p-SUMMABLE COMMUTATORS IN DIMENSION d

WILLIAM ARVESON

Abstract. We show that many invariant subspaces M for d-shifts $(S_1, \dots, S_d)$ of finite rank have the property that the orthogonal projection P_M onto M satisfies

$$P_M S_k - S_k P_M \in \mathcal{L}^p, \qquad 1 \leq k \leq d$$

for every $p > 2d$, $\mathcal{L}^p$ denoting the Schatten-von Neumann class of all compact operators having p-summable singular value lists. In such cases, the d tuple of operators $\bar{T} = (T_1, \dots, T_d)$ obtained by compressing $(S_1, \dots, S_d)$ to $M^\perp$ generates a $*$-algebra whose commutator ideal is contained in $\mathcal{L}^p$ for every $p > d$.

It follows that the C^*-algebra generated by $\{T_1, \dots, T_d\}$ and the identity is commutative modulo compact operators, the Dirac operator associated with $\bar{T}$ is Fredholm, and the index formula for the curvature invariant is stable under compact perturbations and homotopy for this restricted class of finite rank d-contractions. Though this class is limited, we conjecture that the same conclusions persist under much more general circumstances.

1. Introduction

The purpose of this paper is to establish a result in higher dimensional operator theory that supports a general conjecture about the stability of the curvature invariant under compact perturbations and homotopy. Specifically, we show that certain finite rank pure d-contractions $\bar{T} = (T_1, \dots, T_d)$ have the property that the C^*-algebra generated by $\{T_1, \dots, T_n\}$ is commutative modulo compact operators. It follows that the Dirac operator associated with such a d-contraction is Fredholm, a key fact that leads to the desired stability properties for the curvature invariant by way of an index formula, see (1.2) below. These results represent a first step toward developing an effective Fredholm theory of d-contractions, an ingredient necessary for completing the index theorem that was partially established in [Arv02], following up on [Arv00].

We first describe the issues that prompted this work. We use the term *multioperator* (of complex dimension $d = 1, 2, \dots$) to denote a d-tuple $\bar{T} = (T_1, \dots, T_d)$ of mutually commuting bounded operators acting on a common Hilbert space H. Every multioperator $\bar{T}$ gives rise to an associated Dirac operator, whose definition we recall for the reader's convenience (see [Arv02]

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for more detail). Let Z be a complex Hilbert space of dimension d and let ΛZ be the exterior algebra of Z, namely the direct sum of finite-dimensional Hilbert spaces

$$\Lambda Z = \sum_{k=0}^{d} \Lambda^k Z,$$

where $\Lambda^k Z$ denotes the kth exterior power of Z, and where $\Lambda^0 Z$ is defined to be the one-dimensional Hilbert space $\mathbb{C}$. For $k = 1, \dots, d$, $\Lambda^k Z$ is spanned by wedge products of the form $z_1 \wedge z_2 \wedge \cdots \wedge z_k$, $z_i \in Z$, and the inner product in $\Lambda^k Z$ is uniquely determined by the formula

$$\langle z_1 \wedge z_2 \wedge \cdots \wedge z_k, w_1 \wedge w_2 \wedge \cdots \wedge w_k \rangle = \det(\langle z_i, w_j \rangle),$$

the right side denoting the determinant of the $k \times k$ matrix of inner products $\langle z_i, w_j \rangle$, $1 \le i, j \le k$. If we choose an orthonormal basis $e_1, \dots, e_d$ for Z, then there is a sequence of creation operators $C_1, \dots, C_d$ on ΛZ that are defined uniquely by their action on generating vectors as follows

$$C_i : z_1 \wedge \cdots \wedge z_k \mapsto e_i \wedge z_1 \cdots \wedge z_k,$$

$z_1, \dots, z_k \in Z$, $1 \le k \le d$, where each C_i maps $\lambda \in \Lambda^0 Z = \mathbb{C}$ to λe_i and maps the last summand $\Lambda^d Z$ to $\{0\}$. The operators $C_1, \dots, C_d$ satisfy the canonical anticommutation relations

$$C_i C_j + C_j C_i = 0, \qquad C_i^* C_j + C_j C_i^* = \delta_{ij} \mathbf{1}.$$

The Dirac operator of $\bar{T}$ is a self-adjoint operator D acting on the Hilbert space $\tilde{H} = H \otimes \Lambda Z$ as follows: $D = B + B^*$, where B is the sum

$$B = T_1 \otimes C_1 + T_2 \otimes C_2 + \cdots + T_d \otimes C_d.$$

If one replaces $e_1, \dots, e_d$ with a different orthonormal basis for Z, then of course one changes D; but the two Dirac operators are naturally isomorphic in a sense that we will not spell out here (see [Arv02]). Thus the Dirac operator of $\bar{T}$ is uniquely determined by $\bar{T}$ up to isomorphism.

The multioperator $\bar{T}$ is said to be *Fredholm* if its Dirac operator D is a Fredholm operator. Since D is self-adjoint, this simply means that it has closed range and finite-dimensional kernel. In this case there is an integer invariant associated with $\bar{T}$, called the *index*, that is defined as follows. Consider the natural $\mathbb{Z}_2$-grading of $\tilde{H}$, defined by the orthogonal decomposition $\tilde{H} = \tilde{H}_+ \oplus \tilde{H}_-$, where

$$\tilde{H}_+ = \sum_{k \text{ even}} H \otimes \Lambda^k Z, \qquad \tilde{H}_- = \sum_{k \text{ odd}} H \otimes \Lambda^k Z.$$

One finds that D is an odd operator relative to this grading in the sense that $D\tilde{H}_+ \subseteq \tilde{H}_-$, $D\tilde{H}_- \subseteq \tilde{H}_+$. Thus the decomposition $\tilde{H} = \tilde{H}_+ \oplus \tilde{H}_-$ gives rise to a 2×2 matrix representation

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix},$$

D_+ denoting the restriction of D to $\tilde{H}_+$. When D is Fredholm, one finds that $D_+\tilde{H}_+$ is a closed subspace of $\tilde{H}_-$ of finite codimension and D_+ has finite dimensional kernel. Indeed, $\bar{T}$ is a Fredholm multioperator iff D_+ is a Fredholm operator in $\mathcal{B}(\tilde{H}_+, \tilde{H}_-)$. The index of D_+, namely

$$\operatorname{ind}(D_+) = \dim(\ker D \cap \tilde{H}_+) - \dim(\tilde{H}_-/D\tilde{H}_+),$$

is an integer invariant for Fredholm multioperators that is stable under compact perturbations and homotopy.

