Contemporary Mathematics

Four Lectures on Noncommutative Dynamics

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Abstract. These lectures concern basic aspects of the theory of semigroups of endomorphisms of type I factors that relate to causal dynamics, dilation theory, and the problem of classifying E_0-semigroups up to cocycle conjugacy. We give only a few proofs here; full details can be found in the author's upcoming monograph *Noncommutative Dynamics and E-semigroups*, to be published in the Springer series Monographs in Mathematics.

Contents

1. Dynamical Origins: Histories and Interactions

The dynamics of one-parameter groups of automorphisms of $\mathcal{B}(H)$ has been completely understood since the 1930s, following work of Wigner, Stone, and the multiplicity theory of Hahn and Hellinger. After reviewing these basic issues, we show that if one takes into account a natural notion of causality for such dynamical groups, one encounters entirely new phenomena. We describe the basic properties of these "causal" dynamical systems and their connections with E_0-semigroups, we discuss positive results concerning their existence and nontriviality, and we point out some of the basic mathematical problems that remain unsolved.

1.1. The Dynamics of Quantum Systems. Let us recall the basic mathematical setting of quantum physics. The observables of quantum theory are self-adjoint operators acting on a separable Hilbert space H. Observables such as linear or angular momentum arise as generators of one-parameter unitary groups and are typically unbounded and only densely defined. However, there is no essential loss

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in restricting attention to bounded functions of these unbounded operators. It goes without saying that one has to make use of the spectral theorem in order to define bounded functions of unbounded self-adjoint operators, and we do so freely.

The quantum replacement for the distribution of a random variable requires specifying not only an observable X but also a unit vector $\xi \in H$. Once X and ξ are fixed, there is a unique probability measure $\mu_{X,\xi}$ defined on the real line by specifying its integral with respect to bounded Borel functions $f : \mathbb{R} \to \mathbb{R}$ as follows

$$\int_{-\infty}^{\infty} f(t)\, d\mu_{X,\xi}(t) = \langle f(X)\xi, \xi\rangle .$$

For a Borel set $S \subseteq \mathbb{R}$, one interprets $\mu_{X,\xi}(S)$ as representing the probability of finding an observed value of X in the set S, given that the system is in the pure state associated with ξ.

Given two observables X, Y and a unit vector ξ, there is no "joint distribution" $\mu_{X,Y,\xi}$ defined on $\mathbb{R}^2$. On the level of physics, this phenomenon is associated with the theory of measurement and is a consequence of the uncertainty principle. From the point of view of operator theory, since the operators X and Y normally fail to commute there is no way of using them to define a spectral measure on $\mathbb{R}^2$. This nonexistence of joint distributions is one of the fundamental differences between quantum theory and probability theory.

Turning now to dynamics, consider the way the flow of time acts on the algebra of observables. Every symmetry of quantum theory corresponds to either a $*$-automorphism or a $*$-anti-automorphism of the algebra $\mathcal{B}(H)$ of all bounded operators on H. If we are given a one-parameter group of such symmetries, then since each one of them is the square of another it follows that all of the symmetries must be $*$-automorphisms. Thus, the flow of time on a quantum sytem is given by a one-parameter family $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{B}(H)$ such that $\alpha_s \circ \alpha_t = \alpha_{s+t}$, and which satisfies the natural continuity condition: for every $A \in \mathcal{B}(H)$ and every pair of vectors $\xi, \eta \in H$ the function $t \in \mathbb{R} \mapsto \langle \alpha_t(A)\xi, \eta\rangle$ is continuous.

Let us consider the possibilities: How does one classify one-parameter groups of automorphisms of $\mathcal{B}(H)$? In the late 1930s, Eugene Wigner proved that every such one-parameter group is implemented by a strongly continuous one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ in the sense that

$$\alpha_t(A) = U_t A U_t^*, \qquad A \in \mathcal{B}(H), \quad t \in R.$$

Earlier, Marshall Stone had shown that a strongly continuous one-parameter unitary group U is the Fourier transform of a unique spectral measure E defined on the Borel subsets of the real line as follows

$$U_t = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda).$$

Equivalently, Stone's theorem implies that for the unbounded self-adjoint operator $X = \int_{\mathbb{R}} \lambda\, dE(\lambda)$, we have $U_t = e^{itX}$, $t \in \mathbb{R}$.

Together, these two results imply that every one-parameter group α of automorphisms of $\mathcal{B}(H)$ corresponds to an observable X as follows

$$\alpha_t(A) = e^{itX} A e^{-itX}, \qquad A \in \mathcal{B}(H), \quad t \in \mathbb{R}.$$

X is not uniquely determined by α, since replacing X with a scalar translate of the form $X + \lambda\mathbf{1}$ with $\lambda \in \mathbb{R}$ does not change α. However, X is uniquely determined by α up to such scalar perturbations.

Two one-parameter groups α and β of $*$-automorphisms (acting on $\mathcal{B}(H)$ and $\mathcal{B}(K)$ respectively) are said to be *conjugate* if there is a $*$-isomorphism θ of $\mathcal{B}(H) \to \mathcal{B}(K)$ such that

$$\theta(\alpha_t(A)) = \beta_t(\theta(A)), \qquad A \in \mathcal{B}(H), \quad t \in \mathbb{R}.$$

Recalling that such a $*$-isomorphism θ must be implemented by a unitary operator $W : H \to K$ by way of $\theta(A) = WAW^*$, we see that Wigner's theorem completely settles the classification issue for one-parameter groups of automorphisms of $\mathcal{B}(H)$. Indeed, using that result we may find unbounded self-adjoint operators X, Y on the respective Hilbert spaces such that α and β are given by $\alpha_t(A) = e^{itX}Ae^{-itX}$ and $\beta_t(B) = e^{itY}Be^{-itY}$, $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$, $t \in \mathbb{R}$. It is an elementary exercise to show that $\theta(A) = WAW^*$ implements a conjugacy of α and β if and only if there is a real scalar λ such that $WXW^* = Y + \lambda\mathbf{1}$. Thus, the classification of one-parameter groups of automorphisms is reduced to the classification of unbounded self-adjoint operators up to unitary equivalence. By the spectral theorem, this is equivalent to the classification up to unitary equivalence of spectral measures on the real line; and the latter problem is completely understood in terms of the multiplicity theory of Hahn and Hellinger.

