$. Notice that by adjusting ψ appropriately, we may also choose n as large as we wish in this representation for ρ_0. Since the normal part of the dual of $\mathcal{A}$ is norm-closed, it suffices to show that $\rho_0 \circ \phi_t$ belongs to the normal part of $\mathcal{A}'$.

For such a decomposable ρ_0, we have

$$\rho_0 \circ \phi_t = [\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n)] \otimes \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k.$$

Considering the normal linear functional on $\mathcal{B}(K)$

$$\nu = [\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n)] \otimes \bigotimes_{k=n+1}^{\infty} \omega_k$$

and the fact that $\|\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n)\| \leq \|\psi\| = \|\rho_0\|$, we have

$$\|\rho_0 \circ \phi_t - \nu\| \leq \|\rho_0\| \cdot \|(\bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \bigotimes_{k=n+1}^{\infty} \omega_k) \restriction_{\mathcal{A}} \|.$$

Since

$$\|(\bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \bigotimes_{k=n+1}^{\infty} \omega_k) \restriction_{\mathcal{A}} \| \leq \sum_{k=n+1}^{\infty} \|\omega_k \circ \phi_t^k - \omega_k\|$$

and since for $0 \leq t \leq 1$ the series on the right is the tail of a convergent series by (H), it follows that $\|\rho_0 - \nu\|$ can be made arbitrarily small by choosing n sufficiently large. Since ν is normal, $\rho_0 \circ \phi_t$ must be normal, as required.

Thus we may extend each ϕ_t uniquely to a normal completely positive map $\tilde{\phi}_t$ of $\mathcal{B}(K)$. We prove next that the extended semigroup is continuous in the time parameter in the sense that for every normal linear functional ρ on $\mathcal{B}(K)$

$$\lim_{t \to 0} \|\rho \circ \tilde{\phi}_t - \rho\| = 0.$$

Since we can approximate ρ arbitrarily closely in norm with decomposable functionals of the form

$$\rho_0 = \psi \otimes \bigotimes_{k=n+1}^{\infty} \omega_k$$

where $n \in \mathbb{N}$ and $\psi \in \mathcal{A}_n'$ and since $\|\tilde{\phi}_t\| \leq 1$ for every t, it suffices to show that $\|\rho_0 \circ \tilde{\phi}_t - \rho_0\| \to 0$ as $t \to 0$, for ρ_0 of this form.

As in the preceding argument, we may suppose that n is as large as we please; moreover the norm of any normal linear functional on $\mathcal{B}(K)$ agrees with the norm of its restriction to $\mathcal{A}$. Writing

$$\rho_0 \circ \tilde{\phi}_t - \rho_0 = (\rho_0 \circ \tilde{\phi}_t - \psi \otimes \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k) + (\psi \otimes \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \rho_0),$$

we estimate the norm of $(\rho_0 \circ \tilde{\phi}_t - \rho_0) \restriction_{\mathcal{A}}$ as follows

$$\begin{aligned} \|\rho_0 \circ \tilde{\phi}_t - \rho_0\| \leq & \|\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n) - \psi\| \cdot \| \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k \| \\ & + \|\psi\| \cdot \| \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \bigotimes_{k=n+1}^{\infty} \omega_k \| \\ \leq & \|\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n) - \psi\| + \| \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \bigotimes_{k=n+1}^{\infty} \omega_k \| \end{aligned}$$

The second term on the right can be estimated as in the preceding argument,

$$\| \bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi_t^k - \bigotimes_{k=n+1}^{\infty} \omega_k \| \leq \sum_{k=n+1}^{\infty} \|\omega_k \circ \phi_t^k - \omega_k\| \leq \sum_{k=n+1}^{\infty} A_k,$$

because of (H). Since $\phi^1 \otimes \cdots \otimes \phi^n$ is a CP semigroup acting on $\mathcal{B}(H_1 \otimes \cdots \otimes H_n)$, we have

$$\limsup_{t\to 0} \|\rho_0 \circ \tilde{\phi}_t - \rho_0\| \leq \limsup_{t\to 0} \|\psi \circ (\phi_t^1 \otimes \cdots \otimes \phi_t^n) - \psi\| + \sum_{k=n+1}^{\infty} A_k = \sum_{k=n+1}^{\infty} A_k,$$

and since we are free to choose n as large as we please, the right side can be made as small as we please because $\sum_k A_k$ converges. Thus $\tilde{\phi}$ is a CP semigroup.

Assuming now that each ϕ^k is a pure CP semigroup, we prove next that $\tilde{\phi}$ is pure. That result requires that we first discuss the behavior of pure CP semigroups under finite tensor products. For a Hilbert space H, we write $\mathcal{L}_0^1(H)$ for the space of all trace class operators $T \in \mathcal{L}^1(H)$ satisfying trace $T = 0$.

Remark. The significance of the space $\mathcal{L}_0^1(H)$ for pure CP semigroups acting on $\mathcal{B}(H)$ is as follows. Notice first that a trace class operator T on H belongs to $\mathcal{L}_0^1(H)$ iff its associated linear functional $\sigma(A) = \text{trace}(TA)$, $A \in \mathcal{B}(H)$, satisfies $\sigma(\mathbf{1}) = 0$. On the other hand, a normal linear functional σ satisfies $\sigma(\mathbf{1}) = 0$ iff it admits a decomposition

$$\sigma = \lambda \cdot (\rho_1 - \rho_2) + \mu \cdot \sqrt{-1} \cdot (\rho_3 - \rho_4),$$

where λ, μ are real scalars and each ρ_k is a normal state. We conclude: *a CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ is pure iff for every normal linear functional σ satisfying $\sigma(\mathbf{1}) = 0$, we have*

$$\lim_{t\to\infty} \|\sigma \circ \phi_t\| = 0. \tag{2.5}$$

Lemma 2.3. *For Hilbert spaces H_1 and H_2,*

$$\mathcal{L}_0^1(H_1 \otimes H_2) = \overline{span}\{T_1 \otimes S_2 + S_1 \otimes T_2 : T_k \in \mathcal{L}^1(H_k), \quad S_k \in \mathcal{L}_0^1(H_k)\}.$$

proof. The inclusion $\supseteq$ is obvious. For the opposite inclusion, let $A \in \mathcal{B}(H_1 \otimes H_2)$ be a bounded operator satisfying $\text{trace}(RA) = 0$ for every trace class operator R of the form $R = T_1 \otimes S_2$ or $R = S_1 \otimes T_2$ where $S_k \in \mathcal{L}_0^1(H_k)$ and $T_k \in \mathcal{L}^1(H_k)$, $k = 1, 2$. Considering the natural duality between bounded operators and trace class operators, it is enough to show that A is a scalar multiple of the identity.