By analogy with the index theorems of Atiyah and Singer, one might expect that the computation of the index of Fredholm multioperators will lead to important relations between the geometric and analytic properties of multioperators, and perhaps connect with basic issues of algebraic geometry. Such a program requires that one should have effective tools for a) determining when a given Dirac operator is Fredholm and b) computing the index in terms of concrete geometric properties of its underlying multioperator. Some progress has been made in the direction of b), and we will describe that in Remark 1.2 below. However, the problem a) of proving that the natural examples of multioperators are Fredholm remains largely open. It is that problem we want to address in this paper.

Here is a useful sufficient criterion for Fredholmness; we reiterate the proof given in [Arv03] for the reader's convenience.

Proposition 1.1. *Let $\bar{T} = (T_1, \dots, T_d)$ be a multioperator satisfying*

(i) *$\bar{T}$ is essentially normal in that all self-commutators $T_kT_j^* - T_j^*T_k$ are compact, $1 \le j, k \le d$, and*
(ii) *$T_1T_1^* + \cdots + T_dT_d^*$ is a Fredholm operator.*

Then $\bar{T}$ is a Fredholm multioperator.

Proof. Since the Dirac operator D of $\bar{T}$ is self-adjoint, it suffices to show that D^2 is a Fredholm operator. To that end, consider $B = T_1 \otimes C_1 + \cdots + T_d \otimes C_d$. Since T_j commutes with T_k and C_j anticommutes with C_k, a straightforward computation shows that $B^2 = 0$. Hence

$$D^2 = (B + B^*)^2 = B^*B + BB^* = \sum_{k,j=1}^{d} T_k^*T_j \otimes C_k^*C_j + \sum_{k,j=1}^{d} T_jT_k^* \otimes C_jC_k^*.$$

Using $C_jC_k^* = \delta_{jk}\mathbf{1} - C_k^*C_j$, we can write the second term on the right as

$$F \otimes \mathbf{1} - \sum_{k,j=1}^{d} T_jT_k^* \otimes C_k^*C_j,$$

where $F = T_1T_1^* + \cdots + T_dT_d^*$, so that

$$D^2 = F \otimes \mathbf{1} + \sum_{k,j=1}^{d} (T_k^*T_j - T_jT_k^*) \otimes C_k^*C_j.$$

Since $F \otimes \mathbf{1}$ is a Fredholm operator by (ii) and each summand in the second term is compact by (i), it follows that D^2 is a Fredholm operator. □

Remark 1.2 (Finite rank d-contractions). We are primarily concerned with finite rank d-contractions, that is, multioperators $\bar{T} = (T_1, \dots, T_d)$ that define row contractions in the sense that $T_1T_1^* + \cdots + T_dT_d^* \leq \mathbf{1}$, whose defect operators $\mathbf{1} - T_1T_1^* - \cdots - T_dT_d^*$ have finite rank. $\bar{T}$ is said to be *pure* if the powers of the completely positive map $\phi(X) = \sum_k T_kXT_k^*$ satisfy $\phi^n(\mathbf{1}) \downarrow 0$ as $n \to \infty$.

Proposition 1.1 implies that the Dirac operator of a finite rank d-contraction is Fredholm provided that the self-commutators $T_kT_j^* - T_j^*T_k$ are compact for all $1 \leq k, j \leq d$. In that case, the C^*-algebra $C^*(T_1, \dots, T_d)$ generated by $\{T_1, \dots, T_d\}$ and the identity operator is commutative modulo compact operators $\mathcal{K} \subseteq \mathcal{B}(H)$, and we have an exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(T_1, \dots, T_d) + \mathcal{K} \longrightarrow C(X) \longrightarrow 0, \tag{1.1}$$

X being a compact subset of the unit $(2d-1)$-sphere in $\mathbb{C}^d$.

For every finite rank d-contraction $\bar{T}$ it is possible to define a real number $K(\bar{T})$ in the interval $[0, \operatorname{rank} \bar{T}]$, called the curvature invariant. $K(\bar{T})$ is a geometric invariant of $\bar{T}$, defined as the integral of the trace of a certain matrix-valued function over the unit sphere in $\mathbb{C}^d$ (see [Arv00] for more detail). $K(\bar{T})$ was computed for many examples in [Arv00], and it was found to be an integer, namely the Euler characteristic of a certain finitely generated module over the polynomial ring $\mathbb{C}[z_1, \dots, z_d]$. However, it was also shown in [Arv00] that this formula equating $K(\bar{T})$ to an Euler characteristic fails to hold in general. Still, that formula provided enough evidence to lead us to conjecture that $K(\bar{T})$ is an integer in general, and that has now been established by Greene, Richter and Sundberg in [GRS02]. Unfortunately, the integer arising in the proof of the latter result - namely the rank of an almost-everywhere constant rank projection defined on the unit sphere of $\mathbb{C}^d$ - appeared to have no direct connection with spatial properties of the underlying d-tuple of operators $T_1, \dots, T_d$. What was still lacking was a formula that relates $K(\bar{T})$ to some natural integer invariant of $\bar{T}$ that holds in general and is, hopefully, easy to compute.

Such considerations led us to initiate a new approach in [Arv02]. Our aim was to introduce an appropriate notion of Dirac operator and to seek a formula that would relate the curvature invariant to the index of the associated Dirac operator - hopefully in general. Assuming that $\bar{T}$ is a *graded* d-contraction, then the result of [Arv02] is that both $\ker D_+$ and $\ker D_+^*$ are finite-dimensional, and moreover

$$(-1)^d K(\bar{T}) = \dim \ker D_+ - \dim \ker D_+^*. \tag{1.2}$$

Significantly, there are no known exceptions to this formula; in particular, it persists for the (ungraded) examples that violated the previous formula that related $K(\bar{T})$ to an Euler characteristic. As we have described in [Arv02], it is natural to view (1.2) as an operator-theoretic counterpart of the Gauss-Bonnet-Chern formula of Riemannian geometry in its modern dress as an index theorem (see page 311 of [GM91]).

Notice, however, that even in the graded cases the right side of (1.2) is unstable if D is not a Fredholm operator, despite the fact that both subspaces $\ker D_+$ and $\ker D_+^*$ must be finite-dimensional. On the other hand, if D is a Fredholm operator then (1.2) reduces to a stable formula

$$(-1)^d K(\bar{T}) = \operatorname{ind} D_+. \tag{1.3}$$

In view of Proposition 1.1 and the stability properties of the index of Fredholm operators one may conclude: *Within the class of finite rank graded d-contractions $\bar{T}$ whose self-commutators are compact, the curvature invariant $K(\bar{T})$ is stable under compact perturbations and homotopy.*

Unfortunately, it is not known if the self-commutators of pure finite-rank graded d-contractions are always compact, though we believe that they are. More generally, we believe that the Dirac operator of any pure finite rank d-contraction $\bar{T}$ – graded or not – is a Fredholm operator and, moreover, that formula (1.3) continues to hold in that generality. Precise formulations of these conjectures will be found in Section 5. We will prove the most tractable cases of the first of these two conjectures in Sections 2 and 3.