These remarks show that the most basic aspect of quantum dynamics, namely the structure and classification of one-paramter groups of $*$-automorphisms of $\mathcal{B}(H)$, is completely understood. We have seen all of the possibilities, and they are described by self-adjoint operators (or spectral measures on the line) and their multiplicity theory in a completely explicit way.

1.2. Causality: Histories and Interactions. We now show that by introducing a natural notion of causality into such dynamical systems, one encounters entirely new phenomena. These "causal" dynamical systems acting on $\mathcal{B}(H)$ are only partially understood. We have surely not seen all of them, and we have only partial information about how to classify the ones we have seen.

We are concerned with one-parameter groups of $*$-automorphisms, of the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space H, which carry a particular kind of causal structure. More precisely, A *history* is a pair (U, M) consisting of a one-parameter group $U = \{U_t : t \in \mathbb{R}\}$ of unitary operators acting on a separable infinite-dimensional Hilbert space H, together with a type I subfactor $M \subseteq \mathcal{B}(H)$ that is invariant under the automorphisms $\gamma_t(X) = U_t X U_t^*$ for negative t, and which has the following two properties

(i) (irreducibility)

$$(\bigcup_{t\in\mathbb{R}} \gamma_t(M))'' = \mathcal{B}(H),$$

(ii) (trivial infinitely remote past)

$$\bigcap_{t\in\mathbb{R}} \gamma_t(M) = \mathbb{C} \cdot \mathbf{1}.$$

We find it useful to think of the group $\{\gamma_t : t \in \mathbb{R}\}$ as representing the flow of time in the Heisenberg picture, and the von Neumann algebra M as representing bounded observables that are associated with the "past". However, we will be concerned with purely mathematical issues concerning the dynamical properties of histories, with problems concerning their existence and construction, and especially with the issue of nontriviality (to be defined momentarily). Two histories (U, M) (acting on H) and $(\tilde{U}, \tilde{M})$ (acting on $\tilde{H}$) are said to be *isomorphic* if there is a $*$-isomorphism $\theta : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\theta(M) = \tilde{M}$ and $\theta \circ \gamma_t = \tilde{\gamma}_t \circ \theta$ for every $t \in \mathbb{R}$, γ, $\tilde{\gamma}$ denoting the automorphism groups associated with U, $\tilde{U}$. The basic problems addressed in these lectures all bear some relation to the problem of classifying histories. We have already alluded to the fact that the results are far from complete.

An E_0-*semigroup* is a one-parameter semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of unit-preserving $*$-endomorphisms of a type I_∞ factor M, which is continuous in the natural sense. The subfactors $\alpha_t(M)$ decrease as t increases, and α is called *pure* if $\cap_t \alpha_t(M) = \mathbb{C}\mathbf{1}$. There are two E_0-semigroups α^-, α^+ associated with any history, α^- being the one associated with the "past" by restricting γ_{-t} to M for $t \geq 0$ and α^+ being the one associated with the "future" by restricting γ_t to the commutant M' for $t \geq 0$.

By an *interaction* we mean a history with the additional property that there are normal states ω_-, ω_+ of M, M' respectively such that ω_- is invariant under the action of α^- and ω_+ is invariant under the action of α^+. Because of (i) and (ii), both α^- and α^+ are pure E_0-semigroups, and it is not hard to show that if a pure E_0-semigroup α has a normal invariant state ω then that state is an *absorbing* state in the sense that for every other normal state ρ on the domain of α one has

$$\lim_{t \to \infty} \|\rho \circ \alpha_t - \omega\| = 0.$$

In particular, ω_- (resp. ω_+) is the unique normal state of M (resp. M') that is invariant under the action of α^- (resp. α^+).

In particular, it follows from this uniqueness that if one is given two interactions (U, M) and $(\tilde{U}, \tilde{M})$ with respective pairs of normal states ω_+, ω_- and $\tilde{\omega}_+$, $\tilde{\omega}_-$, then an isomorphism of histories $\theta : (U, M) \to (\tilde{U}, \tilde{M})$ must associate $\tilde{\omega}_+$, $\tilde{\omega}_-$ with ω_+, ω_- in the sense that if θ_+ (resp. θ_-) denotes the restriction of θ to M (resp. M'), then one has $\tilde{\omega}_\pm \circ \theta_\pm = \omega_\pm$.

Remark 1.2.1 (Normal Invariant States). Since the state space of a unital C^*-algebra is weak*-compact, the Markov-Kakutani fixed point theorem implies that every E_0-semigroup has invariant states. But there is no reason to expect that there is a *normal* invariant state. Indeed, there are examples of pure E_0-semigroups which have no normal invariant states. Notice too that ω_-, for example, is defined *only* on the algebra M of the past. Of course, ω^- has many extensions to normal states of $\mathcal{B}(H)$, but none of these normal extensions need be invariant under the action of the group γ. In fact, we will see that if there is a *normal* γ-invariant state defined on all of $\mathcal{B}(H)$ then the interaction must be trivial.

In order to discuss the dynamics of interactions we must introduce a C^*-algebra of "local observables". For every compact interval $[s, t] \subseteq \mathbb{R}$ there is an associated von Neumann algebra

$$\mathcal{A}_{[s,t]} = \gamma_t(M) \cap \gamma_s(M)'. \tag{1.1}$$

Notice that since $\gamma_s(M) \subseteq \gamma_t(M)$ are both type I factors, so is the relative commutant $\mathcal{A}_{[s,t]}$. Clearly $\mathcal{A}_I \subseteq \mathcal{A}_J$ if $I \subseteq J$, and for adjacent intervals $[r, s]$, $[s, t]$,

$r \leq s \leq t$ we have

$$\mathcal{A}_{[r,t]} = \mathcal{A}_{[r,s]} \otimes \mathcal{A}_{[s,t]}, \tag{1.2}$$

in the sense that the two factors $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ mutually commute and generate $\mathcal{A}_{[r,t]}$ as a von Neumann algebra. The automorphism group γ permutes the algebras $\mathcal{A}_I$ covariantly,

$$\gamma_t(\mathcal{A}_I) = \mathcal{A}_{I+t}, \qquad t \in \mathbb{R}. \tag{1.3}$$

Finally, we define the local C^*-algebra $\mathcal{A}$ to be the *norm* closure of the union of all the $\mathcal{A}_I$, $I \subseteq \mathbb{R}$. $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(H)$ that is strongly dense and invariant under the action of the automorphism group γ.