Indeed, we claim that from the hypothesis

$$\text{trace}(A(S_1 \otimes T_2)) = 0, \qquad \text{for all } S_1 \in \mathcal{L}_0^1(H_1), \quad T_2 \in \mathcal{L}^1(H_2)$$

it follows that $A \in (\mathcal{B}(H_1) \otimes \mathbf{1}_{H_2})'$. To see that, choose $X \in \mathcal{B}(H_1)$ and consider the commutator $[X \otimes \mathbf{1}_{H_2}, A] = (X \otimes \mathbf{1}_{H_2})A - A(X \otimes \mathbf{1}_{H_2})$. Then for any pair of trace class operators $T_k \in \mathcal{L}^1(H_k)$, $k = 1, 2$ we have

$$\text{trace}([X \otimes \mathbf{1}_{H_2}, A]T_1 \otimes T_2) = \text{trace}(A(T_1X - XT_1) \otimes T_2) = 0,$$

because the commutator $T_1X - XT_1$ belongs to $\mathcal{L}^1_0(H_1)$. Since $\mathcal{L}^1(H_1 \otimes H_2)$ is the closed linear span of $\{T_1 \otimes T_2 : T_k \in \mathcal{L}^1(H_k)\}$, $[X \otimes \mathbf{1}_{H_2}, A] = 0$ follows.

Similarly, the hypothesis

$$\operatorname{trace}(A(T_1 \otimes S_2)) = 0, \qquad \text{for all } T_1 \in \mathcal{L}^1(H_1), \quad S_2 \in \mathcal{L}^1_0(H_2)$$

implies that $A \in (\mathbf{1}_{H_1} \otimes \mathcal{B}(H_2))'$. Hence

$$A \in (\mathcal{B}(H_1) \otimes \mathbf{1}_{H_2})' \cap (\mathbf{1}_{H_1} \otimes \mathcal{B}(H_2))' = (\mathcal{B}(H_1) \otimes \mathcal{B}(H_2))' = \mathbb{C} \cdot \mathbf{1}_{H_1 \otimes H_2},$$

as required. $\square$

Lemma 2.4. *Let ϕ^1 and ϕ^2 be pure CP semigroups acting on $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ respectively. Then the tensor product $\phi^1 \otimes \phi^2$ is a pure CP semigroup acting on $\mathcal{B}(H_1 \otimes H_2)$.*

proof. We deduce this from Lemma 2.3 as follows. According to the characterization (2.5), it is enough to show that for every normal linear functional σ on $\mathcal{B}(H_1 \otimes H_2)$ satisfying $\sigma(\mathbf{1}) = 0$, we have $\|\sigma \circ (\phi^1_t \otimes \phi^2_t)\| \to 0$ as $t \to \infty$. Because of Lemma 2.3, σ can be closely approximated in norm with a finite linear combination of functionals of the form $\rho_1 \otimes \rho_2$ where ρ_k is a normal linear functional on $\mathcal{B}(H_k)$ and where *either* ρ_1 or ρ_2 annihilates the identity operator on its respective space.

But for a functional of the form $\sigma = \rho_1 \otimes \rho_2$ we have

$$\|(\rho_1 \otimes \rho_2) \circ (\phi^1_t \otimes \phi^2_t)\| = \|\rho_1 \circ \phi^1_t\| \cdot \|\rho_2 \circ \phi^2_t\|,$$

so that if either $\rho_1(\mathbf{1}_{H_1}) = 0$ or $\rho_2(\mathbf{1}_{H_2}) = 0$, then because both ϕ^1 and ϕ^2 are pure we conclude from (2.5) that

$$\lim_{t \to \infty} \|\rho_1 \circ \phi^1_t\| \cdot \|\rho_2 \circ \phi^2_t\| = 0.$$

After taking finite linear combinations of such functionals $\rho_1 \otimes \rho_2$ and then passing to norm limits, it follows that $\|\sigma \circ (\phi^1_t \otimes \phi^2_t)\| \to 0$ as $t \to \infty$ for every normal linear functional σ satisfying $\sigma(\mathbf{1}) = 0$. $\square$

Returning now to the proof of Theorem A, suppose that each of the given sequence of CP semigroups ϕ^k is pure. ¿From Lemma 2.4 it follows that every finite tensor product $\phi^1 \otimes \cdots \otimes \phi^n$ is a pure CP semigroup acting on $\mathcal{A}_n$, $n = 1, 2, \dots$. We now show that the infinite tensor product $\tilde{\phi}$ is pure. For that, fix a pair of normal states ρ, σ on $\mathcal{B}(K)$. We have to show that

$$\lim_{t \to \infty} \|\rho \circ \tilde{\phi}_t - \sigma \circ \tilde{\phi}_t\| = 0.$$

Again, we approximate ρ and σ closely in norm with normal product states ρ_0 and σ_0 respectively

$$\rho_0 = \rho^n \otimes \bigotimes_{k=n+1}^{\infty} \omega_k, \quad \sigma_0 = \sigma^n \otimes \bigotimes_{k=n+1}^{\infty} \omega_k$$

where ρ^n and σ^n are states of $\mathcal{A}_n$ and n is a positive integer, as large as we wish. Estimating as above, we have

$$\begin{aligned}\|(\rho_0 - \sigma_0) \circ \tilde{\phi}_t \restriction_{\mathcal{A}}\| &\le \|(\rho^n - \sigma^n) \circ (\phi^1_t \otimes \cdots \otimes \phi^n_t) \restriction_{\mathcal{A}_n}\| \cdot \|\bigotimes_{k=n+1}^{\infty} \omega_k \circ \phi^k_t\| \\ &\le \|(\rho^n - \sigma^n) \circ (\phi^1_t \otimes \cdots \otimes \phi^n_t) \restriction_{\mathcal{A}_n}\|\end{aligned}$$

and the right side tends to zero as $t \to \infty$ because $\phi^1 \otimes \cdots \otimes \phi^n$ is a pure CP semigroup on $\mathcal{A}_n$.

