# EIGENVALUE LISTS OF NONCOMMUTATIVE PROBABILITY DISTRIBUTIONS

William Arveson

Expanded notes for a lecture given 4 May 1999

12 May 1999

Abstract. In probability theory all nonatomic probability measures look the same. That is because any two nonatomic separable measure algebras are isomorphic. Quantum probability theory is different: two normal states of $\mathcal{B}(H)$ are conjugate only when the eigenvalue lists of their density operators are the same. Suppose now that one is given an increasing sequence $M_1 \subseteq M_2 \subseteq \dots$ of type $I$ subfactors of $\mathcal{B}(H)$ whose union is weak$^*$-dense in $\mathcal{B}(H)$. Common sense suggests that if one restricts a normal state $\rho$ of $\mathcal{B}(H)$ to $M_n$ and considers its eigenvalue list $\Lambda_n$ for large $n$, then $\Lambda_n$ should be close to the eigenvalue list of $\rho$ when $n$ is large.

We discuss some natural examples which show that this intuition is wrong, and we attempt to explain the phenomemon by describing the correct asymptotic formula when the sequence $(M_n)$ is "stable". Applications are not discussed here, but are taken up in [1].

A *tower* is an increasing sequence of type $I$ subfactors of $\mathcal{B}(H)$ $M_1 \subseteq M_2 \subseteq \dots$ whose union is weak$^*$-dense in $\mathcal{B}(H)$. Here $H$ is separable and infinite dimensional, and the subfactors $M_n$ may be finite-dimensional (i.e., isomorphic to full matrix algebras) or isomorphic to $\mathcal{B}(H)$ itself. In order to sidestep trivialities we also require that each $M_n$ be of infinite codimension in $\mathcal{B}(H)$.

Let $\rho$ be a normal state defined on a type $I$ factor $M$ (having separable predual). We can associate an *eigenvalue list* with $\rho$ as follows. Noting that $M$ is $*$-isomorphic to $\mathcal{B}(K)$ for some separable Hilbert space $K$ we realize $\rho$ as a normal state on $\mathcal{B}(K)$ and write $R$ for its density operator

$$\rho(A) = \operatorname{trace}(RA), \qquad A \in \mathcal{B}(K).$$

$R$ is a positive operator of trace 1, whose eigenvalues including multiplicity may be enumerated in decreasing order $\{\lambda_1 \geq \lambda_2 \geq \dots\}$. This sequence $\Lambda(\rho)$ is called the eigenvalue list of $\rho$. In case $K$ is finite dimensional we may consider $\Lambda(\rho)$ to be an infinite ordered sequence by extending it out to infinity with zeros; thus, eigenvalue lists of normal states of type $I$ factors $M$ are always infinite sequences of nonnegative real numbers summing to 1, even when $M$ is finite dimensional.

There is a natural metric defined on the space of eigenvalue lists, that is the space of all decreasing sequences $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ of nonnegative real numbers satisfying $\sum_n \lambda_n = 1$. For $\Lambda = \{\lambda_1, \lambda_2, \dots\}$, $\Lambda' = \{\lambda_1', \lambda_2', \dots\}$ we set

$$\|\Lambda - \Lambda'\| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_n'|.$$

The space of eigenvalue lists is a closed subset of the unit ball of $\ell^1$.

The basic properties of eigenvalue lists are summarized as follows

(1) If $\omega_\xi$ is a vector state of $\mathcal{B}(K)$ then $\Lambda(\omega_\xi) = \{1, 0, 0, \dots\}$.
(2) If $\rho$ and $\sigma$ are normal states on $M$ and $N$ respectively which are conjugate in the sense that there is a $*$-isomorphism $\theta : M \to N$ such that $\sigma \circ \theta = \rho$, then $\Lambda(\rho) = \Lambda(\sigma)$.
(3) Weyl's inequality: If $\rho$, $\sigma$ are normal states of the same type $I$ factor $M$ then
$$\|\Lambda(\rho) - \Lambda(\sigma)\| \leq \|\rho - \sigma\|.$$
(4) If $\rho_k$ is a normal state of a type $I_\infty$ factor $M_k$, $k = 1, 2$, then there is a sequence of $*$-isomorphisms $\theta_k : M_1 \to M_2$ such that
$$\|\Lambda(\rho_1) - \Lambda(\rho_2)\| = \lim_{n\to\infty} \|\rho_1 - \rho_2 \circ \theta_n\|.$$

