Does quantum field theory exist?

Final Lecture, Mathematics 189, Spring 1998
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We have seen how to quantize a finite dimensional classical mechanical system (this will be reveiwed briefly below). The purpose of this lecture is to formulate a *mathematical* problem associated with the quantization of interacting Boson fields, to describe the serious mathematical problems assiciated with finding a (mathematically sound) solution to the problem, and to make some remarks about the nature of the difficulties. The problem remains unsolved to this day.

Consider a nonlinear wave equation in four spacetime dimensions of the form

$$\Box\phi + \alpha\phi + \beta\phi^3 = 0. \tag{1}$$

Here, $\phi = \phi(t, x, y, z)$ is a function defined on $\mathbb{R}\times\mathbb{R}^3$, α and β are positive constants, and $\Box$ is the wave operator

$$\Box\phi = \frac{\partial^2\phi}{\partial t^2} - \Delta\phi = \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2}.$$

For example, in equation (1) the term $\beta\phi^3$ stands for the function $\beta\phi(t, x, y, z)^3$. One may consider equation (1) as defining a problem in classical nonlinear PDE (acronym service: PDE = Partial Differential Equations), and in that context one would ideally like to know that given initial data (i.e., two functions $u(x, y, z)$ and $v(x, y, y)$ defined over $\mathbb{R}^3$), there is a unique solution ϕ to equation (1) which satisfies the "initial conditions"

$$\phi(0, x, y, z) = u(x, y, z), \qquad \frac{\partial\phi}{\partial t}(0, x, y, z) = v(x, y, z).$$

In order to fit this into a context resembling classical mechanics, we can reformulate equation (1) as a second order ordinary differential equation on an infinite dimensional configuration space as follows. Consider the space $\mathcal{D}$ consisting of all infinitely differentiable functions $f : \mathbb{R}^3 \to \mathbb{R}$. The space $\mathcal{D}$ occupies the position of the configuration space of classical mechanics (actually, we will have to refine this a bit below). Consider too the operator $A : \mathcal{D} \to \mathcal{D}$ defined by

$$A(u) = \Delta u - \alpha u - \beta u^3.$$

Notice that A is defined explicitly on a concrete infinite dimensional space, but it is not a *linear* operator. Rather, it is a polynomial of degree 3. As we will see, it is this nonlinearity that causes most of the mathematical problems in the quantized version of (1); at the same time (and this is one of the paradoxes of the subject) the physics underlying this problem shows that it is the nonlinearity of A that is *necessary* for a quantum field to model nontrivial interactions.

The problem of solving (1) now becomes the problem of finding a smooth function $t \in \mathbb{R} \to \phi(t) \in \mathcal{D}$ which satisfies the "ordinary differential equation"

$$\frac{d^2}{dt^2}\phi(t) = A(\phi(t)), \qquad t \in \mathbb{R}. \tag{2}$$

Let us continue to think of (2) as an ODE (acronym service: ODE = Ordinary Differential Equations) and proceed as if we were doing classical mechanics. Think of the operator A as a vector field, $u \in \mathcal{D} \to A(u) \in \mathcal{D}$. We can find a "potential" having A as its gradient as follows.

We first cut down the space of test functions a bit so that we can define a natural inner product on it. Let $\mathcal{K}$ denote all functions in $\mathcal{D}$ which vanish outside some cube (which may depend on the function of course). On this smaller space $\mathcal{K}$ we can define a natural inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^3} u(x,y,z)v(x,y,z)\,dx\,dy\,dz.$$

Note that A maps $\mathcal{K}$ into itself. We want to find a "potential" function V; i.e., a function $V : \mathcal{K} \to \mathbb{R}$ such that $-\nabla V = A$. Since we now have an inner product on $\mathcal{K}$, we can imitate what we did in class when we gave a coordinate-free definition of the gradient of a function defined on an finite dimensional inner product space. Thus we want to find a function $V : \mathcal{K} \to \mathbb{R}$ with the property that for every pair of functions $u, v \in \mathcal{K}$ we have

$$\langle A(u), v\rangle = -\frac{d}{ds}V(u+sv)|_{s=0}. \tag{3}$$

Notice that the left side of (3) is

$$\int_{\mathbb{R}^3} (\Delta u(x,y,z) - \alpha u(x,y,z) - \beta u(x,y,z)^3)v(x,y,z)\,dx\,dy\,dz,$$

and the right side of (3) is $-\langle \nabla V(u), v\rangle$ (this is simply the *definition* of ∇V as a function from $\mathcal{K}$ to $\mathcal{K}$).