REMARK 1.2.2. It may be of interest to compare the local structure of the C^*-algebra $\mathcal{A}$ to its commutative counterpart, namely the local algebras associated with a stationary random distribution with independent values at every point [**GV64**]. More precisely, suppose that we are given a random distribution ϕ; i.e., a linear map from the space of real-valued test functions on $\mathbb{R}$ to the space of real-valued random variables on some probability space (Ω, P). With every compact interval $I = [s,t]$ with $s < t$ one may consider the weak*-closed subalgebra $\mathcal{A}_I$ of $L^\infty(\Omega, P)$ generated by random variables of the form $e^{i\phi(f)}$, f ranging over all test functions supported in I. When the random distribution ϕ is stationary and has independent values at every point, this family of subalgebras of $L^\infty(\Omega, P)$ has properties analogous to (1.2) and (1.3), in that there is a one-parameter group of measure preserving automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of $L^\infty(\Omega, P)$ which satisfies (1.3), and instead of (1.2) we have the assertion that the algebras $\mathcal{A}_{[r,s]}$ and $\mathcal{A}_{[s,t]}$ are *probabilistically independent* and generate $\mathcal{A}_{[r,t]}$ as a weak*-closed algebra.

One should keep in mind, however, that this commutative analogy has serious limitations. For example, we have already pointed out that in the case of interactions there is typically no normal γ-invariant state on $\mathcal{B}(H)$, and there is no reason to expect any normal state of $\mathcal{B}(H)$ to decompose as a product state relative to the decompositions of (1.2).

There is also some common ground with the Boolean algebras of type I factors of Araki and Woods [**AW69**], but here too there are significant differences. For example, the local algebras of (1.1) and (1.2) are associated with intervals (and more generally with finite unions of intervals), but not with more general Borel sets as in [**AW69**]. Moreover, here the translation group acts as automorphisms of the given structure whereas in [**AW69**] there is no hypothesis of symmetry with respect to translations.

1.3. Dynamics of Interactions. The C^*-algebra $\mathcal{A}$ of local observables is important because it provides a way of comparing ω_- and ω_+. Indeed, both states ω_- and ω_+ extend *uniquely* to γ-invariant states $\bar{\omega}_-$ and $\bar{\omega}_+$ of $\mathcal{A}$. We sketch the proof for ω_-.

PROPOSITION 1.3.1. *There is a unique γ-invariant state $\bar{\omega}_-$ of $\mathcal{A}$ such that*

$$\bar{\omega}_- \restriction_{\mathcal{A}_I} = \omega_- \restriction_{\mathcal{A}_I}$$

for every compact interval $I \subseteq (-\infty, 0]$.

PROOF. For existence of the extension, choose any compact interval $I = [a,b]$ and any operator $X \in \mathcal{A}_I$. Then for sufficiently large $s > 0$ we have $I - s \subseteq (-\infty, 0]$

and for these values of s $\omega_-(\gamma_{-s}(X))$ does not depend on s because ω_- is invariant under the action of $\{\gamma_t : t \leq 0\}$. Thus we can define $\bar\omega_-(X)$ unambiguously by

$$\bar\omega_-(X) = \lim_{t\to-\infty} \omega_-(\gamma_t(X)).$$

This defines a positive linear functional $\bar\omega_-$ on the unital $*$-algebra $\cup_I \mathcal{A}_I$, and now we extend $\bar\omega_-$ to all of $\mathcal{A}$ be norm-continuity. The extended state is clearly invariant under the action of γ_t, $t \in \mathbb{R}$.

The uniqueness of the state $\bar\omega_-$ is apparent. □

It is clear from the proof of Proposition 1.3.1 that these extensions of ω_- and ω_+ are *locally normal* in the sense that their restrictions to any localized subalgebra $\mathcal{A}_I$ define normal states on that type I factor. Thus, the local C^*-algebra $\mathcal{A}$ has a definite "state of the past" and a definite "state of the future" in the following sense:

Proposition 1.3.2. *For every $X \in \mathcal{A}$ and every normal state ρ of $\mathcal{B}(H)$ we have*

$$\lim_{t\to-\infty} \rho(\gamma_t(X)) = \bar\omega_-(X), \qquad \lim_{t\to+\infty} \rho(\gamma_t(X)) = \bar\omega_+(X)$$

Proof. Consider the first limit formula. The set of all $X \in \mathcal{A}$ for which this formula holds is clearly closed in the operator norm, hence it suffices to show that it contains $\mathcal{A}_I$ for every compact interval $I \subseteq \mathbb{R}$.

We will make use of the fact (discussed more fully at the beginning of section 5) that if ρ is any normal state of M and A is an operator in M then

$$\lim_{t\to-\infty} \rho(\gamma_t(A)) = \omega_-(A),$$

see formula (4.1). Choosing a real number T sufficiently negative that $I + T \subseteq (-\infty, 0]$, the preceding remark shows that for the operator $A = \gamma_T(X) \in M$ we have $\lim_{t\to-\infty} \rho(\gamma_t(A)) = \omega_-(A)$, and hence

$$\lim_{t\to-\infty} \rho(\gamma_t(X)) = \lim_{t\to-\infty} \rho(\gamma_{t-T}(\gamma_T(X))) = \omega_-(\gamma_T(X)) = \bar\omega_-(X).$$

The proof of the second limit formula is similar. □

Definition 1.3.3. The interaction (U, M), with past and future states ω_- and ω_+, is said to be trivial if $\bar\omega_- = \bar\omega_+$.