The converse assertion, namely that purity of $\tilde{\phi}$ on $\mathcal{B}(K)$ implies that each ϕ^k is a pure CP semigroup on $\mathcal{B}(H_k)$ is straightforward, and we omit the proof. $\square$

When the Hilbert spaces H_k are all finite dimensional, there is a more concrete criterion in terms of the generators of the individual semigroups ϕ^k on $\mathcal{B}(H_k)$.

Corollary 1. *For every $k = 1, 2, \dots$ let ϕ^k be a CP semigroup acting on a full matrix algebra $\mathcal{M}_k$ which has the normalized trace on $\mathcal{M}_k$ as an absorbing state, and let ω_k be a pure state of $\mathcal{M}_k$. Let $\bar{\omega} = \omega_1 \otimes \omega_2 \otimes \dots$ be the asociated product state on the UHF algebra $\mathcal{A} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \dots$ and let $\phi = \{\phi_t : t \geq 0\}$ be the product semigroup, acting on $\mathcal{A}$ by $\phi_t = \phi_t^1 \otimes \phi_t^2 \otimes \dots$, $t \geq 0$.*

Let $L_k : \mathcal{M}_k \to \mathcal{M}_k$ be the generator of ϕ^k, $\phi_t^k = \exp tL_k$, $t \geq 0$, and assume that

$$\sum_{k=1}^{\infty} \|\omega_k \circ L_k\| < \infty. \tag{2.6}$$

The GNS construction applied to $\bar{\omega}$ realizes $\mathcal{A}$ as an irreducible C^*-algebra acting on a Hilbert space K with the property that ϕ extends uniquely to a normal CP semigroup $\tilde{\phi}$ acting on $\mathcal{B}(K)$ which is pure and has no normal invariant states.

Remarks. Regarding the summability hypothesis (2.6), consider any CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on a matrix algebra $\mathcal{M}$ and let L be the generator of ϕ, $\phi_t = \exp tL$, $t \geq 0$. A state ω is invariant in the sense that $\omega \circ \phi_t = \omega$ for all $t \geq 0$ iff $\omega \circ L = 0$. Thus one may regard ω as being "approximately" invariant when $\|\omega \circ L\|$ is small. In the setting of Corollary 1, the tracial state of $\mathcal{M}_k$ is the unique ϕ^k-invariant state of $\mathcal{M}_k$, thus no pure state can be invariant. However, the hypothesis (2.6) means that the sequence of pure states ω_k should be chosen so that they are "approximately" invariant in the sense that the sum (2.6) converges. Notice however that (2.6) will be satisfied with an *arbitrary* sequence of pure states $\omega_1, \omega_2, \dots$ whenever $\|L_k\|$ tends to zero fast enough that $\sum_k \|L_k\|$ converges.

proof of Corollary 1. We show first that $\tilde{\phi}$ extends uniquely to a pure CP semigroup on $\mathcal{B}(K)$. In view of Theorem A it suffices to show that the hypothesis (2.6) implies that (H) is satisfied. For that, we claim that for any CP semigroup acting on a matrix algebra $\mathcal{M}$ with generator L, one has

$$\sup_{0 \leq t \leq 1} \|\omega \circ \phi_t - \omega\| \leq \|\omega \circ L\|, \tag{2.7}$$

for every state ω of $\mathcal{M}$. Indeed, for fixed $A \in \mathcal{M}$ satisfying $\|A\| \leq 1$,

$$\omega \circ \phi_t(A) - \omega(A) = \int_0^t \frac{d}{ds}\omega \circ \phi_s(A)\ , ds = \omega \circ L(\int_0^t \phi_s(A)\, ds).$$

Since

$$\| \int_0^t \phi_s(A)\, ds\| \leq \int_0^t \|\phi_s(A)\|\, ds \leq t \cdot \|A\| \leq t,$$

we have $|\omega \circ \phi_t(A) - \omega(A)| \leq \|\omega \circ L\|$ for every $t \in [0, 1]$, and (2.7) follows.

It remains only to show that $\tilde{\phi}$ has no normal invariant state, and for that we claim that the normalized trace τ on the UHF algebra $\mathcal{A} = \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \dots$ is the unique state of $\mathcal{A}$ which is invariant under the action of the product semigroup $\phi^1 \otimes \phi^2 \dots$ acting on $\mathcal{A}$. Indeed, Lemma 2.3 implies that for every $n = 1, 2, \dots$, the restriction of $\phi^1 \otimes \phi^2 \otimes \dots$ to $\mathcal{A}_n = \mathcal{B}(H_1 \otimes \dots \otimes H_n)$ is pure, and since by hypothesis each ϕ^k leaves the trace of $\mathcal{B}(H_k)$ invariant, it follows that the trace of $\mathcal{B}(H_1 \otimes \dots \otimes H_n)$ is invariant under the action of this restricted CP semigroup, hence the trace on $\mathcal{B}(H_1 \otimes \dots \otimes H_n)$ is an absorbing state. In particular it is the unique invariant state. It follows that τ is the unique invariant state of $\mathcal{A}$.