2. Statement of Results

Let $\bar{S} = (S_1, \dots, S_d)$ be the d-shift of rank one, i.e., the multioperator that acts on the symmetric Fock space $H^2(\mathbb{C}^d)$ over the d-dimensional one-particle space $\mathbb{C}^d$ by symmetric tensoring with a fixed orthonormal basis $e_1, \dots, e_d$ for $\mathbb{C}^d$. We lighten notation by writing H^2 for $H^2(\mathbb{C}^d)$, the dimension d being a positive integer (normally larger than 1) that will be fixed throughout. The elements of H^2 can be realized as certain holomorphic functions defined in the open unit ball

$$B_d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z| = (|z_1|^2 + \cdots + |z_d|^2)^{1/2} < 1\},$$

and in this function-theoretic realization the rank one d-shift is the d-tuple of multiplication operators $S_k : f(z) \mapsto z_k f(z)$, $1 \le k \le d$ (see [Arv98]).

Let r be a positive integer, let E be a Hilbert space of dimension r, and consider the d-tuple of operators defined defined on $H^2 \otimes E$ by

$$S_k \otimes \mathbf{1}_E : f \otimes \zeta \mapsto z_k f \otimes \zeta, \qquad 1 \le k \le d.$$

It will be convenient to overwork notation by writing $\bar{S} = (S_1, \dots, S_d)$ for these operators as well, and to refer to that multioperator as the d-shift of rank r. Thus the d-shift $(S_1, \dots, S_d)$ of rank r acts as follows on elements of the form $f \otimes \zeta$, with $f \in H^2$ and $\zeta \in E$:

$$S_k : f \otimes \zeta \mapsto z_k f \otimes \zeta, \qquad 1 \le k \le d.$$

The d-shift is known to be universal in the class of pure d-contractions in the sense that *every pure d-contraction $\bar{T} = (T_1, \dots, T_d)$ of rank r is* *unitarily equivalent to one obtained by compressing the d-shift of rank r to the orthogonal complement of an invariant subspace* (see [Arv98]).

We will make use of the natural partial ordering on the discrete abelian group $\mathbb{Z}^d$; for $m = (m_1, \dots, m_d), n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ we write $m \leq n$ if $m_k \leq n_k$ for every $k = 1, \dots, d$. For every $n \geq 0$ in $\mathbb{Z}^d$ there is a monomial in H^2 defined by

$$z^n = z_1^{n_1} z_2^{n_2} \dots z_d^{n_d}.$$

The set of all monomials $\{z^n : n \geq 0\}$ form an orthogonal (but not orthonormal) set which spans H^2. Similarly, an element $\xi \in H^2 \otimes E$ having the particular form $\xi = z^n \otimes \zeta$ where $n \geq 0$ and $\zeta \in E$ is called a *monomial* in $H^2 \otimes E$. Notice that monomials of the form $z^m \otimes \eta$ and $z^n \otimes \zeta$ are orthogonal if $m \neq n$, or if $m = n$ and $\eta \perp \zeta$. Obviously, $H^2 \otimes E$ is spanned by monomials.

Theorem 2.1. *Let $M \subseteq H^2 \otimes \mathbb{C}^r$ be an invariant subspace for the d-shift of rank r that is generated as an invariant subspace by any set of monomials, and let P_M be the orthogonal projection onto M. Then for every $p > 2d$,*

$$P_M S_k - S_k P_M \in \mathcal{L}^p, \qquad 1 \leq k \leq d.$$

Corollary 2.2. *Let $M \subseteq H^2 \otimes \mathbb{C}^r$ be an invariant subspace satisfying the hypotheses of Theorem 2.1, let $\bar{T} = (T_1, \dots, T_d)$ be the d-contraction obtained by compressing $\bar{S}$ to the Hilbert space $(H^2 \otimes E)/M \cong M^\perp$, and let $\mathcal{A}$ be the $*$-algebra generated by $\{T_1, \dots, T_d\}$ and the identity. Then*

(i) *the commutator ideal of $\mathcal{A}$ is contained in $\mathcal{L}^p$ for every $p > d$, and*
(ii) *$\bar{T}$ is a Fredholm multioperator.*

Remark 2.3 (Structure of quotient modules). Consider the two dimensional rank one case $d = 2$, $E = \mathbb{C}$. The simplest nontrivial example of an invariant subspace satisfying the hypotheses of Theorem 2.1 is the subspace $M \subseteq H^2(\mathbb{C}^2)$ generated by the single monomial $f(z_1, z_2) = z_1 z_2$. In this case it is possible to compute the operators $T_1, T_2 \in \mathcal{B}(H^2/M)$ in explicit terms. Once that is done, one can verify directly that for this example the self-commutators $T_i^* T_j - T_j T_i^*$, $1 \leq i, j \leq 2$, are in $\mathcal{L}^p$ for every $p > 2$, so that Proposition 1.1 implies that (T_1, T_2) is a Fredholm pair.

However, for more general examples the quotient modules H^2/M are not recognizable (consider the invariant subspace $M \subseteq H^2(\mathbb{C}^3)$ generated by the two monomials $z_1 z_2$ and $z_2 z_3^2$) . Thus our proof of Theorem 2.1 and Corollary 2.2 is based on different ideas.

Remark 2.4 (Structure of the algebraic set X). Let $f_1, \dots, f_s$ be a set of homogeneous polynomials in $\mathbb{C}[z_1, \dots, z_d]$ and consider the closed invariant subspace $M \subseteq H^2(\mathbb{C}^d)$ that they generate. Let $\bar{T} = (T_1, \dots, T_d)$ be the rank-one d-contraction associated with the quotient H^2/M. Assuming that the self-commutators $T_j^* T_k - T_k T_j^*$ are compact, $1 \leq j, k \leq d$, then we have an exact sequence of C^*-algebras (1.1) that terminates in $C(X)$, where X is the projective algebraic set defined by the common zeros of $f_1, \dots, f_s$.

However, if the f_k are all monomials then X is trivial - a union of coordinate axes. Thus the Fredholm d-tuples provided by Corollary 2.2 fail to make significant connections with algebraic geometry.