It is a good exercise to verify that there is such a function $V : \mathcal{K} \to \mathbb{R}$; V is unique up to an additive constant, and is given explicitly as follows:

$$V(u) = \int_{\mathbb{R}^3} (1/2\|\nabla u\|^2 + \alpha u^2/2 + \beta u^4/4)\,dx\,dy\,dz. \tag{4}$$

I strongly urge you to verify that the function $V : \mathcal{K} \to \mathbb{R}$ defined by (4) actually satisfies equation (3). In the integrand, the expression $\|\nabla u\|$ means the Euclidean norm of the usual three dimensional gradient vector $\nabla u(x,y,z)$. Notice the powers of u in the integrand: we started with a combination of u and u^3 in the definition of A and now we have quadratic and quartic terms in the formula for the potential V. That is why this is called the "ϕ-fourth" or "quartic" interaction.

To summarize: we have reformulated the nonlinear PDE (1) as the following problem in "infinite dimensional" classical mechanics: we have an inner product $\langle\cdot,\cdot\rangle$ on the configuration space $\mathcal{K}$, we have a smooth function $V : \mathcal{K} \to \mathbb{R}$, and we want to find the flow associated with the second order differential equation

$$\frac{d^2\phi}{dt^2} = -\nabla V(\phi). \tag{5}$$

Of course, showing the existence of a flow for (1) is a hard problem in PDE, and we have only reformulated it into another hard problem that looks more familiar.

What we really want to do is to show how equation (5) should be "quantized". To do that, we try to follow the method discussed in class, and we will watch carefully for things that appear to work and things that don't.

We now take up the problem of quantizing equation (5). Let us first recall what we did for the situation in which we had the finite dimensional vector space $\mathbb{R}^n$, the inner product $\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n$, a potential function $V : \mathbb{R}^n \to \mathbb{R}$, and a differential equaation

$$\frac{d^2x}{dt^2} = -\nabla V(x), \qquad x \in \mathbb{R}^n \tag{6}$$

In that case, the problem of quantization amounted to finding an irreducible representation of the commuatation relations on a Hilbert space H, and a one parameter group of automorphisms of the algebra $\mathcal{B}(H)$ which satisfies the requirements of Ehrenfest's theorem. Using the theorem of Stone and von Neumann (which asserts that every irreducible representation of the cannonical commutation relations is unitarily equivalent to the Schrödinger representation), and Wigner's theorem (asserting that every one-parameter group of autormorphisms of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H is implemented by a one-parameter group of unitary operators of the form $U_t = e^{itH}$), we reduced the problem to to a problem involving operators on the concrete Hilbert space $L^2(\mathbb{R}^n)$. We found that, because of the Stone- von Neumann theorem, we could assume that the operators of the commutation relations were the concrete operators defined on $L^2(\mathbb{R}^n)$ defined by

$$\begin{aligned} Q_i &= \text{multiplication by } x_i, \\ P_i &= \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_i}. \end{aligned}$$

We also found that the requirements of Ehrenfest's theorem implied that the "Hamiltonian" H should satisfy

$$\begin{aligned} \sqrt{-1}(HQ_i - Q_iH) &= P_i, \\ \sqrt{-1}(HP_i - P_iH) &= F_i(Q_1, \ldots, Q_n) \end{aligned}$$

where $F_1, \ldots, F_n$ are the components of the gradient $-\nabla V$ of the potential V.

Under these conditions we found that the quantum mechanical Hamiltonian H *must* have the form

$$\begin{aligned} H &= 1/2(P_1^2 + \cdots + P_n^2) + V(Q_1, \ldots, Q_n) \\ &= -1/2\Delta + \text{multiplication by } V. \end{aligned}$$

This is the correct Schrödinger operator that is commonly used for carrying out the calculations of quantum mechanics.

The mathematical problem remaining for quantum mechanics is that of showing that the differential operator defined on $L^2(\mathbb{R}^n)$ by the preceding equation actually does generate a one-parameter unitary group (this is a significant mathematical problem even today, depending on the nature of the potential V, but it is a problem that has been solved for many...perhaps most...of the important cases).

Let us now try to carry out the same procedures for the infinite dimensional system described by (5). To do that, we must first find an irreducible representation of the cannonical commutation relations that goes with the inner product space $(\mathcal{K}, \langle\cdot,\cdot\rangle)$.

This has to be formulated carefully, since $\mathcal{K}$ is infinite dimensional. Weyl's form of the CCRs (acronym service: CCR = Canonical Commutation Relations) must be used here in the following way. We seek a pair of mappings $f \in \mathcal{K} \to U(f)$, $f \in \mathcal{K} \to V(f)$ from $\mathcal{K}$ to the unitary operators on a Hilbert space H such that $U(f+g) = U(f)U(g)$, $V(f+g) = V(f)V(g)$, and which satisfy Weyl's form of the CCRs

$$V(f)U(g) = e^{i\langle f,g\rangle}U(g)V(f). \tag{7}$$

We also want the set of operators $\{U(f), V(g) : f, g \in \mathcal{K}\}$ to be irreducible in an appropriate sense.