More generally, the norm $\|\bar\omega_- - \bar\omega_+\|$ gives some measure of the "strength" of the interaction, and of course we have $0 \leq \|\bar\omega_- - \bar\omega_+\| \leq 2$.

If there is a normal state ρ of $\mathcal{B}(H)$ that is invariant under the action of γ, then since ω_- (resp. ω_+) is the unique normal invariant state of α_- (resp. α_+) we must have $\rho\restriction_M = \omega_-$, $\rho\restriction_{M'} = \omega_+$, and hence $\bar\omega_- = \bar\omega_+ = \rho\restriction_{\mathcal{A}}$ by the uniqueness part of Proposition 1.3.1. In particular, *if the interaction is nontrivial then neither $\bar\omega_-$ nor $\bar\omega_+$ can be extended from $\mathcal{A}$ to a normal state of its strong closure $\mathcal{B}(H)$.*

Thus, whatever (normal) state ρ one chooses to watch evolve over time on operators in $\mathcal{A}$, it settles down to become $\bar\omega_+$ in the distant future, it must have come from $\bar\omega_-$ in the remote past, and the limit states do not depend on the choice of ρ. For a trivial interaction, nothing happens over the long term: for fixed X and ρ the function $t \in \mathbb{R} \mapsto \rho(\gamma_t(X))$ starts out very near some value (namely $\bar\omega_-(X)$), exhibits transient fluctuations over some period of time, and then settles down near the same value again. For a nontrivial interaction, there will be a definite change from the limit at $-\infty$ to the limit at $+\infty$ - at least for some choices of $X \in \mathcal{A}$.

REMARK 1.3.4 (Existence of Nontrivial Interactions). We have seen that every interaction gives rise to a pair of pure E_0-semigroups α^-, α^+, representing its "past" and its "future". However, we have seen no examples of interactions, and in particular, we have said nothing to indicate that nontrivial interactions exist. In the remainder of this lecture, we describe a body of results that address this key issue of existence in the simplest cases, namely where both past and future E_0-semigroups are cocycle perturbations of the CAR/CCR flows.

In order to exhibit examples of interactions with such properties, one has to address three questions. First, how does one construct examples of cocycle perturbations of CAR/CCR flows which are a) pure, and b) have a normal invariant state with specified properties? Second, given such a pair of E_0-semigroups, how does one determine when they can be "assembled" into an interaction so that one represents its future and the other represents its past? Third, given that there is an interaction assembled from a pair of E_0-semigroups α^- and α^+ in this way, how does one determine if that interaction is nontrivial? In sections 1.4, 1.5, 1.6 we discuss these three questions in turn.

1.4. Cocycle Perturbations of CAR/CCR flows. By a *cocycle* for an E_0-semigroup α acting on a type I factor M we mean a *strongly continuous* family of unitary operators $U = \{U_t : t \geq 0\}$ in M satisfying

$$U_{s+t} = U_s \alpha_s(U_t), \qquad s, t \geq 0. \tag{1.4}$$

Notice that (1.4) implies that $U_0 = \mathbf{1}$. The condition also implies that the family of endomorphisms $\beta = \{\beta_t : t \geq 0\}$ defined by

$$\beta_t(A) = U_t \alpha_t(A) U_t^*, \qquad t \geq 0 \tag{1.5}$$

satisfies the semigroup property $\beta_{s+t} = \beta_s \circ \beta_t$. The notion of cocycle perturbation is useful in the more general context of semigroups acting on arbitrary factors (see [**Arv03**]), but here we confine the discussion to the case where $M \sim \mathcal{B}(H)$.

DEFINITION 1.4.1 (Cocycle Perturbations). An E_0-semigroup β of the form (1.5) called a *cocycle perturbation* of α. Two E_0-semigroups are said to be *cocycle conjugate* if one of them is conjugate to a cocycle perturbation of the other.

The fundamental problem in the theory of E_0-semigroups is to find an effective classification up to cocycle conjugacy for E_0-semigroups acting on $\mathcal{B}(H)$. We will have more to say about cocycle perturbations, cocycle conjugacy, and the classification problem in Section 3.

1.4.1. *Numerical Index.* The numerical index is the simplest example of a cocycle conjugacy invariant for E_0-semigroups acting on $\mathcal{B}(H)$, and it is defined as follows. By a *unit* for an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ we mean a strongly continuous semigroup $T = \{t_t : t \geq 0\}$ of bounded operators on H such that $T_0 = \mathbf{1}$ and which satisfies the following commutation relation

$$\alpha_t(A) T_t = T_t A, \qquad t \geq 0, \quad A \in \mathcal{B}(H). \tag{1.6}$$

The set $\mathcal{U}_\alpha$ of all units is empty if α is of type III, but it is nonempty otherwise (see Lecture 3). In the latter cases, a simple argument based on the commutation formula leads to the conclusion that for every pair $S, T \in \mathcal{U}_\alpha$ there is a unique complex number $c(S,T)$ with the property

$$T_t^* S_t = e^{tc(S,T)} \mathbf{1}, \qquad t \geq 0.$$

The function $c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}$ is called the covariance function of α. The covariance function is conditionally positive definite in the sense that for every finite set $T_1, \dots, T_n$ of units and every set $\lambda_1, \dots, \lambda_n$ of complex numbers satisfying $\lambda_1 + \dots + \lambda_n = 0$, one has

$$\sum_{j,k=1}^{n} \lambda_j \bar{\lambda}_k c(T_j, T_k) \geq 0.$$

A familiar construction based on these inequalities produces a complex Hilbert space $H(\mathcal{U}_\alpha, c)$, and it is possible to show that this Hilbert space is separable whenever $\mathcal{U}_\alpha \neq \emptyset$. The index of α is defined as follows

$$\operatorname{index} \alpha = \begin{cases} \dim H(\mathcal{U}_\alpha, c), & if\ \mathcal{U}_\alpha \neq \emptyset \\ 2^{\aleph_0}, & if\ \mathcal{U}_\alpha = \emptyset. \end{cases}$$