3. Proofs.

Turning now to the proof of Theorem 2.1 and Corollary 2.2, let $E = \mathbb{C}^r$ and let $M \subseteq H^2 \otimes E$ be an invariant subspace generated by some set of monomials. We have to show that the commutators

$$[P_M, S_k] = P_M S_k - S_k P_M, \qquad 1 \leq k \leq d$$

belong to $\mathcal{L}^p$ for $p > 2d$. Because of the obvious symmetry we treat only the case $k = 1$. We will make repeated use of the following elementary property of the Schatten-von Neumann classes $\mathcal{L}^p$: For any pair of Hilbert spaces H_1, H_2, any operator $B \in \mathcal{B}(H_1, H_2)$, and any $p \geq 1$ one has

$$B \in \mathcal{L}^{2p} \iff B^* B \in \mathcal{L}^p \iff BB^* \in \mathcal{L}^p. \tag{3.1}$$

Thus, in order to prove Theorem 2.1 it suffices to show that the operator

$$B = [P_M, S_1]^* = S_1^* P_M - P_M S_1^* = (\mathbf{1} - P_M) S_1^* P_M$$

satisfies $B^* B \in \mathcal{L}^p$ for every $p > d$. Equivalently, we will prove:

Theorem 3.1. *Let A be the restriction of the operator $(\mathbf{1} - P)S_1^*$ to M. Then $A^*A \in \mathcal{L}^p$ for every $p > d$.*

Consider the natural decomposition of M induced by S_1

$$M = \overline{S_1 M} \oplus (M \ominus S_1 M).$$

Since $S_1 M \perp M \ominus S_1 M$, we have

$$S_1^*(M \ominus S_1 M) \subseteq M^\perp, \tag{3.2}$$

and therefore both S_1^* and $(\mathbf{1} - P_M)S_1^*$ restrict to the same operator on $M \ominus S_1 M$. Thus it suffices to establish the following two results, the principal one being Lemma 3.3.

Lemma 3.2. *$S_1^* S_1$ leaves M invariant, hence the restriction of $(\mathbf{1} - P_M)S_1^*$ to $\overline{S_1 M}$ is zero.*

Lemma 3.3. *The restriction of S_1^* to $M \ominus S_1 M$ belongs to $\mathcal{L}^{2p}$ for every $p > d$.*

We first bring in an action of the compact group $\mathbb{T}^d$ that will be useful. The full unitary group of the one-particle space $\mathbb{C}^d$ acts naturally as unitary operators on $H^2 = H^2(\mathbb{C}^d)$, and by restricting that representation to the abelian subgroup of all unitary operators which are diagonal relative to the usual orthonormal basis for $\mathbb{C}^d$, one obtains a strongly continuous unitary representation of the d-dimensional torus $\Gamma_0 : \mathbb{T}^d \mapsto \mathcal{B}(H^2)$. In more explicit terms, if we consider the elements of $H^2(\mathbb{C}^d)$ as holomorphic functions defined on the open unit ball $B_d \subseteq \mathbb{C}^d$, the action of Γ_0 is given by

$$\Gamma_0(\lambda) : f(z_1, \ldots, z_d) \mapsto f(\lambda_1 z_1, \ldots, \lambda_d z_d), \qquad \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{T}^d.$$

By increasing the multiplicity appropriately, we obtain a corresponding representation $\Gamma : \mathbb{T}^d \to \mathcal{B}(H^2 \otimes E)$,

$$\Gamma(\lambda) = \Gamma_0(\lambda) \otimes \mathbf{1}_E, \qquad \lambda \in \mathbb{T}^d. \tag{3.3}$$

We have the following relations between Γ and the rank r d-shift

$$\Gamma(\lambda) S_k \Gamma(\lambda)^* = \lambda_k S_k, \qquad k = 1, \dots, d, \quad \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{T}^d. \tag{3.4}$$

The character group of $\mathbb{T}^d$ is the discrete abelian group $\mathbb{Z}^d$, an element $n = (n_1, \dots, n_d)$ in $\mathbb{Z}^d$ being associated with the following character of $\mathbb{T}^d$,

$$\lambda \mapsto \lambda^n = \lambda_1^{n_1} \dots \lambda_d^{n_d} \in \mathbb{T}, \qquad \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{T}^d.$$

Notice that every monomial $z^n \otimes \zeta$, $n \geq 0$ is an eigenvector of Γ,

$$\Gamma(\lambda)(z^n \otimes \zeta) = \lambda^n \cdot z^n \otimes \zeta, \qquad \lambda \in \mathbb{T}^d.$$

Conversely, if $\xi \in H^2 \otimes E$ satisfies $\Gamma(\lambda)\xi = \lambda^n \xi$ for every $\lambda \in \mathbb{T}^d$ and if $\xi \neq 0$ then we must have $n \geq 0$ and ξ must be a monomial of degree n.

The role of Γ is described in the following proposition.

Proposition 3.4. *For any closed subspace $M \subseteq H^2 \otimes E$ that is invariant under $S_1, \dots, S_d$, the following are equivalent.*

(i) *M is generated (as a closed $\{S_1, \dots, S_d\}$-invariant subspace) by a set of monomials in $H^2 \otimes E$.*
(ii) *$\Gamma(\lambda) M \subseteq M$, for every $\lambda \in \mathbb{T}^d$.*

Proof. (i) $\implies$ (ii): Obviously, operators of the form $S_1^{n_1} S_2^{n_2} \dots S_d^{n_d}$, with $n_1, \dots, n_k \geq 0$ must map monomials in $H^2 \otimes E$ to other monomials in $H^2 \otimes E$, and since monomials in $H^2 \otimes E$ form one-dimensional Γ-invariant subspaces, it follows that any invariant subspace M that is generated by a set of monomials must also be invariant under the action of Γ.

(ii) $\implies$ (i): Any closed linear subspace $M \subseteq H^2 \otimes E$ that is invariant under the action of Γ must be spanned by its spectral subspaces

$$M(n) = \{\xi \in M : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}^d\},$$

for $n \in \mathbb{Z}^d$. We have already pointed out that if such a subspace $M(n)$ is not $\{0\}$ then one must have $n \geq 0$, and that it must have the form

$$M(n) = z^n \otimes E_0 = \{z^n \otimes \zeta : \zeta \in E_0\},$$

E_0 being some subspace of E. Thus M is spanned by the monomials it contains, and in particular it is generated as in (i). □

proof of Lemma 3.2. Proposition 3.4 implies that M is invariant under the von Neumann algebra generated by the range $\Gamma(\mathbb{T}^d)$ of Γ, and thus it suffices to show that $S_1^* S_1$ belongs to that algebra. Because of the double commutant theorem it is enough to show that for every operator T satisfying

$$\Gamma(\lambda) T = T \Gamma(\lambda), \qquad \lambda \in \mathbb{T}^d,$$

we have $T S_1^* S_1 = S_1^* S_1 T$.