As it happens, such representations exist. But here we have the first serious problem: unlike the case of finite dimensions in which we have the Stone-von Neumann theorem asserting that there is only one irreducible representation of the CCRs (up to unitary equivalence), in the infinite dimensional case the Stone von Neumann theorem is false: there are infinitely many essentially inequivalent irreducible representations of the CCRs that satisfy (7). Which one should we choose? Notice too that the "Schrödinger" representation makes no sense here because $\mathcal{K}$ is infinite dimensional. For example, when configuration space is finite dimensional like $\mathbb{R}^n$ then we know what $L^2(\mathbb{R}^n)$ means: it means complex-valued functions $f : \mathbb{R}^n \to \mathbb{C}$ which are measurable and satisfy the condition

$$\int_{\mathbb{R}^n} |f(x_1, \dots, x_n)|^2 \, dx_1 \, \dots \, dx_n < \infty.$$

But what could $L^2(\mathcal{K})$ possibly mean? What is the analogue of Lebesgue measure on an infinite dimensional space like $\mathcal{K}$ and what could $L^2(\mathcal{K})$ be?

Let us put these serious problems aside, settle on a particular irreducible representation of the CCRs somehow (as we have pointed out, they do exist), and simply push on as if we had not just had premonitions of disaster. Thus we are given an irreducible representation of the relations (7) on a Hilbert space H. The arguments we have given for deriving the form of the quantum Hamiltonian H (by way of Ehrenfest's theorem, for example) are still valid here, so let us try to see how H should be defined.

Consider the CCRs as defined in equation (7), and let us proceed formally in order to get a feeling for the problems we face. Given a pair U, V satisfying (7) it can be proved that there are linear mappings Q and P, from functions in $\mathcal{K}$ to unbounded self-adjoint operators on H such that

$$U(f) = e^{iQ(f)}, \qquad V(f) = e^{iP(f)}, \qquad f \in \mathcal{K}.$$

$Q(f)$ and $P(f)$ satisfy the CCRs in Heisenberg's form:

$$\sqrt{-1}(P(f)Q(g) - Q(g)P(f)) = \langle f, g\rangle \, \mathbf{1}, \tag{8}$$

where $\langle\cdot,\cdot\rangle$ represents the given inner product in $\mathcal{K}$

$$\langle u, v\rangle = \int_{\mathbb{R}^3} u(x,y,z)v(x,y,z)\,dx\,dy\,dz.$$

This much can be made completely rigorous. But if we examine equation (8) more carefully, we find unexpected surprises.

For example, let us assume for the moment that the linear mappings Q and P can be expressed in terms of operator functions $Q(x,y,z)$ and $P(x,y,z)$ defined on $\mathbb{R}^3$ in the following way:

$$Q(f) = \int_{\mathbb{R}^3} f(x,y,z)Q(x,y,z)\,dx\,dy\,dz$$
$$P(f) = \int_{\mathbb{R}^3} f(x,y,z)P(x,y,z)\,dx\,dy\,dz.$$

A formal computation shows that the only way these "functions" can satisfy equation (8) is if they satisfy

$$\sqrt{-1}(P(x,y,z)Q(x',y',z') - Q(x',y',z')P(x,y,z)) = \delta(x-x', y-y', z-z')\mathbf{1}, \tag{9}$$

where δ denotes the delta function in three variables.

Equation (9) is very troublesome. It implies that $Q(x,y,z)$ and $P(x,y,z)$ cannot exist as operator valued functions...even unbounded operator valued functions. They can exist only as "operator valued distributions". For our purposes, what the latter means is that it is impossible to assign a "value" to $Q(x,y,z)$ at a particular point (x,y,z) of $\mathbb{R}^3$; it is only meaningful to assign a value to an integral

$$\int_{\mathbb{R}^3} f(x,y,z)Q(x,y,z)\,dx\,dy\,dz$$

where f is a smooth function concentrated near the given point.

Finally, let us now go to the problem of writing down a formula for the Hamiltonian corresponding to this quantum field. If we think through what we did for quantum mechanics and the formula for the potential that we found in equation (4), then we arrive at the following "formula" for the Hamiltonian that goes with the given representation Q, P of the CCRs:

$$H = \int_{\mathbb{R}^3} [1/2P(x,y,z)^2 + 1/2\nabla Q(x,y,z)^2 + \alpha Q(x,y,z)^2/2 + \beta Q(x,y,z)^4/4]\,dx\,dy\,dz, \tag{10}$$

where $\nabla Q(x,y,z)^2$ denotes the sum of the squares of the three partial derivatives of the "function" $Q(x,y,z)$.