The possible values of the index are $\{0, 1, 2, \dots, \infty = \aleph_0, c = 2^{\aleph_0}\}$, and the index is denumerable iff α is not of type III. It is a nontrivial fact that we have unrestricted validity of the logarithmic addition formula

$$\operatorname{index} \alpha \otimes \beta = \operatorname{index} \alpha + \operatorname{index} \beta.$$

Notice that, in order to calculate the index of an E_0-semigroup, one has to calculate the entire set $\mathcal{U}_\alpha$ of its units, as well as the covariance function

$$c : \mathcal{U}_\alpha \times \mathcal{U}_\alpha \to \mathbb{C}.$$

It is significant that these calculations can be carried out for specific examples, and in particular, we will see in Lecture 3 that the index of the CAR/CCR flow of rank $r = 1, 2, \dots, \infty$ is its rank. From that calculation it follows, for example, that the CAR/CCR flow of rank 2 is not conjugate to any cocycle perturbation of the CAR/CCR flow of rank 1.

1.4.2. *Eigenvalue Lists of Normal States.* We have already pointed out that one fundamental way that quantum probability differs from classical probability is that in quantum theory, there is no sensible notion of joint distribution. Another fundamental difference is that while in probability theory all nonatomic probability measures "look the same", that is not so in quantum theory.

More precisely, if $(X, \mathcal{A}, P)$ and $(Y, \mathcal{B}, Q)$ are two probability spaces based on standard Borel spaces $(X, \mathcal{A})$, $(Y, \mathcal{B})$ and if in both cases the probability of finding any singleton $\{p\}$ is zero, then the two probability spaces are isomorphic in the sense that there is a Borel isomorphism $\phi : X \to Y$ that transforms one measure into the other: $P(\phi^{-1}(F)) = Q(F)$ for every $F \in \mathcal{B}$. The quantum analogue of a nonatomic probability measure is a normal state of $\mathcal{B}(H)$, and if one is given two normal states ρ, ρ' defined on $\mathcal{B}(H)$, $\mathcal{B}(H')$ respectively, then there may or may not exist a $*$-isomorphism $\theta : \mathcal{B}(H) \to \mathcal{B}(H')$ which carries one to the other in the sense that

$$\rho'(\theta(A)) = \rho(A), \qquad A \in \mathcal{B}(H). \tag{1.7}$$

Indeed, for ρ and ρ' to be so related it is necessary for them to have the same eigenvalue list; and we now elaborate on this important invariant. By an *eigenvalue list* we mean a decreasing sequence of nonnegative real numbers $\lambda_1 \geq \lambda_2 \geq \dots$ with finite sum. Every normal state ω of a type I factor is associated with a positive operator of trace 1, whose eigenvalues counting multiplicity can be arranged into an eigenvalue list which will be denoted $\Lambda(\omega)$. If the factor is finite dimensional, we still consider $\Lambda(\omega)$ to be an infinite list by adjoining zeros in the obvious way. Given two eigenvalue lists $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ and $\Lambda' = \{\lambda'_1 \geq \lambda'_2 \geq \dots\}$, we will write

$$\|\Lambda - \Lambda'\| = \sum_{k=1}^{\infty} |\lambda_k - \lambda'_k|$$

for the ℓ^1-distance from one list to the other. A classical result of Hermann Weyl implies that if ρ and σ are normal states of a type I factor M, then we have

$$\|\Lambda(\rho) - \Lambda(\sigma)\| \leq \|\rho - \sigma\|.$$

Now suppose that ρ and ρ' are normal states of $\mathcal{B}(H)$ and $\mathcal{B}(H')$ respectively. Since a $*$-isomorphism $\theta : \mathcal{B}(H) \to \mathcal{B}(H')$ must be implemented by a unitary operator from H to H', it follows that ρ and ρ' are conjugate as in (1.7) only when $\Lambda(\rho) = \Lambda(\rho')$. More generally, assuming that both H and H' are of dimension $\aleph_0$, there is a sequence of $*$-isomorphisms $\theta_n : \mathcal{B}(H) \to \mathcal{B}(H')$ such that

$$\lim_{n\to\infty} \|\rho' \circ \theta_n - \rho\| = 0$$

iff ρ and ρ' have the same eigenvalue list.

Finally, note that the eigenvalue list of a vector state of $\mathcal{B}(H)$ has the form $\{1, 0, 0, \dots\}$; and more generally, the eigenvalue list of a state of $\mathcal{B}(H)$ has only a finite number of nonzero terms iff the state is continuous in the weak operator topology of $\mathcal{B}(H)$.

THEOREM 1.4.2 (Existence of Cocycle Perturbations). *Let $N = 1, 2, \dots, \infty$ and let $\Lambda = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ be an eigenvalue list with only a finite number of nonzero terms, such that $\lambda_1 + \lambda_2 + \dots = 1$. There is a cocycle perturbation α of the CAR/CCR flow of index N which is pure, and which has an absorbing state with eigenvalue list Λ.*

The proof of Theorem 1.4.2 is very indirect, and we merely outline the four key ideas behind the argument. Starting with the finite sequence $\lambda_1 \geq \dots \geq \lambda_n > 0$ of positive terms of the given eigenvalue list, one first constructs a semigroup of unital completely positive maps $P = \{P_t : t \geq 0\}$ acting on the matrix algebra $M_n(\mathbb{C})$ which satisfies $P_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, which is *pure* in a sense that is appropriate for CP semigroups, and which leaves invariant a state of $M_n(\mathbb{C})$ with precisely the eigenvalue list $\{\lambda_1, \dots, \lambda_n\}$. The second step appeals to the Dilation Theory of Lecture 2 in order to find a minimal dilation of P to an E_0-semigroup α that acts on a Hilbert space H. Making use of minimality, it is possible to show that α is a pure E_0-semigroup which acts on a type I factor and has a normal invariant state with an eigenvalue list whose nonzero terms are precisely $\lambda_1, \dots, \lambda_n$ - and by a further adjustment one can arrange that the index of α is the given integer N. Third, one appeals to results of [**Arv99**] which imply that α is completely spatial, and finally, one may appeal to the classification results of [**Arv89a**] for completely spatial E_0-semigroups to infer that α is conjugate to a cocycle perturbation of the CAR/CCR flow of index N.