Now $H^2 \otimes E$ decomposes into an orthogonal direct sum of spectral subspaces for Γ, namely the subspaces of the form $z^n \otimes E$ where $n \in \mathbb{Z}^d$ satisfies $n \geq 0$, and by virtue of its commutation relation with Γ, T must leave each of these subspaces invariant. Thus there is a sequence of operators $T_n \in \mathcal{B}(E)$, $n \in \mathbb{Z}^d$, $n \geq 0$ such that the restriction of T to $z^n \otimes E$ is given by

$$T(z^n \otimes \zeta) = z^n \otimes T_n\zeta, \qquad \zeta \in E. \tag{3.5}$$

Consider now the action of $S_1^* S_1$ on the spectral subspace $z^n \otimes E$. Writing $n = (n_1, \dots, n_d)$ with $n_k \geq 0$ we have

$$S_1^* S_1(z^n \otimes \zeta) = S_1^*(z_1^{n_1+1} z_2^{n_2} \dots z_d^{n_d} \otimes \zeta) = (S_1^*(z_1^{n_1+1} z_2^{n_2} \dots z_d^{n_d})) \otimes \zeta.$$

Using formula (3.9) of [Arv98] we have

$$S_1^*(z_1^{n_1+1} z_2^{n_2} \dots z_d^{n_d}) = \frac{n_1 + 1}{|n| + 1} \cdot z_1^{n_1} z_2^{n_2} \dots z_d^{n_d},$$

where $|n|$ denotes $n_1 + n_2 + \dots + n_d$. Thus

$$S_1^* S_1(z^n \otimes \zeta) = \frac{n_1 + 1}{|n| + 1} \cdot (z^n \otimes \zeta).$$

Thus the restriction of $S_1^* S_1$ to each spectral subspace of Γ is a scalar multiple of the identity; because of (3.5), $S_1^* S_1$ must commute with T. $\square$

proof of Lemma 3.3. We prove Lemma 3.3 in two assertions as follows.

Assertion 1. *There is a positive integer q such that*

$$M \ominus S_1 M \subseteq \overline{span}\{z_1^{n_1} z_2^{n_2} \dots z_d^{n_d} \otimes \zeta : (n_1, n_2, \dots, n_d) \geq 0,\ n_1 \leq q,\ \zeta \in E\}.$$

Assertion 2. *For every positive integer q the restriction B of S_1^* to*

$$\overline{span}\{z^n \otimes \zeta : n \geq 0, \quad n_1 \leq q, \quad \zeta \in E\}$$

satisfies $B^* B \in \mathcal{L}^p$ for every $p > d$.

proof of Assertion 1. We remark first that M is finitely generated in the sense that there is a finite set F of monomials in M such that

$$M = \overline{\text{span}}\{f(S_1, \dots, S_d)\xi : \xi \in F, \quad f \in \mathbb{C}[z_1, \dots, z_d]\}.$$

Indeed, let M_0 be the (nonclosed) linear span of the monomials in M. M_0 is dense in M and invariant under the action of all polynomials in $S_1, \dots, S_d$. Thus M_0 is a submodule of $\mathbb{C}[z_1, \dots, z_d] \otimes E$ (where the latter is considered a finitely generated module over the algebra of polynomials $\mathbb{C}[z_1, \dots, z_d]$). By Hilbert's basis theorem (in the form which asserts that a submodule of a finitely generated $\mathbb{C}[z_1, \dots, z_d]$-module is finitely generated), it follows that there is a set of polynomials $f_1, \dots, f_s$ and a set $\zeta_1, \dots, \zeta_s \in E$ such that M_0 is generated by $\{f_1 \otimes \zeta_1, \dots, f_s \otimes \zeta_s\}$. Using the invariance of M_0 under Γ we can decompose each $f_j \otimes \zeta_j$ into a finite sum of monomials in M_0 to obtain the required finite set of generators for M.

We conclude that there is a finite set of d-tuples $\{\nu_1, \dots, \nu_s\}$ in $\mathbb{Z}^d$, satisfying $\nu_j \geq 0$, $\nu_i \neq \nu_j$ for $i \neq j$, and a finite set of subspaces $E_1, \dots, E_s \subseteq E$ with the property that M is generated as follows

$$M = \overline{span}\{f_1(z)z^{\nu_1} \otimes \zeta_1 + \dots + f_s(z)z^{\nu_s} \otimes \zeta_s\}, \tag{3.6}$$

where $f_1, \dots, f_s$ range over $\mathbb{C}[z_1, \dots, z_d]$ and $\zeta_j \in E_j$.

We must identify the spectral subspaces of M

$$M(n) = \{\xi \in M : \Gamma(\lambda)\xi = \lambda^n \xi, \quad \lambda \in \mathbb{T}^d\}, \qquad n \in \mathbb{Z}^d$$

in terms of $\nu_1, \dots, \nu_s$ and $E_1, \dots, E_s$. This requires some care since the spaces $E_1, \dots, E_s$ need not be mutually orthogonal.

Lemma 3.5. *For every $n \in \mathbb{Z}^d$ and every $k = 1, \dots, s$ define $E_k(n) = E_k$ if $\nu_k \leq n$, $E_k(n) = \{0\}$ if $\nu_k \not\leq n$, and let $E(n) = E_1(n) + \dots + E_s(n)$. Then $M(n) = \{0\}$ except when $n \geq 0$, and for $n \geq 0$ we have*

$$M(n) = z^n \otimes E(n) = \{z^n \otimes \zeta : \zeta \in E(n)\}.$$

Proof. Clearly $M(n) = \{0\}$ except when $n \geq 0$, so fix $n \geq 0$ and let $Q(n)$ be the orthogonal projection of M onto $M(n)$. We show that $Q(n)M = z^n \otimes E(n)$.

For each $k = 1, \dots, s$ let $M_k \subseteq M$ be the invariant subspace generated by $z^{\nu_k} \otimes E_k$,

$$M_k = \overline{span}\{z^p \otimes E_k : p \geq \nu_k\}.$$

If k is such that $\nu_k \leq n$ then M_k contains $z^n \otimes E_k$ and in fact $Q(n)M_k = z^n \otimes E_k$. If on the other hand k is such that $\nu_k \not\leq n$ then n cannot belong to the set $\{p \in \mathbb{Z}^d : p \geq \nu_k\}$ and hence $z^n \otimes E$ must be orthogonal to M_k. It follows that $Q(n)M_k = \{0\}$ when $\nu_k \not\leq n$. We conclude that

$$Q(n)(M_1 + \dots + M_s) = \sum_{k=1}^{s} z^n \otimes E_k(n) = z^n \otimes E(n).$$

After one notes that $M_1 + \dots + M_s$ must be dense in M because of (3.6), the proof is complete. □

Consider now the subspace $\overline{S_1 M}$. Letting $e_1 \in \mathbb{Z}^d$ be the d-tuple with components $(1, 0, \dots, 0)$ we see from (3.6) that $\overline{S_1 M}$ is generated by

$$z^{\nu_1 + e_1} \otimes E_1, \dots, z^{\nu_s + e_1} \otimes E_s,$$

and Lemma 3.5 implies that its spectral subspaces are given by

$$(\overline{S_1 M})(n) = z^n \otimes F(n), \qquad n \geq 0,$$

where $F(n)$ is defined as the sum $F_1(n) + \dots + F_s(n)$ where $F_k(n) = \{0\}$ if $\nu_k + e_k \not\leq n$ and $F_k(n) = E_k$ if $\nu_k + e_1 \leq n$.