In fact, the previous formula is obviously the "correct" Hamiltonian, in the sense that it satisfies the same formal requirements as the Schrödinger operator of quantum mechanics. However, the formula is meaningless! We have seen above that the values of $Q(x,y,z)$ and $P(x,y,z)$ do not really exist at a point $(x,y,z) \in \mathbb{R}^3$, and only exist if they are "smoothed" by integrating them with a function (in technical language, Q and P exist as operator valued distributions but do not exist as operator valued functions). How does one raise $Q(x, y, z)$ to the fourth power if $Q(x, y, z)$ does not exist? But even if $Q(x, y, z)^4$ did exist as an operator valued function, what could an integral like

$$\int_{\mathbb{R}^3} Q(x, y, z)^4 \, dx \, dy \, dz$$

possibly mean? Surely $Q(x, y, z)^4$ does not die out when (x, y, z) is large...which it would have to do if its integral is to make sense. Thus, even if $Q(x, y, z)^4$ were to exist (which it doesn't), it couldn't be integrable.

The implications of these observations are very serious: quantities like $Q(x, y, z)^2$ and $Q(x, y, z)^4$ in the integrand are meaningless, their integrals are meaningless, and therefore the preceding integral "formula" for H makes no sense at all. Thus there is no Hamiltonian operator H. Since there is no Hamiltonian, there is no chance of proving that it generates a one-parameter group of unitary operators, hence there is no dynamical group and no quantum field.

Here is a good problem for 21st century mathematicians: show that this program for quantizing nonlinear PDEs like equation (1) can be carried out in four spacetime dimensions. In other words, show that Quantum Field Theory exists in rigorous mathematical terms.

Physics changes constantly, occasionally abruptly and profoundly. It is conceivable that physics will change so much that it will no longer be necessary to solve mathematical problems like this one...simply because the "new" physics will go around them. I suppose that is conceivable, but nature has not been that accommodating in the past. Rather, the history of physics has been otherwise; nature has caused us to rethink the fundamental principles we use to understand nature in order to explain the paradoxes that nature presents.

I will conclude with a few comments about the history of work on this problem of "constructive quantum field theory". I think it is fair to say that there is no evidence that Quantum Field Theory exists as a mathematical object. We have demonstrated the source and nature of the problems above. It is and was possible for physicists to carry out calculations that predicted the results of experiments accurately, based largely on formalism (involving so-called Feynman integrals and the diagrams that are associated with them which relate to formula (10) and which are "obviously" correct...but which lack mathematical foundation). But the fundamental mathematical ideas are missing. What is a particle? What is a wave? What is a quantum field? What is the dynamical group of a quantum field? Should it be implemented by a one-paramteter group of unitary operators acting on a Hilbert space, or should it be simply a one-paramter group of autormophisms of a C^*-algebra that somehow represents the algebra of "local observables"?

Irving Segal began to ask these hard questions in the 1950s. He asked for firm mathematical definitions, and sought to give complete proofs (constructive proofs, if that was appropriate) for everything. It is interesting to me that few mathematicians noticed the importance of this early work of Segal's on the most fundamental problems connecting mathematics with physics. Nowadays, it is very fashionable for mathematicians to pay lip service to physics; and it is especially interesting to me that the fundamental problems involving the Hilbert space quantization of nonlinear classical fields are still there; and many of the mathematicians who nod in this particular way to physics do not address these fundamental issues.

In the sixties, several talented mathematicians began to work on Segal's program for providing a basis for QFT (acronym service: QFT = quantum field theory). I am personally most familiar with the work of James Glimm, Arthur Jaffe, and Edward Nelson. Nelson's work, by analytically continuing real time into imaginary time, opened the door to a mathematically sound "Euclidean" approach to QFT, which remains a powerful method applicable far beyond its original goal...to quantum gravity for example. Glimm and Jaffe proved, by way of very ingenious arguments, that the Hamiltonian associated with a quartic interaction in two spacetime dimensions is self-adjoint and therefore generates a one-paramter unitary group (and a quantum field). There have been many other talented people who have made significant contributions, and this work continues.

To date, however, the fundamental problem remains completely open. The work of Glimm and Jaffe provides a rigorous mathematical proof of the existence of quartic interactions in two dimensions (one time dimension one space dimension), and there is some progress in three spacetime dimensions. But for the physically realistic case of four spacetime dimensions, there is (essentially) nothing.

Disclaimer and Acknowledgements. These notes have been hastily thrown together, and have not been carefully proofread. Nor have I made any effort to make systematic references to important work. If I have offended colleagues by these sins of omission, I am truly sorry. Finally, I want to thank Paul Chernoff, who read a preliminary version of these notes carefully and stopped me from making some foolish mistakes.