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Bill Arveson, Editorial Staff

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The purpose of these remarks is to report some news concerning type III E_0-semigroups. I've been working through Bob Powers' remarkable paper [1], hoping to understand his construction in terms as simple as possible and to make connections with other parts of operator theory (like Wiener-Hopf operators and Hankel operators) where that seemed helpful. The following discussion summarizes what I have learned, and appears to clarify things to some extent.

For notation, $\hat{\mathbb{R}}$ will denote the dual group of the additive group $\mathbb{R}$ of real numbers, and I choose normalizations of Lebesgue measure on $\mathbb{R}$ and $\hat{\mathbb{R}}$ which cause irritating factors of $\sqrt{2\pi}$ and its reciprocal, that put in unwanted appearances when taking Fourier transforms, to disappear. This causes the unit interval to have peculiar length; but that problem is secondary since we do no numerical calculations. Following the physicists, I will remind you of where we are by using the letters x, y for elements of $\mathbb{R}$ and $p, q \in \hat{\mathbb{R}}$ for elements of $\hat{\mathbb{R}}$. $C(\hat{\mathbb{R}})$ will denote all continuous functions $f : \hat{\mathbb{R}} \to \mathbb{C}$ which have a limit at the point at ∞,

$$f(\infty) = \lim_{p\to\infty} f(p).$$

For $N = 1, 2, \dots,$ $L^2(\mathbb{R}) \otimes \mathbb{C}^N$ denotes all vector functions

$$\xi : \mathbb{R} \to \mathbb{C}^N$$

which are square integrable in the usual sense (and of course we regard $\mathbb{C}^N$ as an N dimensional Hilbert space in the usual way). $L^2[0, \infty) \otimes \mathbb{C}^N$ denotes the subspace of functions vanishing almost everywhere on the negative axis, and

$$P_+ : L^2(\mathbb{R}) \otimes \mathbb{C}^N \to L^2[0, \infty) \otimes \mathbb{C}^N$$

is the natural projection. Finally, $M_N(C(\hat{\mathbb{R}}))$ denotes the $C*$-algebra of all $N \times N$ matrices over $C(\hat{\mathbb{R}})$...whose elements we often consider as matrix-valued functions defined on $\hat{\mathbb{R}}$.

For every $\Phi \in M_N(C(\hat{\mathbb{R}}))$ we define a generalized "convolution" operator C_Φ on $L^2(\mathbb{R}) \otimes \mathbb{C}^N$ by

$$\widehat{C_\Phi \xi}(p) = \Phi(p)\hat{\xi}(p), \qquad p \in \hat{\mathbb{R}},$$

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where the hat denotes the Fourier transform

$$\hat{\xi}(p) = \int_{-\infty}^{\infty} e^{-ipx}\xi(x)\,dx.$$

C_Φ is a singular integral operator, but we require only elementary technology here. Let R be the natural reflection operator defined on $\mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^N$ by $R\xi(x) = \xi(-x)$, $x \in \mathbb{R}$.

Definition. *For every $\Phi \in M_N(C(\hat{\mathbb{R}}))$, the operator H_Φ defined on $L^2[0,\infty) \otimes \mathbb{C}^N$ by*

$$H_\Phi = P_+ R C_\Phi \restriction_{L^2[0,\infty)\otimes\mathbb{C}^N}$$

is called the Hankel operator with symbol Φ.

Remarks. I am calling H_Φ a Hankel operator because it satisfies the abstract definition of Hankel operator on $L^2([0,\infty)) \otimes \mathbb{C}^N$, namely $H_\Phi S_t = S_t^* H_\Phi$, $t \geq 0$, where $S = \{S_t : t \geq 0\}$ is the shift semigroup

$$S_t\xi(x) = \begin{cases} \xi(x-t), & \text{if } x \geq t \\ 0, & \text{if } 0 \leq x < t. \end{cases}$$

If Φ happens to be integrable in the sense that $\int_{\hat{\mathbb{R}}} \|\Phi(p)\|\,dp < \infty$, then one can check that C_Φ is an integral operator with a "Hankel" kernel

$$C_\Phi\xi(x) = \int_0^\infty \hat{\Phi}(x+y)\xi(y)\,dy, \qquad x \geq 0,$$

where $\hat{\Phi} \in M_N(C(\mathbb{R}))$ is the Fourier transform of Φ

$$\hat{\Phi}(x) = \int_{-\infty}^{\infty} e^{ipx}\Phi(p)\,dp.$$

Moreover, in this case one can verify by routine changes-of-variables that the following are equivalent

(1) H_Φ and H_{Φ^*} are Hilbert-Schmidt operators.
(2) $\int_{-\infty}^{\infty} |x|\,\mathrm{trace}|\hat{\Phi}(x)|^2\,dx < \infty$,

where as usual $\mathrm{trace}|A|^2$ denotes $\mathrm{trace}(A^*A)$. Note that the condition (1) involves H_{Φ^*} and not the adjoint of H_Φ.

Here are the two results I was able to squeeze out of Bob's work.