Obviously $F_k(n) \subseteq E_k(n)$ for every $k = 1, \dots, s$, so that $F(n) \subseteq E(n)$. Thus $M \ominus S_1 M$ decomposes into an orthogonal direct sum

$$M \ominus S_1 M = \sum_{n \geq 0} E(n) \ominus F(n).$$

Finally, let $q_k \in \mathbb{Z}^+$ be the first component of $\nu_k = (q_k, *, *, \dots)$, $k = 1\dots, s$. We claim that any $n \geq 0$ for which $E(n) \ominus F(n) \neq \{0\}$ must have its first component in the set $\{q_1, \dots, q_s\}$. Indeed, for such an n there must be a $k = 1, \dots, s$ such that $F_k(n) \neq E_k(n)$, and this implies that $F_k(n) = \{0\}$ and $E_k(n) = E_k$. The first condition implies that $\nu_k + e_1 \not\leq n$ and the second implies that $\nu_k \leq n$; hence the first component of n must agree with the first component q_k of ν_k.

Setting $q = \max(q_1, \dots, q_s)$, we conclude that

$$M \ominus S_1 M \subseteq \overline{span}\{z^n \otimes E : n = (n_1, \dots, n_d) \geq 0, \quad 0 \leq n_1 \leq q\},$$

and Assertion 1 is proved. □

proof of Assertion 2. It is pointed out in ([Arv98], Corollary of Proposition 5.3) that the operators S_k are hyponormal, $S_k S_k^* \leq S_k^* S_k$, $1 \leq k \leq d$. Thus it suffices to show that the restriction C of S_1 to

$$K = \overline{span}\{z^n \otimes \zeta : n_1 \leq q, \quad \zeta \in E\}$$

satisfies $C^*C \in \mathcal{L}^p$ for every $p > d$. C^*C is the compression of $S_1^* S_1$ to K. We have seen in the proof of Lemma 3.2 that monomials are eigenvectors for $S_1^* S_1$,

$$S_1^* S_1 : z^n \otimes \zeta \mapsto \frac{n_1 + 1}{|n| + 1} \cdot z^n \otimes \zeta, \qquad n = (n_1, \dots, n_d), \tag{3.7}$$

where $|n| = n_1 + \cdots + n_d$. Let $\zeta_1, \dots, \zeta_r$ be an orthonormal basis for E. Then K is spanned by the orthogonal set of all monomials of the form $z^n \otimes \zeta_j$ with $n_1 \leq q$, $1 \leq j \leq r$, and from (3.7) it follows that

$$0 \leq P_K S_1^* S_1 P_K \leq (q+1) P_K ((\mathbf{1} + N)^{-1} \otimes \mathbf{1}_E) P_K, \tag{3.8}$$

where N is the number operator of H^2, the unbounded self-adjoint operator having the set of monomials as eigenvectors as follows: $N : z^n \mapsto |n| z^n$, $n \geq 0$. It is known that $(\mathbf{1}_{H^2} + N)^{-1}$ belongs to $\mathcal{L}^p$ for every $p > d$ (see formula (5.2) of [Arv98]). Since E is finite dimensional we have

$$(\mathbf{1} + N \otimes 1_E)^{-1} = (\mathbf{1}_{H^2} + N)^{-1} \otimes \mathbf{1}_E \in \mathcal{L}^p, \qquad p > d.$$

In view of (3.8), we conclude that $P_K S_1^* S_1 P_K \in \mathcal{L}^p$ for every $p > d$. □

Lemma 3.3 follows from Assertions 1 and 2, thereby completing the proof of Proposition 3.1 and Theorem 2.1. □

It remains only to deduce Corollary 2.2. We sketch the argument as follows. Let $(T_1, \dots, T_d)$ be the d-tuple acting on $M^\perp$ by compression

$$T_k = (\mathbf{1} - P_M) S_k \restriction_{M^\perp}, \quad k = 1, \dots, d,$$

and let $\mathcal{A}$ be the $*$-algebra generated by $T_1, \dots, T_d$ and the identity operator. A straightforward argument (that we omit) shows that the set of commutators $\{AB - BA : A, B \in \mathcal{A}\}$ is contained in $\mathcal{L}^p$ iff all the self-commutators $T_i^* T_j - T_j T_i^*$, $1 \leq i, j \leq d$, belong to $\mathcal{L}^p$.

Thus it suffices to show that the self-commutators all belong to $\mathcal{L}^p$ for $p > d$. Writing P for the projection onto M, $P^\perp$ for $\mathbf{1} - P$ and $T_k = P^\perp S_k P^\perp$ we have

$$T_i^* T_j = P^\perp S_i^* (\mathbf{1} - P) S_j P^\perp = P^\perp S_i^* S_j P^\perp - P^\perp S_i^* P S_j P^\perp \tag{3.9}$$

$$= P^\perp S_i^* S_j P^\perp - A_i A_j^*, \tag{3.10}$$

where $A_i = P^\perp S_i^* P$, and $T_j T_i^* = P^\perp S_j P^\perp S_i^* P^\perp = P^\perp S_j S_i^* P^\perp$. Thus

$$[T_i^*, T_j] = P^\perp [S_i^*, S_j] P^\perp - A_i A_j^*.$$

Proposition 3.1 implies that $A_i \in \mathcal{L}^p$ for $p > 2d$, and hence $A_i A_j^* \in \mathcal{L}^p$ for $p > d$. Finally, according to Proposition 5.3 of [Arv98] we have $[S_i^*, S_j] \in \mathcal{L}^p$ for $p > d$, and the desired conclusion follows.

4. Submodules and Quotients

Every d-contraction $\bar{A} = (A_1, \dots, A_d)$ has a defect operator

$$\Delta_{\bar{A}} = \mathbf{1} - (A_1 A_1^* + \cdots + A_d A_d^*),$$

and one has $0 \le \Delta_{\bar{A}} \le \mathbf{1}$. While this notation differs from that of [Arv98] where $\Delta_{\bar{A}}$ was defined as the square root of $\mathbf{1} - (A_1 A_1^* + \cdots + A_d A_d^*)$, it is better suited for our purposes here. We use the traditional notation $[X, Y]$ to denote the commutator $XY - YX$ of two operators X, Y.