*Remarks.* Together, properties (3) and (4) are essentially reformulations of the following more familiar assertion about positive trace class operators $A$, $B$ acting on a separable infinite dimensional Hilbert space $H$. Let $\{\lambda_1, \lambda_2, \dots\}$ and $\{\mu_1, \mu_2, \dots\}$ be the eigenvalue lists of $A$ and $B$ respectively, and let $\mathcal{U}(H)$ denote the unitary group in $\mathcal{B}(H)$. Then
$$\sum_{n=1}^{\infty} |\lambda_n - \mu_n| = \inf_{U \in \mathcal{U}(H)} \operatorname{trace}|A - UBU^{-1}|.$$

Given a tower $(M_n)$ in $\mathcal{B}(H)$ and a normal state $\rho$ on $\mathcal{B}(H)$, we are interested in the asymptotic behavior of the eigenvalue lists of the restrictions $\rho\restriction_{M_n}$ when $n$ is large, and especially we want to compute $\lim_n \Lambda(\rho\restriction_{M_n})$ when this limit exists. There are many examples of towers for which such limits do not exist. The hypothesis we require is formulated not in terms of the sequence $M_n$ but rather the sequence $M_n'$ of commutants.

**Definition.** *A tower $(M_n)$ in $\mathcal{B}(H)$ is called stable if there is a normal state $\omega$ of $\mathcal{B}(H)$ and an eigenvalue list $\Lambda_\infty$ such that*
$$\lim_{n\to\infty} \Lambda(\omega\restriction_{M_n'}) = \Lambda_\infty$$
*in the metric of eigenvalue lists.*

Notice that for any tower $(M_n)$ in $\mathcal{B}(H)$ and for any two normal states $\rho_1$ and $\rho_2$ of $\mathcal{B}(H)$, we must have
$$\lim_{n\to\infty} \|(\rho_1 - \rho_2)\restriction_{M_n'}\| = \|(\rho_1 - \rho_2)\restriction_{\mathbb{C}\cdot\mathbf{1}}\| = 0$$
since the commutants $M_n'$ decrease to $\mathbb{C} \cdot \mathbf{1}$. By (3) above it follows that
$$\lim_{n\to\infty} \|\Lambda(\rho_1\restriction_{M_n'}) - \Lambda(\rho_2\restriction_{M_n'})\| = 0.$$

We conclude that the limit list $\Lambda_\infty$ is a uniquely defined asymptotic invariant of stable towers in the following sense.

**Observation.** *Let $(M_n)$ be a stable tower with limit list $\Lambda_\infty$. Then for any normal state $\rho$ of $\mathcal{B}(H)$ we have*

$$\lim_{n\to\infty} \Lambda(\rho \restriction_{M_n'}) = \Lambda_\infty.$$

**Examples.** Our main examples come from dynamics, and the simplest of these is as follows. Let $\alpha$ be an endomorphsim of $\mathcal{B}(H)$ (by that we mean a normal $*$-homomorphism of $\mathcal{B}(H)$ into itself for which $\alpha(\mathbf{1}) = \mathbf{1}$). For each $n = 1, 2, \dots$ let $M_n$ be the *commutant* of $\alpha^n(\mathcal{B}(H))$. It is obvious that $M_1 \subseteq M_2 \subseteq \dots$, and if $\alpha$ is *pure* in the sense that $\cap_n \alpha^n(\mathcal{B}(H)) = \mathbb{C} \cdot \mathbf{1}$, then $(M_n)$ is a tower. Suppose, in addition, that there is a normal state $\omega$ of $\mathcal{B}(H)$ which is invariant under $\alpha$: $\omega \circ \alpha = \omega$. Notice that property (2) above implies that $\Lambda(\omega \restriction_{M_n'}) = \Lambda(\omega)$ for every $n = 1, 2, \dots$. We conclude that $M_n = \alpha^n(\mathcal{B}(H))'$ *defines a stable tower whenever $\alpha$ is a pure endomorphism of $\mathcal{B}(H)$ with a normal invariant state.*

In the analysis of interactions [1] one has a similar situation, but instead of a sequence $(M_n)$ one has a one-parameter family of type $I$ subfactors $M_t = \alpha_t(\mathcal{B}(H))'$, $\alpha = \{\alpha_t : t \geq 0\}$ being a pure $E_0$-semigroup. The basic problem reduces to establishing the existence of limits $\lim_{t\to\infty} \Lambda(\rho \restriction_{M_t})$ and calculating their values, where $\rho$ is a normal state of $\mathcal{B}(H)$. We now take up these issues in the somewhat simpler case of stable towers.