The finiteness hypothesis on the eigenvalue list allowed us to work with matrix algebras in constructing the initial CP semigroup P. We conjecture that similar constructions can be carried out with CP semigroups having bounded generators that act on an infinite dimensional type I factor, but this has not been proved. As a test problem for such techniques, we propose

Problem: Can the hypothesis that Λ is finitely nonzero be dropped from Theorem 1.4.2?

Theorem 1.4.2 asserts that for every type I E_0-semigroup α and every finitely nonzero eigenvalue list Λ, there is a cocycle perturbation of α that is pure and which has an absorbing state with a specified eigenvalue list. It seems reasonable to ask if the same is true if one drops the hypothesis that α should be of type I. As a limited step in that direction, we conjecture that the following question has an affirmative answer.

Problem: Can an arbitrary E_0-semigroup be perturbed by a cocycle into a pure E_0-semigroup?

1.5. Existence of Interactions. Given a pair of pure E_0-semigroups α^-, α^+ acting, respectively on $\mathcal{B}(H^-)$, $\mathcal{B}(H^+)$, each of which is a cocycle perturbation of a CAR/CCR flow, it is natural to ask when they can be assembled into a history. Thus, we seek a simple test for determining when there is a one-parameter unitary group $U = \{U_t : t \geq 0\}$ acting on $H^- \otimes H^+$ with the property that its associated automorphsim group $\gamma_t = U_t \cdot U_t^*$ should satisfy

$$\gamma_{-t}(A \otimes \mathbf{1}_{H^+}) = \alpha_t^-(A) \otimes \mathbf{1}_{H_-}, \qquad \gamma_t(\mathbf{1}_{H_-} \otimes B) = \mathbf{1}_{H_-} \otimes \alpha_t^+(B),$$

for all $t \geq 0$, $A \in \mathcal{B}(H^-)$, $B \in \mathcal{B}(H^+)$.

More generally, let $\mathcal{M}$ be a type I subfactor of $\mathcal{B}(H)$, and let α, β be two E_0-semigroups acting, respectively, on $\mathcal{M}$ and its commutant $\mathcal{M}'$. In Lecture 3 we will discuss a general result - Theorem 3.6.1 - which provides a necessary and sufficient condition for the existence of a one-parameter unitary group $\{U_t : t \in \mathbb{R}\}$ acting on H whose associated automorphism group $\gamma_t(A) = U_t A U_t^*$ has α as its past and β as its future in the sense that

$$\gamma_{-t} \restriction_{\mathcal{M}} = \alpha_t, \qquad \gamma_t \restriction_{\mathcal{M}'} = \beta_t, \qquad t \geq 0, \tag{1.8}$$

the condition being that the product systems of α and β are *anti-isomorphic.*

What we require for the current discussion is the following corollary of Theorem 3.6.1, which can be viewed as a counterpart for noncommutative dynamics of von Neumann's theorem on the existence of self-adjoint extensions of symmetric operators in terms of their deficiency indices.

Theorem 1.5.1. *Let α, β be two E_0-semigroups acting, respectively, on $\mathcal{B}(H)$, $\mathcal{B}(K)$, each of which is a cocycle perturbation of a CCR/CAR flow. The following are equivalent.*

(i) *There is a one-parameter group $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{B}(H \otimes K)$ satisfying*

$$\gamma_{-t}(A \otimes \mathbf{1}) = \alpha_t(A) \otimes \mathbf{1}, \qquad \gamma_t(\mathbf{1} \otimes B) = \mathbf{1} \otimes \beta_t(B),$$

for all $t \geq 0$, $A \in \mathcal{B}(H)$, $B \in \mathcal{B}(K)$.

(ii) $\operatorname{index} \alpha = \operatorname{index} \beta$.

In Section 3 we will show how Theorem 1.5.1 follows from the results of that section concerning the relation between the structure of product systems and the dynamics of histories. We emphasize that when the group γ of Theorem 1.5.1 exists, it is not uniquely determined by α and β, and that issue will also be discussed in Lecture 3. Together, Theorems 1.4.2 and 1.5.1 lead one to the following conclusion:

THEOREM 1.5.2 (Existence of Interactions). *Let $n = 1, 2, \ldots, \infty$ and let Λ_- and Λ_+ be two eigenvalue lists having only a finite number of nonzero terms. There is an interaction $(U, \mathcal{M})$ whose past and future E_0-semigroups are cocycle perturbations of the CAR/CCR flow of index n, and whose absorbing states have eigenvalue lists Λ_-, Λ_+ respectively.*

PROOF OF THEOREM 1.5.2. Fix $n = 1, 2, \ldots, \infty$ and let Λ_+, Λ_- be eigenvalue lists with finitely many nonzero terms. By Theorem 1.4.2, there is a pure cocycle perturbation α^+ of the CAR/CCR flow of index n which has an absorbing state with eigenvalue list Λ_+. Similarly, we find a pure cocycle perturbation α_- of the same CAR/CCR flow which has an absorbing state with list Λ_-. Since both α^- and α^+ are pure, we conclude from Theorem 1.5.1 that there is a history with past and future E_0-semigroups given, respectively, by α^- and α^+. This history must be an interaction since both α^- and α^+ have normal invariant states. □

REMARK 1.5.3 (On the Existence and Nonexistence of Dynamics). Given an arbitrary E_0-semigroup α, it is natural to ask if cocycle perturbations of α can represent both the past and future of some history, as can cocycle perturbations of the CAR/CCR flows. More precisely, is there a history whose past and future E_0-semigroups are *both* conjugate to cocycle perturbations of α?