Now if ρ is a $\tilde{\phi}$-invariant normal state of $\mathcal{B}(K)$ then the restriction of ρ to $\mathcal{A}$ is an invariant state of $\mathcal{A}$ which, by the preceding paragraph, must be the normalized trace τ of $\mathcal{A}$. But τ cannot belong to the normal part of the dual of $\mathcal{A}$ in any irreducible representation of $\mathcal{A}$, contradicting the existence of ρ. □

Corollary 2. *Let $\mathcal{M} = M_N(\mathbb{C})$ be a matrix algebra and let $\theta = \{\theta_t : t \geq 0\}$ be a CP semigroup on $\mathcal{M}$ which has the normalized trace as an absorbing state. Let $\lambda_1, \lambda_2, \dots$ be a sequence of positive numbers satisfying $\sum_n \lambda_n < \infty$. Consider the UHF algebra $\mathcal{A} = \mathcal{M} \otimes \mathcal{M} \otimes \dots$ and the CP semigroup acting on it as follows*

$$\phi_t = \theta_{\lambda_1 t} \otimes \theta_{\lambda_2 t} \otimes \dots .$$

Then for every sequence of pure state ω_k of $\mathcal{M}$ the product state $\omega_1 \otimes \omega_2 \otimes \dots$ gives rise to an irreducible representation of $\mathcal{A}$ on a Hilbert space H with the property that ϕ extends uniquely to a CP semigroup acting on $\mathcal{B}(H)$ which has no normal invariant states.

proof. Letting L_k be the generator of $\phi_t^k = \theta_{\lambda_k t}$, one sees that L_k is related to the generator M of θ by $L_k = \lambda_k M$, and hence

$$\sum_{k=1}^{\infty} \|\omega_k \circ L_k\| \leq \sum_{k=1}^{\infty} \|L_k\| = \|M\| \cdot \sum_{k=1}^{\infty} \lambda_k < \infty,$$

and the conclusion is immediate from Corollary 1. □

3. Examples.

In order to apply the results of the preceding section one must start with a sequence of CP semigroups acting on matrix algebras, each of which is pure and has the normalized trace as an absorbing state. In this section we give concrete examples of the latter and discuss how such examples can be constructed in general.

Remarks. We have already pointed out that if a CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ has an absorbing state, then it must be pure. Significantly, in the simplest cases in which H is finite dimensional, the converse is true as well. Indeed, any pure CP semigroup acting on a matrix algebra $\mathcal{M}$ must have an invariant state ω, by the Markov-Kakutani fixed point theorem. It follows that for any state ρ on $\mathcal{M}$ we have

$$\lim_{t \to \infty} \|\rho \circ \phi_t - \omega\| = \lim_{t \to \infty} \|\rho \circ \phi_t - \omega \circ \phi_t\| = 0,$$

because ϕ is pure. Hence ω must be an absorbing state.

We begin with a concrete example and then put that into a more general context.

Example 3.1. In [Bi] P. Biane defines a CP semigroup $B = \{B_t : t \geq 0\}$, called an Ornstein-Uhlenbeck semigroup, on finite-dimensional C^*-algebras $\mathcal{A}$ generated by finitely many hermitian unitary elements $u_1, u_2, \dots, u_n$, $n \in \mathbb{N}$, which pairwise either commute or anticommute. The action of the semigroup on ordered words $w = u_{k_1} u_{k_2} \dots u_{k_j}, k_1 < k_2 < \dots < k_j$ in $\mathcal{A}$ is as follows, $B_t(w) = \exp(-tj)w$. Assuming that $n = 2N$ is even and the u_k anticommute, their generated C^*-algebra $\mathcal{A}$ is isomorphic to the matrix algebra $M_{2^N}(\mathbb{C})$; and since for every nontrivial word w we have

$$\lim_{t \to \infty} B_t(w) = 0,$$

it follows that for every $A \in \mathcal{A}$ we have

$$\lim_{t \to \infty} B_t(A) = \tau(A)\mathbf{1}.$$

Thus, B is a pure CP semigroup acting on $\mathcal{A}$ which has the normalized trace as an absorbing state.

The following results show how to construct many such examples.

Proposition 3.1. *Let $\mathcal{M}$ be a matrix algebra and let Q be a unit-preserving completely positive map of $\mathcal{M}$ into itself which preserves the trace in the sense that $\tau \circ Q = \tau$ and is ergodic in the sense that*

$$Q(A) = A \implies A \in \mathbb{C} \cdot \mathbf{1}. \tag{3.1}$$

Then $L(A) = Q(A) - A$ is the generator of a pure CP semigroup which has the normalized trace as an absorbing state.

Proof. Let $\phi_t = \exp tL$ where $L(A) = Q(A) - A$. Since $\phi_t = e^{-t} \exp tQ$, it is clear that ϕ is a CP semigroup, and it leaves the trace invariant because $\tau \circ L = 0$. Condition (3.1) implies that the only operators A satisfying $\phi_t(A) = A$ for all t are scalars, i.e., ϕ acts ergodically on $\mathcal{M}$. By Theorem 4.4 of [A2] ϕ is pure, and thus τ must be an absorbing state. □

Remark. It is well-known [EL] that the most general unit-preserving completely positive linear map on a matrix algebra $\mathcal{M}$ may be constructed as follows

$$Q(A) = V_1 A V_1^* + \dots + V_r A V_r^*,$$

where $V_1, \dots, V_r$ is an arbitrary sequence of elements of $\mathcal{M}$ satisfying $V_1 V_1^* + \dots + V_r V_r^* = \mathbf{1}$. For such a map Q the ergodicity requirement (3.1) is satisfied iff the set $\{V_k\}$ is irreducible in the sense that

$$\{V_1, V_1^*, \dots, V_r, V_r^*\}' = \mathbb{C}\mathbf{1}.$$

This follows, for example, from Theorem 4.4 of [A2].