Given an invariant subspace $M \subseteq H$ for a d-contraction $\bar{A}$, the restriction of $\bar{A}$ to M and the compression of $\bar{A}$ to the quotient H/M define two new d-contractions. In this section we examine the relationships between these three multioperators. We identify the quotient Hilbert space H/M with the orthocomplement $M^\perp$ of M in H, and its associated d-contraction with the d-tuple obtained by compressing $(A_1, \dots, A_d)$ to $M^\perp$.

Proposition 4.1. *Let $\bar{A} = (A_1, \dots, A_d)$ be a d-contraction acting on a Hilbert space H, let M be a closed $\bar{A}$-invariant subspace with projection $P : H \to M$, and let $\bar{B} = (B_1, \dots, B_d)$ and $\bar{C} = (C_1, \dots, C_d)$ be the d-contractions obtained, respectively, by restricting $\bar{A}$ to M and compressing $\bar{A}$ to $M^\perp$. Writing $P^\perp$ for the projection onto the subspace $M^\perp \subseteq H$, we have the following formulas relating various commutators and the three defect operators $\Delta_{\bar{A}}$, $\Delta_{\bar{B}}$, $\Delta_{\bar{C}}$.*

$$[B_j, B_k^*] P = -[P, A_j][P, A_k]^* + P[A_j, A_k^*] P \tag{4.1}$$

$$[C_j, C_k^*] P^\perp = [P, A_k]^* [P, A_j] + P^\perp [A_j, A_k^*] P^\perp \tag{4.2}$$

$$\Delta_{\bar{B}} P = P \Delta_{\bar{A}} P + \sum_{k=1}^{d} [P, A_k][P, A_k]^* \tag{4.3}$$

$$\Delta_{\bar{C}} P^\perp = P^\perp \Delta_{\bar{A}} P^\perp. \tag{4.4}$$

Proof. To verify (4.1), we write

$$\begin{aligned}[B_j, B_k^*]P =& A_j P A_k^* P - P A_k^* A_j P = A_j P A_k^* P - P A_j A_k^* P + P[A_j, A_k^*]P \\ =& - P A_j P^\perp A_k^* P + P[A_j, A_k^*]P.\end{aligned}$$

Since $PA_jP^\perp = PA_j - A_jP$, we have $PA_jP^\perp A_k^*P = [P, A_j][P, A_k]^*$, and (4.1) follows.

(4.2) follows similarly, after using $P^\perp A_j P^\perp = P^\perp A_j$ to write

$$\begin{aligned}[C_j, C_k^*]P^\perp &= P^\perp A_j A_k^* P^\perp - P^\perp A_k^* P^\perp A_j P^\perp \\ &= P^\perp A_k^* A_j P^\perp - P^\perp A_k^* P^\perp A_j P^\perp + P^\perp [A_j, A_k^*] P^\perp \\ &= P^\perp A_k^* P A_j P^\perp + P^\perp [A_j, A_k^*] P^\perp.\end{aligned}$$

(4.2) follows after one notes that $P^\perp A_k^* P A_j P^\perp = [P, A_k]^*[P, A_j]$.

To prove (4.3), one writes $\Delta_{\bar B} P$ as follows,

$$\begin{aligned}P - \sum_{k=1}^d A_k P A_k^* &= P\Delta_{\bar A} P + \sum_{k=1}^d P A_k(\mathbf{1} - P)A_k^* P \\ &= P\Delta_{\bar A} P + \sum_{k=1}^d [P, A_k][P, A_k]^*,\end{aligned}$$

and (4.4) follows similarly, since

$$P^\perp - \sum_{k=1}^d P^\perp A_k P^\perp A_k^* P^\perp = P^\perp \Delta_{\bar A} P^\perp + \sum_{k=1}^d P^\perp A_k P A_k^* P^\perp = P^\perp \Delta_{\bar A} P^\perp.$$

That completes the proof. □

Corollary 4.2. *Let $\bar A$, $\bar B$, $\bar C$ satisfy the hypotheses of Proposition 4.1. Then for every p satisfying $1 \le p \le \infty$, the following are equivalent:*

(i) *Both defect operators $\Delta_{\bar B}$ and $\Delta_{\bar C}$ belong to $\mathcal{L}^p$.*
(ii) *$\Delta_{\bar A}$ belongs to $\mathcal{L}^p$ and $[P_M, A_k] \in \mathcal{L}^{2p}$. $1 \le k \le d$.*

Proof. The implication (ii) $\Longrightarrow$ (i) is an immediate consequence of the formulas (4.3) and (4.4).

(i) $\Longrightarrow$ (ii): We write P for P_M. ¿From (4.3) and (4.4), together with the fact that the right side of (4.3) is a sum of positive operators, we may conclude that all of the operators

$$P\Delta_{\bar A} P,\ P^\perp \Delta_{\bar A} P^\perp,\ [P, A_1][P, A_1]^*, \dots, [P, A_d][P, A_d]^*$$

belong to $\mathcal{L}^p$. By (3.1) we have $[P, A_k] \in \mathcal{L}^{2p}$, $1 \le k \le d$. Another application of (3.1) shows that both $\sqrt{\Delta_{\bar A}} P$ and $\sqrt{\Delta_{\bar A}} P^\perp$ belong to $\mathcal{L}^{2p}$. The latter two operators sum to $\sqrt{\Delta_{\bar A}} \in \mathcal{L}^{2p}$, and therefore $\Delta_{\bar A} \in \mathcal{L}^p$. □

We now apply Proposition 4.1 to obtain concrete information about the examples of greatest interest for us, namely the submodules and quotients that are associated with pure finite rank d-contractions.

Theorem 4.3. *Let $M \subseteq H^2 \otimes \mathbb{C}^r$ be an invariant subspace of the d-shift $\bar S = (S_1, \dots, S_d)$ of finite rank r and let $\bar B$ and $\bar C$ be, respectively, the restriction of $\bar S$ to M and the compression of $\bar S$ to $M^\perp$. Then for every p satisfying $d < p \le \infty$, the following are equivalent:*

(i) *The defect operator of $\bar B$ belongs to $\mathcal{L}^p$.*
(ii) $[B_j, B_k^*] \in \mathcal{L}^p$, $1 \le j, k \le d$.
(iii) $[C_j, C_k^*] \in \mathcal{L}^p$, $1 \le j, k \le d$.
(iv) $[P_M, S_k] \in \mathcal{L}^{2p}$, $1 \le k \le d$.