Significantly, the answer can be no. Boris Tsirelson has given examples of product systems which are *not* anti-isomorphic to themselves [**Tsi00a**]. By Theorem 4.5.2 below, such a product system is isomorphic to the product system of some E_0-semigroup α, and it follows that there are E_0-semigroups α whose product systems are not anti-isomorphic themselves. Since cocycle perturbations of E_0-semigroups must have isomorphic product systems, the general criteria of Theorem 3.6.1 implies that such E_0-semigroups cannot serve as both the past and future of any history.

On the other hand, every E_0-semigroup α acting on $\mathcal{B}(H)$ can serve as the "past" of some automorphism group. To see why, let E be the product system of α and let E^{op} be the product system opposite to E. Theorem 4.5.2 implies that there is an E_0-semigroup β, acting on $\mathcal{B}(K)$, whose product system is isomorphic to E^{op} and therefore anti-isomorphic to E. We may conclude from the Theorem 3.6.1 that there is a one parameter group of automorphisms γ acting on $\mathcal{B}(H \otimes K)$ which satisfies (1.8) by having $\alpha \otimes \mathbf{1}$ (acting on $\mathcal{B}(H) \otimes \mathbf{1}$) as its past and $\mathbf{1} \otimes \beta$ (acting on $\mathbf{1} \otimes \mathcal{B}(K)$) as its future. We point out that a more elementary proof of this extension result, based on the theory of spectral subspaces, can be found in [**AK92**].

1.6. The Interaction Inequality. Given a pair of eigenvalue lists Λ_-, Λ_+, each of which has only a finite number of nonzero terms, and given a positive integer $n = 1, 2, \ldots, \infty$, Theorem 1.5.2 implies that there is an interaction whose past and future absorbing states ω_-, ω_+, have eigenvalue lists Λ_- and Λ_+ respectively. One might expect that such an interaction should be nontrivial when $\Lambda_- \neq \Lambda_+$, but this is is far from self-evident.

More precisely, we seek a method for computing, or at least estimating, the quantity $\|\bar\omega_- - \bar\omega_+\|$ in terms of the eigenvalue lists of ω_- and ω_+. We now describe a general solution of this problem involving an inequality that appears to be of some interest in its own right.

THEOREM 1.6.1 (Interaction Inequality). *Let (U, M) be an interaction with past and future states ω_-, ω_+ on M, M' respectively, and let $\bar\omega_-$ and $\bar\omega_+$ denote their* *extensions to γ-invariant states of $\mathcal{A}$. Then*

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda(\omega_- \otimes \omega_-) - \Lambda(\omega_+ \otimes \omega_+)\|.$$

REMARK 1.6.2. Notice the tensor product of states on the right. For example, $\Lambda(\omega_- \otimes \omega_-)$ is obtained from the eigenvalue list $\Lambda(\omega_-) = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ of ω_- by rearranging the doubly infinite sequence of all products $\lambda_i\lambda_j$, $i, j = 1, 2, \dots$ into decreasing order. It can be an unpleasant combinatorial chore to calculate $\Lambda(\omega_- \otimes \omega_-)$ even when $\Lambda(\omega_-)$ is relatively simple and finitely nonzero; but it is also possible to show that if A and B are two positive trace class operators such that $\Lambda(A \otimes A) = \Lambda(B \otimes B)$, then $\Lambda(A) = \Lambda(B)$.

Note too that we do not assume that the eigenvalue lists of $\omega_\pm$ are finitely nonzero in Theorem 1.6.1. The reader is referred to [**Arv03**] for these details, as well as for the proof of Theorem 1.6.1.

The results of the preceding three sections are summarized as follows:

COROLLARY 1.6.3 (Nontriviality of Interactions). *Let $n = 1, 2, \dots, \infty$ be a positive integer and let Λ_-, Λ_+ be two eigenvalue lists with finitely many nonzero terms. Then there is an interaction whose past and future E_0-semigroups are cocycle perturbations of the CAR/CCR flow of index n, and whose past and future absorbing states ω_-, ω_+ have eigenvalue lists Λ_-, Λ_+ respectively.*

Such an interaction in nontrivial whenever $\Lambda_- \neq \Lambda_+$, and in general we have

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda_- \otimes \Lambda_- - \Lambda_+ \otimes \Lambda_+\|.$$

REMARK 1.6.4 (Existence of Strong Interactions). To illustrate Corollary 1.6.3, we use it to show that strong interactions exist in the sense that for every $\epsilon > 0$ there is an interaction with the property

$$\|\bar{\omega}_- - \bar{\omega}_+\| > 2 - \epsilon. \tag{1.9}$$

To see that, choose positive integers $p < q$ and consider the eigenvalue lists

$$\begin{aligned}\Lambda_- &= \{1/p, 1/p, \dots, 1/p, 0, 0, \dots\}\\ \Lambda_+ &= \{1/q, 1/q, \dots, 1/q, 0, 0, \dots\},\end{aligned}$$

where $1/p$ is repeated p times and $1/q$ is repeated q times. Corollary 1.6.3 implies that there is an interaction (U, M) whose past and future E_0-semigroups are cocycle perturbations of the CAR/CCR flow of index 1, whose past and future absorbing states satisfy $\Lambda(\omega_-) = \Lambda_-$ and $\Lambda(\omega_+) = \Lambda_+$, and we have

$$\|\bar{\omega}_- - \bar{\omega}_+\| \geq \|\Lambda_- \otimes \Lambda_- - \Lambda_+ \otimes \Lambda_+\|.$$

If we neglect zeros, the eigenvalue list of $\Lambda_- \otimes \Lambda_-$ consists of the single eigenvalue $1/p^2$, repeated p^2 times, and that of $\Lambda_+ \otimes \Lambda_+$ consists of $1/q^2$ repeated q^2 times. Thus

$$\|\Lambda_- \otimes \Lambda_- - \Lambda_+ \otimes \Lambda_+\| = p^2(1/p^2 - 1/q^2) + (q^2 - p^2)/q^2 = 2 - 2p^2/q^2,$$

and the inequality (1.9) follows whenever q and p satisfy $q > p\sqrt{2/\epsilon}$.