Remark. As the simplest nontrivial example of the semigroups of Example 3.1 above, consider the semigroup $B = \{B_t : t \geq 0\}$ acting on the C^*-algebra generated by a pair of anticommuting self-adjoint unitaries u_1, u_2. Up to conjugacy one may take the C^*-algebra to be $M_2(\mathbb{C})$, and for the generator of the semigroup $\{B_t : t \geq 0\}$ one can take the map $L(A) = Q(A) - A$ where Q is the composition $\theta \circ E$ of the natural conditional expectation E of $M_2(\mathbb{C})$ onto the diagonal subalgebra and θ is the order 2 automorphism $\theta(A) = RAR$,

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We omit these computations, but one can check quite easily that the map $Q = \theta \circ E$ satisfies the hypotheses of Proposition 3.1.

4. Applications to E_0-semigroups.

In this section we show how the CP semigroups constructed in section 2 can be dilated so as to give examples of pure E_0-semigroups which have no normal invariant states.

It is important to distinguish between the dilation theory that is associated with CP semigroups and the simpler dilation theory associated with Stinespring's theorem. In order to discuss the issue, we take the simplest case in which ϕ is a normal completely positive map acting on $\mathcal{B}(H)$ satisfying $\phi(\mathbf{1}) = \mathbf{1}$. Stinespring's theorem implies that there is a Hilbert space $\tilde{H}$ containing H and a representation $\pi : \mathcal{B}(H) \to \mathcal{B}(\tilde{H})$ satisfying $\pi(\mathbf{1}_H) = \mathbf{1}_{\tilde{H}}$, such that for $P = P_H$ we have

$$\phi(A) = P\pi(A) \restriction_H, \qquad A \in \mathcal{B}(H).$$

Since ϕ is normal, one can arrange that π is a normal representation. We abuse notation slightly by writing $\phi(A) = P\pi(A)P$, after identifying $\mathcal{B}(H)$ with the corner $P\mathcal{B}(\tilde{H})P$ of $\mathcal{B}(\tilde{H})$.

Useful as the Stinespring representation is, it is inadequate for our purposes. Instead, we require a Hilbert space $\tilde{H} \supseteq H$ and a normal $*$-endomorphism α of $\mathcal{B}(\tilde{H})$ into itself, such that $\alpha(\mathbf{1}) = \mathbf{1}$, $\alpha(P) \geq P$ for $P = P_H$ and

$$\phi(A) = P\alpha(A)P, \qquad A \in \mathcal{B}(H) = P\mathcal{B}(\tilde{H})P. \tag{4.1}$$

One verifies readily that the condition $\alpha(P) \geq P$ implies that (4.1) is valid for all powers in the sense that

$$\phi^n(A) = P\alpha^n(A)P, \qquad n = 0, 1, 2, \dots. \tag{4.2}$$

Such a representation of ϕ cannot be deduced from Stinespring's theorem. It can be deduced from a general dilation theorem for (finite or infinite) sequences of (perhaps noncommuting) operators $V_1, V_2, \dots \in \mathcal{B}(H)$ which satisfy

$$V_1V_1^* + V_2V_2^* + \dots = \mathbf{1}_H.$$

That dilation theorem implies that there is a sequence of isometries $U_1, U_2, \dots$ on a larger Hilbert space $\tilde{H} \supseteq H$ satisfying $U_1U_1^* + U_2U_2^* + \dots = \mathbf{1}_{\tilde{H}}$, with the property that $U_k^*H \subseteq H$ and V_k is the compression of U_k to H, for every k. The relation between all of these operators, ϕ, and α is

$$\begin{aligned}\phi(A) &= V_1AV_1^* + V_2AV_2^* + \dots, \quad A \in \mathcal{B}(H),\\ \alpha(B) &= U_1BU_1^* + U_2BU_2^* + \dots \quad B \in \mathcal{B}(\tilde{H}).\end{aligned}$$

We remark that the existence of operator dilations of this type originated in papers of Frazho [Fr] and Bunce [Bu], and have been extensively studied.

In [B1] (also see [B2], [B3], [BF]) B. V. R. Bhat proved a general result which implies that every CP semigroup acting on $\mathcal{B}(H)$ can be dilated to an E_0-semigroup acting on $\mathcal{B}(\tilde{H})$ in a sense parallel to (4.2) above. There is a notion of "minimal" dilation, and a minimal dilation of a CP semigroup is unique up to conjugacy. The details are as follows.

We follow the approach of [A5]. Let $\tilde{H}$ be a Hilbert space, and let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(\tilde{H})$. A projection $P \in \mathcal{B}(\tilde{H})$ is said to be *increasing* if $P \leq \alpha_t(P)$ for all $t \geq 0$. Assuming that P is increasing, it is not hard to show that the family of completely positive maps $\phi = \{\phi_t : t \geq 0\}$ defined on $P\mathcal{B}(\tilde{H})P = \mathcal{B}(P\tilde{H})$ by

$$\phi_t(x) = P\alpha_t(a)P, \qquad t \geq 0$$

satisfies the semigroup property, thus it defines a CP semigroup on $\mathcal{B}(P\tilde{H})$. ϕ is called a *compression* of α and α is called a *dilation* of ϕ. α is said to be a *minimal* dilation of ϕ if the identity is the only increasing projection $Q \in \mathcal{B}(\tilde{H})$, $Q \geq P$, for which the compression of α to $Q\mathcal{B}(\tilde{H})Q$ is an E_0-semigroup. In this context, Bhat's theorem has the following consequence (see [A5] for this formulation).

Theorem 4.1. *Let ϕ be a CP semigroup on $\mathcal{B}(H)$. There exists a dilation α of ϕ on $\mathcal{B}(\tilde{H})$, for some Hilbert space $\tilde{H}$ containing H, such that α is a minimal dilation of ϕ. Any two minimal dilations of ϕ are conjugate.*

Thus we may speak of *the* minimal dilation of a CP semigroup. As shown in Proposition 3.5 of [A2], a minimal dilation α of a pure CP semigroup ϕ inherits from ϕ the property of being pure. If ϕ, in addition to being pure, has no invariant normal states, then α has no invariant normal states. Combining these results with Theorem A of Section 2 yields the following.