Theorem A. *Let $N = 1, 2, \dots$, let $\Phi \in M_N(C(\hat{\mathbb{R}}))$ satisfy $0 \leq \Phi \leq \mathbf{1}$, and let A be the Wiener-Hopf operator defined on the Hilbert space $H = L^2[0,\infty) \otimes \mathbb{C}^N$ by*

$$A = P_+ C_\Phi \restriction_{L^2[0,\infty)\otimes\mathbb{C}^N} .$$

Let ω_A be the gauge-invariant quasifree state defined on the CAR algebra $\mathcal{A}$ over the one-particle space H having two-point operator A. For $f \in H$ we write $c(f)$ for the creation operator associated with $f \in H$. Then the following are equivalent.

(1) *ω_A is a type I factor state and for π the GNS representation of $\mathcal{A}$ associated with ω_A, there is a unique E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\pi(\mathcal{A})''$ such that*

$$\alpha_t(\pi(c(f))) = \pi(c(S_t f)), \qquad t \geq 0, \quad f \in H.$$

(2) *$\Phi(p)$ is a projection for every $p \in \hat{\mathbb{R}}$ and the Hankel operator H_Φ is of Hilbert-Schmidt class.*

Remarks. The Hilbert-Schmidt condition can be characterized as follows. Let $\phi_{ij}(p)$ be the ijth component of the $N \times N$ matrix $\Phi(p)$. The Fourier transform $\hat{\phi_{ij}}$ of ϕ_{ij} is a tempered distribution on $\mathbb{R}$, and H_Φ and H_{Φ^*} are Hilbert-Schmidt operators iff for every $1 \leq i, j \leq N$ $\hat{\phi_{ij}}(x)$ is a function away from $x = 0$ (more precisely, its singular set is a subset of $\{0\}$), and we have

$$\sup_{\epsilon>0} \int_{\mathbb{R}\setminus[-\epsilon,\epsilon]} |x| \cdot |\hat{\phi_{ij}}(x)|^2 \, dx < \infty. \tag{3}$$

In [1], Powers takes $N = 2$ and the symbol

$$\Phi(p) = 1/2 \begin{pmatrix} 1 & e^{i\theta(p)} \\ e^{-i\theta(p)} & 1 \end{pmatrix} \tag{4}$$

where θ is the function

$$\theta(p) = \frac{1}{(1+p^2)^{1/5}}, \qquad p \in \hat{\mathbb{R}}, \tag{5}$$

and he proves (3) more-or-less directly for this particular Φ, as part of his Lemma 4.2. Note too that the $\hat{\phi_{ij}}$ associated with (4), (5) are distributions, not functions.

Of course, the real issue here is the existence of units for the E_0-semigroup α. Here is my perturbation of what Bob does in section 4 of [1]: it gets rid of the somewhat mysterious contradiction employed in the proof of his Lemma 4.6, and gives a concrete condition that the symbol Φ must satisfy.

Theorem B. *Let $\Phi \in M_N(C(\hat{\mathbb{R}}))$ satisfy the conditions of Theorem A (so that we obtain an E_0-semigroup α). If α has a unit, then Φ has the following property:*

$$\int_{-\infty}^{\infty} \mathit{trace}|\Phi(p) - \Phi(\infty)|^2 \, dp < \infty. \tag{6}$$

Remarks. Notice that in example (4), (5) we have

$$\Phi(p) - \Phi(\infty) = 1/2 \begin{pmatrix} 0 & e^{i\theta(p)} - 1 \\ e^{-i\theta(p)} - 1 & 0 \end{pmatrix}$$

so that

$$\mathrm{trace}|\Phi(p) - \Phi(\infty)|^2 = 1/2|e^{i\theta(p)} - 1|^2 = 1 - \cos\theta(p).$$

When $|p|$ is large $\theta(p)$ tends to 0 and we have

$$1 - \cos\theta(p) \sim \theta(p)^2/2 = \frac{1}{2(1+p^2)^{2/5}} \sim \frac{1}{2|p|^{4/5}}.$$

But since $4/5 < 1$, our calculus students know that $\int_1^\infty |p|^{-4/5} \, dp$ diverges, hence the integral on the left side of (6) is infinite. Conclusion: By theorem B the associated E_0-semigroup has no units.

The condition of Theorem B shows that there are many examples of type III E_0-semigroups that arise out of Bob's construction of Theorem A. But what we are still lacking is a criteria for cocycle conjugacy (in terms of the symbol Φ).

Conjecture. *The condition (6) is equivalent to the existence of units for the E_0-semigroup α.*

The conjucture seems reasonable to me, but after going through all of this in a seminar I have lost some of my steam on these issues. If somebody out there is motivated to attack the conjecture, I can supply the details that are only alluded to above (but only as handwritten lecture notes).

References

1. ______, *A non-spatial continuous semigroup os $*$-endomorphisms of* $\mathcal{B}(H)$, Publ. RIMS (Kyoto University) **23** (1987), 1053–1069.