If (i)–(iv) are satisfied for some $p \in (d, \infty]$, then both $\bar B$ and $\bar C$ are Fredholm multioperators, and the indices of their Dirac operators are related by

$$\operatorname{ind} D_{\bar B +} + \operatorname{ind} D_{\bar C +} = (-1)^d \cdot r. \tag{4.5}$$

Proof. It was shown in [Arv98] that the self-commutators $[S_j, S_k^*]$ belong to $\mathcal{L}^q$ for every $q > d$; and of course, the defect operator of $\bar S$ is a projection of rank r, belonging to $\mathcal{L}^q$ for every $q \ge 1$. With these observations in hand, one sees from (4.3) that $\Delta_{\bar B} \in \mathcal{L}^p$ if and only if

$$\sum_{k=1}^{d} [P, S_k][P, S_k]^* \in \mathcal{L}^p, \tag{4.6}$$

where P denotes P_M. Similarly, (4.1) and (4.2) show that the assertions (ii) and (iii) are equivalent, respectively, to the assertions

$$[P, S_j][P, S_k]^* \in \mathcal{L}^p, \qquad 1 \le j, k \le d \tag{4.7}$$

$$[P, S_k]^*[P, S_j] \in \mathcal{L}^p, \qquad 1 \le j, k \le d. \tag{4.8}$$

Thus, the problem of showing that (i)–(iv) are equivalent is reduced to that of showing that each of the assertions (4.6), (4.7) and (4.8) is equivalent to the assertion $[P, S_k] \in \mathcal{L}^{2p}$, $1 \le k \le d$. That is a straightforward consequence of the elementary equivalences (3.1).

To sketch the proof of (4.5), note first that Proposition 1.1, together with (ii), (iii), and the known essential normality of the d-shift, imply that the three Dirac operators $D_{\bar S}$, $D_{\bar B}$ and $D_{\bar C}$ are Fredholm. By property (iv), the commutators $PS_k - S_k P$, $1 \le k \le d$, are compact. It follows that the d-tuple $\bar S$ is unitarily equivalent to a compact perturbation of the direct sum of d-tuples $\bar B \oplus \bar C$. In turn, this implies that $D_{\bar S +}$ is unitarily equivalent to a compact perturbation of the direct sum of Fredholm operators $D_{\bar B +} \oplus D_{\bar C +}$. By stability of the Fredholm index under compact perturbations, we have

$$\operatorname{ind} D_{\bar B +} + \operatorname{ind} D_{\bar C +} = \operatorname{ind} D_{\bar S +}.$$

Finally, from Theorem B of [Arv02] that relates the index of a finite rank graded d-contraction to its curvature invariant, we can compute the right side of the preceding formula

$$\operatorname{ind} D_{\bar S +} = (-1)^d K(\bar S).$$

Since the curvature of a finite direct sum of copies of the d-shift is known to be its rank [Arv00], formula (4.5) follows. $\square$

5. Concluding Remarks and Conjectures

We expect that some variation of Corollary 2.2 should hold under much more general circumstances, and we now discuss these issues.

Conjecture A. *Let M be a closed invariant subspace for the d-shift $\bar{S} = (S_1, \dots, S_d)$ of finite rank r, acting on $H^2 \otimes \mathbb{C}^r$. Assume that M is generated by a set of vector polynomials in $\mathbb{C}[z_1, \dots, z_d] \otimes \mathbb{C}^r$, each of which is homogeneous of some degree. Then $P_M S_k - S_k P_M$ is compact for every $k = 1, \dots, d$.*

Note that because of Hilbert's basis theorem, one may assume that M is generated by a *finite* set of homogeneous vector polynomials .

By Theorem 4.3, Conjecture A implies that the pure d-contraction $\bar{T} = (T_1, \dots, T_d)$ obtained by compressing $\bar{S}$ to $M^\perp$ is a Fredholm multioperator, and as we have seen in Section 1, the index formula (1.3) implies that the curvature invariant $K(\bar{T})$ is stable in these cases. The space X appearing in the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(T_1, \dots, T_d) + \mathcal{K} \longrightarrow C(X) \longrightarrow 0$$

would now be associated with a nontrivial algebraic set in projective space.

Remark 5.1 (Evidence for Conjecture A)*.* Theorem 2.1 implies that Conjecture A is true when the homogeneous polynomials are monomials. Moreover, Conjecture A is true in two dimensions. Indeed, a recent result of Kunyu Guo (Theorem 2.4 of [Guo03]) implies that, in the context of Conjecture A for dimension $d = 2$, the 2-contraction obtained by restricting (S_1, S_2) to M has the property that its defect operator belongs to $\mathcal{L}^p$ for every $p > 1$. By Theorem 4.3, Conjecture A is true when $d = 2$.

Finally, there are a few other classes of (unpublished) examples in arbitrary dimension d involving homogeneous polynomials for which one can decide the issue, and these too support Conjecture A.

Stephen Parrott has shown [Par00] that a pure finite rank single contraction is a Fredholm operator and (1.2) holds; R. N. Levy improved this in [Lev00]. However, some of the examples that occur in this one-dimensional setting are not essentially normal. Thus, one cannot expect the conclusion of Conjecture A to hold for arbitrary invariant subspaces $M \subseteq H^2 \otimes \mathbb{C}^r$ in higher dimensional cases $d > 1$. However, we believe that the following two "ungraded" relatives of Conjecture A are well-founded.

Conjecture B. *Let $M \subseteq H^2 \otimes \mathbb{C}^r$ be an invariant subspace of the d-shift of finite rank r that is generated by a set of vector polynomials. Then $P_M S_k - S_k P_M$ is compact for $1 \leq k \leq d$.*

Assuming the result of Conjecture B, one obtains a Fredholm multioperator by compressing the d-shift to $M^\perp$. In such cases, one would expect the following conjecture to hold; the result would generalize the index formula (1.2) to the ungraded case.

Conjecture C. *The index formula (1.2) holds for the finite rank d-contraction $\bar{T}$ obtained by compressing the d-shift of rank r to $M^{\perp}$, whenever M is generated by vector polynomials and $\bar{T}$ is essentially normal.*

More generally, it is natural to ask if every finite rank pure d-contraction is Fredholm and satisfies the index formula (1.2). While there is scant evidence to illuminate these questions in general, Parrott's work [Par00] implies that both answers are yes in the one-dimensional cases $d = 1$:

Problem D. *Let $\bar{T}$ be a finite rank d-contraction. Is $\bar{T}$ is a Fredholm d-tuple? Does the index formula (1.2) hold?*

We expect that significant progress on Problem D will require further development of the theory of Fredholm multioperators.

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Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: `arveson@math.berkeley.edu`