Theorem 4.2. *Let α be the minimal dilation of a CP semigroup $\phi = \{\phi_t : t \geq 0\}$ on $\mathcal{B}(H)$ which arises from an infinite tensor product construction satisfying the hypotheses of Theorem A. Then α is pure and has no invariant normal states.*

We conclude this section by showing that the minimal dilation α of ϕ has a realization as an infinite tensor product of E_0-semigroups (hence of CP semigroups) in the sense of section 2. We show also that in certain situations α is completely spatial. This result depends upon the fact, [A6], [P1] that the minimal dilation of a CP semigroup on a matrix algebra is completely spatial.

Let (ϕ^k) be a sequence of CP semigroups on matrix algebras M_k satisfying (H). We view M_k as $\mathcal{B}(H_k)$ for a finite dimensional Hilbert space H_k. For each $k \in \mathbb{N}$ let α^k, acting on the algebra $\mathcal{B}(\tilde{H}_k)$, be a minimal dilation of ϕ^k.

First we observe that the pure state $\omega_k = \langle \cdot f_k, f_k \rangle$ on $\mathcal{B}(H_k)$ extends in an obvious way to a pure state (which we'll also call ω_k) on $\mathcal{B}(\tilde{H}_k)$ because $\tilde{H}_k \supseteq H_k$, and if the ϕ^k satisfy (H) then so do the α^k. For fixed k, let p be the projection of $\tilde{H}_k$ onto H_k. We have

$$\|\omega_k - \omega_k \circ \alpha_t^k\| = \sup_{x \in \mathcal{B}(\tilde{H}_k)_1} |\omega_k(x) - \langle \alpha_t^k(x) f_k, f_k \rangle|.$$

Since p is increasing for α^k it follows that $\phi_t^k(pxp) = p\alpha_t^k(pxp)p = p\alpha_t^k(x)p$, and since $f_k = pf_k$, the right side of the preceding formula is

$$\sup_{x \in \mathcal{B}(\tilde{H}_k)_1} |\omega_k(x) - \langle \phi_t^{(k)}(pxp) f_k, f_k \rangle| \leq \|(\omega_k - \omega_k \circ \phi_t^{(k)}) \restriction_{\mathcal{M}_k} \| \|x\| \leq A_k,$$

where A_k is the constant in (H) which dominates $\|(\omega_k - \omega_k \circ \phi_t^k) \restriction_{M_k} \|$ for $0 \leq t \leq 1$. Continuing the argument along the lines of section 2, we conclude that the infinite tensor product $\overset{\infty}{\underset{k=1}{\otimes}} \alpha_k$ exists as a CP semigroup. It is obviously an E_0-semigroup which serves as a dilation of the CP semigroup $\phi = \overset{\infty}{\underset{k=1}{\otimes}} \phi_k$. In fact, we have the following.

Theorem 4.3. *The dilation α on $\mathcal{B}(\tilde{H})$ is a minimal dilation of ϕ.*

Remark. It is straightforward to show, by using the criteria for minimality obtained in Theorem B of [A5], that the finite tensor product of minimal dilations of CP semigroups is itself a minimal dilation. The proof below relies on this result to establish the case for infinite tensor products.

proof. Let $P \in \mathcal{B}(\tilde{H})$ be the projection onto the subspace H on which ϕ acts. Then P compresses α down to ϕ. Let P_+ be the orthogonal projection in $\tilde{H}$ onto the closed linear span H_+ of the vectors of the form

$$\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots\alpha_{t_n}(a_n)\xi, \tag{4.3}$$

where ξ runs over H and the a_j's lie in $\mathcal{B}(H)$. By [A5] P_+ compresses α to a minimal dilation of ϕ. To show that α is minimal, then, it suffices to show that P_+ is the identity in $\mathcal{B}(\tilde{H})$, and to establish this it suffices to show that all product vectors of the form $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m \otimes f_{m+1} \otimes f_{m+2} \otimes \cdots$ are in the range of P_+ for all vectors $\xi_k \in \tilde{H}_k$. Set $\xi = \overset{\infty}{\underset{k=1}{\otimes}} f_k$. For fixed $m \in \mathbb{N}$ consider the linear span of all terms of the form (4.3) where $a_j \in (\overset{m}{\underset{k=1}{\otimes}} \mathcal{M}_k) \otimes (I \otimes I \otimes \cdots)$ for all j. By the remark above $\overset{m}{\underset{k=1}{\otimes}} \alpha^{(k)}$ on $\overset{m}{\underset{k=1}{\otimes}} \mathcal{B}(\tilde{H}_k)$ is a minimal dilation of $\overset{m}{\underset{k=1}{\otimes}} \phi^{(k)}$ on $\overset{m}{\underset{k=1}{\otimes}} \mathcal{B}(H_k)$, and therefore $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m \otimes f_{m+1} \otimes f_{m+2} \otimes \cdots$ is in the range of P_+. But then, since m is arbitrary, all of $\tilde{H}$ is in range of P_+, so $p_+ = 1$ and α is a minimal dilation of ϕ. $\square$

We conjecture that the minimal dilations of CP semigroups constructed as in section 2 are all are all conjugate to a cocycle perturbation of a CAR/CCR flow. The following confirms the conjecture for a large family of CP semigroups satisfying the hypotheses of Corollary 2.

Theorem 4.4. *Given a sequence of positive numbers $\lambda_1, \lambda_2, \ldots,$ let ϕ be a CP semigroup acting on a matrix algebra $\mathcal{M}$. Let $\tilde{\phi}$ be the infinite tensor product*

$$\tilde{\phi}_t = \phi_{\lambda_1 t} \otimes \phi_{\lambda_2 t} \otimes \ldots$$

acting on the UHF algebra $\mathcal{A} = \mathcal{M} \otimes \mathcal{M} \otimes \ldots$. Let $\omega = \omega_1 \otimes \omega_2 \otimes \cdots$ be a pure product state on $\mathcal{A}$ which induces an irreducible representation of $\mathcal{A}$ on $\mathcal{B}(H)$. If $\sum_n \lambda_n^{1/2} < \infty$, then the minimal dilation $\tilde{\alpha}$ of the unique extension of $\tilde{\phi}$ to $\mathcal{B}(H)$ is a cocycle perturbation of a CCR/CAR flow.