The following result characterizes stability for towers in terms that can be related to the above examples based on dynamics.

**Proposition.** *Let $(M_n)$ be an arbitrary tower in $\mathcal{B}(H)$. The following are equivalent.*

(1) *$(M_n)$ is stable.*

(2) *There is a normal state $\omega$ of $\mathcal{B}(H)$ and a sequence of endomorphisms $\alpha_1, \alpha_2, \dots$ of $\mathcal{B}(H)$ such that $\alpha_n(\mathcal{B}(H)) = M_n'$ and*

$$\lim_{n\to\infty} \|\omega \circ \alpha_n - \omega\| = 0.$$

(3) *There is an eigenvalue list $\Lambda_\infty$ with the following property. For every normal state $\omega$ of $\mathcal{B}(H)$ with $\Lambda(\omega) = \Lambda_\infty$ there is a sequence of endomorphisms $\alpha_1, \alpha_2, \dots$ of $\mathcal{B}(H)$ such that $\alpha_n(\mathcal{B}(H)) = M_n'$ and*

$$\lim_{n\to\infty} \|\omega \circ \alpha_n - \omega\| = 0.$$

*proof.* The implications (3) $\implies$ (2) and (2) $\implies$ (1) are completely straightforward. We prove (1) $\implies$ (3). Let $\Lambda_\infty$ be the limit list of a stable tower $(M_n)$ in $\mathcal{B}(H)$, and choose any normal state $\omega$ of $\mathcal{B}(H)$ with $\Lambda(\omega) = \Lambda_\infty$. Since $M_n'$ is a type $I_\infty$ factor, there is a $*$-isomorphism $\pi_n$ of $\mathcal{B}(H)$ onto $M_n'$. By property (2) of eigenvalue lists we have

$$\|\Lambda(\omega \circ \pi_n) - \Lambda(\omega)\| = \|\Lambda(\omega \restriction_{M_n'}) - \Lambda_\infty\| \to 0$$

as $n \to \infty$. Property (4) of eigenvalue lists implies that there is a sequence $\theta_1, \theta_2, \dots$ of $*$-automorphisms of $\mathcal{B}(H)$ such that

$$\|\omega \circ \pi_n - \omega \circ \theta_n\| \leq \|\Lambda(\omega \circ \pi_n) - \Lambda(\omega)\| + 2^{-n}$$

for every $n = 1, 2, \dots$, and the assertion (3) of the Theorem follows by taking $\alpha_n = \pi_n \circ \theta_n^{-1}$ and noting that $\|\omega \circ \alpha_n - \omega\| \to 0$ as $n \to \infty$. ∎

Here is the main result on the convergence of eigenvalue lists along a tower.

**Theorem A.** *Let* $(M_n)$ *be a stable tower in* $\mathcal{B}(H)$ *with limit list* $\Lambda_\infty$*. Then for every normal state* $\rho$ *of* $\mathcal{B}(H)$*,*

$$\lim_{n\to\infty} \Lambda(\rho\restriction_{M_n}) = \Lambda(\rho)\otimes\Lambda_\infty$$

*in the metric of eigenvalue lists.*

*Remarks.* The tensor product of two eigenvalue lists $\{\lambda_1, \lambda_2, \dots\}$ and $\{\mu_1, \mu_2, \dots\}$ is the eigenvalue list obtained by rearranging the double sequence of products $\{\lambda_k\mu_j : k, j = 1, 2, \dots\}$ into decreasing order.

One might expect that, since the sequence of subfactors $M_n$ increases to $\mathcal{B}(H)$, the eigenvalue lists of restrictions of $\rho\restriction_{M_n}$ should be close to the list of $\rho$ itself when $n$ is large. Indeed, if the limit list $\Lambda_\infty$ happens to be the eigenvalue list of a vector state, $\Lambda_\infty = \{1, 0, 0, \dots\}$, then $\Lambda(\rho)\otimes\Lambda_\infty = \Lambda(\rho)$, and hence $\Lambda(\rho\restriction_{M_n})$ converges to $\Lambda(\rho)$.

On the other hand, if $\Lambda_\infty$ has more than one nonzero element then $\Lambda(\rho)\otimes\Lambda_\infty$ looks quite different from the eigenvalue list of $\rho$, and the above intuition is misleading.

The proof of Theorem A follows exactly along the lines of the proof of Theorem C of [1] (see pp. 14–19) by nothing more than a change in notation, and we do not repeat it here.

## References

1. Arveson, W., *Interactions in noncommutative dynamics*, preprint (Summer 1999).