Remark. In [A1] one of us characterized the CAR/CCR flows as being those E_0-semigroups which are completely spatial (see the following definition). Our proof of the theorem rests on the verification that E_0-semigroups satisfying the hypotheses of the theorem are completely spatial. First we recall the following relevant definitions from [A1].

Definition 4.1. A strongly continuous one-parameter semigroup $U = \{U_t : t \geq 0\}$ of operators in $\mathcal{B}(\tilde{H})$ is called a *unit* of an E_0-semigroup α on $\mathcal{B}(\tilde{H})$ if for all $A \in \mathcal{B}(\tilde{H})$ and all $t \geq 0$,

$$U_t A = \alpha_t(A) U_t.$$

Remark. We shall denote by $\mathcal{U}_\alpha$ the family of units of α. It is not difficult to show that any $U \in \mathcal{U}_\alpha$ is, up to perturbation by a semigroup of scalar operators, a one-parameter semigroup of isometries.

Definition 4.2. An E_0-semigroup α on $\mathcal{B}(\tilde{H})$ is said to be *completely spatial* if for all $t \geq 0$ the Hilbert space $\tilde{H}$ coincides with the closed linear span of all vectors in $\tilde{H}$ of the form

$$U^{j_1}_{s_1} U^{j_2}_{s_2} \cdots U^{j_n}_{s_n} \xi, \tag{4.4}$$

with $U^{j_\ell} \in \mathcal{U}_\alpha, j = 1, 2, \ldots, n$, with $\xi \in \mathcal{B}(\tilde{H})$, and with $\sum_{r=1}^n s_r = t$.

proof of theorem. Let α be a minimal dilation of ϕ. Since ϕ is a CP semigroup on a matrix algebra its minimal dilation is completely spatial, by [A6] or [P1]. For each $k \in \mathbb{N}$ let α^k be the E_0-semigroup defined by $\alpha^k_t = \alpha_{t\lambda_k}$. α^k is clearly a minimal dilation of the CP semigroup ϕ^k related to ϕ by $\phi^k_t = \phi_{t\lambda_k}$. Hence each α^k is completely spatial.

Using the notation of section 2, for some $m \in \mathbb{N}$, let $h = \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m \otimes f_{m+1} \otimes f_{m+2} \otimes \cdots$, where $\xi_k \in \tilde{H}_k$ is arbitrary and $f_k, k = 1, 2, \ldots$ are the distinguished unit vectors used in the construction of the tensor product Hilbert space H. Since linear combinations of such vectors are dense in H we will be done if we can show that such vectors are in the closed linear span of vectors of the form (4.4). First note that the finite tensor product of completely spatial E_0-semigroups is completely spatial. One sees this by recalling that the finite tensor product of CAR flows is a CAR flow [P3] and then using the equivalence, up to cocycle conjugacy, between CAR flows and completely spatial semigroups, [A1]. ¿From this observation it follows that the product vector $\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m$ lies in the closed linear span of vectors of the form $U^{j_1}_{s_1} U^{j_2}_{s_2} \cdots U^{j_n}_{s_n} \xi$, where each U^{j_ℓ} is a unit of $\alpha^1 \otimes \cdots \otimes \alpha^m$ and $\xi \in \bigotimes_{k=1}^{m} H_k$.

Now let W be any fixed isometric unit of α. Note then that the semigroup W^k of isometries given by $W^k_t = W_{t\lambda_k}, t \geq 0$ is a unit of α^k. We wish to make sense of the infinite tensor product $W^{m+1} \otimes W^{m+2} \otimes \cdots$ and to show it is a unit of the E_0-semigroup $\bigotimes_{k=m+1}^{\infty} \alpha^k$. To see this, let P be the orthogonal projection in $\mathcal{B}(\tilde{H})$, with range H, which compresses $\tilde{\alpha}$ to $\tilde{\phi}$. Then by [A4] $C = \{C_t = PW_tP : t \geq 0\}$ is a contraction semigroup on H. For any unit vector f in H,

$$\|W_s f - f\|^2 = 2(\|f\|^2 - \Re \langle W_s f, f\rangle) = 2\Re(\|f\|^2 - \langle C_s f, f\rangle) \leq 2|1 - \langle C_s f, f\rangle|.$$

Using the fact that $C_s = e^{sA}$ for a bounded operator A we have

$$|1 - \langle C_s f, f\rangle| = |\int_0^s \frac{d}{ds} \langle C_s f, f\rangle \, ds| = |\int_0^s \langle C_s A f, f\rangle \, ds| \leq \|A\| \cdot s.$$

Taking square roots we arrive at the estimate

$$\|W_s f - f\| \leq c \cdot \sqrt{s}, \tag{E}$$

where c is a constant. It will follow from this calculation that for fixed $s \geq 0$ the sequence of operators $V_q(s) = I \otimes I \otimes \cdots \otimes I \otimes W_s^{m+1} \otimes W_s^{m+2} \otimes \cdots \otimes W_s^q \otimes I \otimes I \otimes \cdots$, $q = m+1, m+2, \dots$, converges strongly in $\mathcal{B}(\tilde{H})$. To verify this it suffices to show, since the sequence of operators is bounded in norm, that the sequence $\{V_q(s)g : q \geq m+1\}$ is Cauchy for each g in some dense subset $\tilde{H}_0$ of $\tilde{H}$. We may take $\tilde{H}_0$ to be the set of finite linear combinations of tensor products of unit vectors of the form $g = g_1 \otimes g_2 \otimes \cdots g_r \otimes f_{r+1} \otimes f_{r+2} \otimes \cdots$, $r \in \mathbb{N}$, $g_k \in \tilde{H}_k, k = 1, 2, \dots r$. It follows that it suffices to show that $\{V_q(s)g : q \geq m+1\}$ is Cauchy for tensor product vectors of this form. Using (E) we have, for $r < q_1 < q_2$,

$$\begin{aligned}\|V_{q_2}(s)g - V_{q_1}(s)g\| &\leq \sum_{k=q_1+1}^{q_2} \|V_k(s)g - V_{k-1}(s)g\| \\ &\leq \sum_{k=q_1+1}^{q_2} \|W_s^k f_k - f_k\| \\ &\leq \sum_{k=q_1+1}^{q_2} c \cdot \sqrt{s} \cdot \sqrt{\lambda_k}\end{aligned}$$

This establishes our claim about the existence of the tensor product $I \otimes I \otimes \cdots I \otimes \bigotimes_{k=m+1}^{\infty} W^k$ in $\mathcal{B}(\tilde{H}_1 \otimes \tilde{H}_2 \otimes \cdots \tilde{H}_m)' \cap \mathcal{B}(\tilde{H})$ of the isometric units W^k. A similar calculation to the one above shows that this semigroup is strongly continuous in the real variable s. Moreover, following the construction in section 2 it is not difficult to show that for fixed $m \in \mathbb{N}$, $\bar{W} = W^{m+1} \otimes W^{m+2} \otimes \cdots$ is a unit of $\bigotimes_{k=m+1}^{\infty} \alpha^k$. Hence if U^ℓ is a unit of $\bigotimes_{k=1}^{m} \alpha^k$, then $U^\ell \otimes \bar{W}$ is a unit of $\bigotimes_{k=1}^{\infty} \alpha^k$.

Finally let h be the product vector given above, let $t \geq 0$ be fixed, and for $\epsilon > 0$ let m sufficiently large that $\|\bar{W}_t(\bigotimes_{k=m+1}^{\infty} f_k) - \bigotimes_{k=m+1}^{\infty} f_k\| < \epsilon/2$. Since $\bigotimes_{k=1}^{m} \alpha^k$ is completely spatial there are units U^ℓ in $\mathcal{B}(\tilde{H}_1 \otimes \tilde{H}_2 \otimes \cdots \tilde{H}_m)$ of this E_0-semigroup such that some linear combination F of vectors of the form $U_{s_1}^{j_1} U_{s_2}^{j_2} \cdots U_{s_n}^{j_n} \xi$ satisfies $\|F - \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_m\| < \epsilon/2$. Noting that $U^\ell \otimes \bar{W}$ is a unit of $\tilde{\alpha}$ for all units U^ℓ of $\bigotimes_{k=1}^{m} \alpha^k$, and noting that $\|(F \otimes \bar{W}_t(f_{m+1} \otimes f_{m+2} \otimes \cdots)) - h\| < \epsilon$ we have verified that $\tilde{\alpha}$ is completely spatial.

References

[A1] W. B. Arveson, *Continuous analogues of Fock space*, Mem. Amer. Math. Soc. **80** (1989).

[A2] ______, *E_0-semigroups and absorbing states*, Comm. Math. Phys. **187** (1997), 19–43.

[A3] ______, *The heat flow of the CCR algebra*, 2000 preprint.

[A4] ______, *The index of a quantum dynamical semigroup*, J. Funct. Anal. **146** (1997), 557–588.

[A5] ______, *Minimal E_0-semigroups*, Operator algebras and their applications, Fields Inst. Commun., vol. 13, Amer. Math. Soc., 1997, pp. 1–12.

[A6] ______, *On the index and dilations of completely positive semigroups*, Internat. J. Math. **10** (1999), 791–823.

[A7] W. B. Arveson, *Interactions in Noncommutative Dynamics*, Comm. Math. Phys. **211** (2000), 63–83.

[B1] B. V. R. Bhat, *An index theory for quantum dynamical semigroups*, Trans. Amer. Math. Soc. **348** (1996), 561–583.

[B2] B. V. R. Bhat, *Minimal dilations of quantum dynamical semigroups to semigroups of endomorphisms of C^*-algebras*, J. Ramanujan Math. Soc. **14, no. 2** (1999), 109–124.

[B3] ———, *Cocycles of CCR flows*, to appear, Memoirs, AMS.

[BF] B. V. R. Bhat and F. Fagnola, *On minimality of Evans-Hudson flows*, Bull. dell' Unione Mat. Ital. **IX-A-3p., Serie VII** (1997), 671–683.

[Bi] P. Biane, *Free hypercontractivity*, Commun. Math. Phys. **184** (1997), 457–474.

[Bu] Bunce, John W., *Models for n-tuples of noncommuting operators*, J. Funct. Anal. **57** (1984), 21–30.

[EL] D. E. Evans and J. T. Lewis, *Dilations of irreversible evolutions in algebraic quantum theory*, Comm. Dublin Inst. Adv. Studies Ser. A (1977), 1–104.

[Fr] Frazho, Arthur E., *Models for noncommuting operators*, J. Funct. Anal. **48** (1982), 1–11.

[G] A. Guichardet, *Produits tensoriels et représentations des relations d'anticommutation*, Ann. Sci. École Norm. Sup. **83** (1966), 1–52.

[vN] J. von Neumann, *On infinite direct products*, Comp. Math. **6** (1936), 1–77.

[P1] R. T. Powers, *Induction of semigroups of endomorphisms of $\mathcal{B}(\mathcal{H})$ from completely positive semigroups of n by n matrix algebras*, Intl. J. Math..

[P2] ———, *New examples of continuous spatial semigroups of $*$-endomorphisms of $\mathcal{B}(\mathfrak{H})$*, Int. J. Math. **10, no. 2** (1999), 215–288.

[P3] ———, *An index theory for semigroups of $*$-endomorphisms of $\mathcal{B}(H)$ and type II_1 factors*, Canad. J. Math. **40**, 86–114.