2. Generators of Dynamics and Dilation Theory

In this lecture we describe a new approach to the dilation theory of quantum dynamical semigroups that is based on the notion of an A-dynamical system. These objects provide the C^*-algebraic structure that underlies much of noncommutative dynamics, whether it takes place in C^*-algebra or a von Neumann algebra, independently of issues relating to dilation theory. After describing the general properties of A-dynamical systems, we introduce α-expectations and noncommutative moment polynomials, and show how these objects enter into the construction of C^*-dilations.

Once one is in possession of this C^*-algebraic infrastructure, one can establish the existence and uniqueness of dilations for quantum dynamical semigroups acting on von Neumann algebras in a natural way.

2.1. Generators of Noncommutative Dynamics. The flow of time in quantum theory is represented by a one-parameter group of $*$-automorphisms $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of a C^*-algebra B. There is often a C^*-subalgebra $A \subseteq B$ that can be singled out from physical considerations which, together with its time translates, generates B. For example, in a nonrelativistic quantum mechanical system with n degrees of freedom the flow of time is represented by a one-parameter group of automorphisms of $\mathcal{B}(L^2(\mathbb{R}^n))$ of the form $\alpha_t(T) = e^{itH}Te^{-itH}$, $t \in \mathbb{R}$, where H is a self-adjoint Schrödinger operator of the form

$$H = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + V(X_1, \dots, X_n).$$

Here, $X_1, \dots, X_n$ denote the configuration observables at time 0

$$X_k : \xi(x_1, \dots, x_n) \mapsto x_k \xi(x_1, \dots, x_n),$$

defined appropriately on a common dense domain in $L^2(\mathbb{R}^n)$, and V denotes the potential associated with the interaction forces. The functional calculus provides a faithful representation of the commutative C^*-algebra $C_0(\mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$ by way of $f \mapsto f(X_1, \dots, X_n)$, and these functions of the configuration operators form a commutative C^*-subalgebra $A \subseteq \mathcal{B}(H)$. It is not hard to see that the set of all time translates $\{\alpha_t(A) : t \geq 0\}$ of A generates an irreducible C^*-subalgebra B of $\mathcal{B}(H)$. In particular, for different times $t_1 \neq t_2$, the C^*-algebras $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ fail to commute with each other. Indeed, *no nontrivial relations appear to exist between* $\alpha_{t_1}(A)$ *and* $\alpha_{t_2}(A)$ *when* $t_1 \neq t_2$.

We now look closely at this phenomenon in general. Throughout this lecture A be denote an arbitrary but fixed C^*-algebra.

Definition 2.1.1. An A-dynamical system is a triple (ι, B, α) consisting of a semigroup $\alpha = \{\alpha_t : t \geq 0\}$ of $*$-endomorphisms acting on a C^*-algebra B and an injective $*$-homomorphism $\iota : A \to B$, such that B is generated by $\cup_{t\geq 0}\alpha_t(\iota(A))$.

Notice that we impose no continuity requirement on the semigroup α_t in its time parameter t. We lighten notation by identifying A with its image $\iota(A)$ in B, thereby replacing ι with the inclusion map $\iota : A \subseteq B$. Thus, an A-dynamical system is a dynamical system (B, α) that contains A as a C^*-subalgebra in a specified way, with the property that B is the norm-closed linear span of finite products

$$B = \overline{\operatorname{span}}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots\alpha_{t_k}(a_k)\} \tag{2.1}$$

where $t_1, \dots, t_k \geq 0$, $a_1, \dots, a_k \in A$, $k = 1, 2, \dots$

Our aim is to say something sensible about the class of *all* A-dynamical systems, and to obtain more detailed information about certain of its members. The examples described in the preceding paragraphs illustrate the fact that in even the simplest cases where A is $C(X)$, the structure of individual A-dynamical systems can be very complex.

There is a natural hierarchy in the class of all A-dynamical systems, defined by $(\iota, B, \alpha) \geq (\tilde{\iota}, \tilde{B}, \tilde{\alpha})$ iff there is a $*$-homomorphism $\theta : B \to \tilde{B}$ satisfying $\theta \circ \alpha_t = \tilde{\alpha}_t \circ \theta$, $t \geq 0$, and $\theta(a) = a$ for $a \in A$. Since θ fixes A, it follows from (2.1) that θ must be surjective, $\theta(B) = \tilde{B}$, hence $(\tilde{\iota}, \tilde{B}, \tilde{\alpha})$ is a *quotient* of (ι, B, α). Two A-dynamical systems are said to be *equivalent* if there is a map θ as above that is an isomorphism of C^*-algebras. This will be the case iff each of the A-dynamical systems dominates the other. One may also think of the class of all A-dynamical systems as a category, whose objects are A-dynamical systems and whose maps θ are the ones just described.

Remark 2.1.2 (The Universal A-dynamical System). There is a largest equivalence class in this hierarchy, whose representatives are called *universal* A-dynamical systems. We exhibit one as follows. Consider the free product of an infinite family of copies of A indexed by the nonnegative real numbers

$$\mathcal{P}A = *_{t \geq 0} A_t, \qquad A_t = A.$$

Thus, we have a family of $*$-homomorhisms θ_t of A into the C^*-algebra $\mathcal{P}A$ such $\mathcal{P}A$ is generated by $\cup\{\theta_t(A) : t \geq 0\}$ and such that the following universal property is satisfied: for every family $\bar{\pi} = \{\pi_t : t \geq 0\}$ of $*$-homomorphisms of A into some other C^*-algebra B, there is a unique $*$-homomorphism $\rho : \mathcal{P}A \to B$ such that $\pi_t = \rho \circ \theta_t$, $t \geq 0$. Nondegenerate representations of $\mathcal{P}A$ correspond to families $\bar{\pi} = \{\pi_t : t \geq 0\}$ of representations $\pi_t : A \to \mathcal{B}(H)$ of A on a common Hilbert space H, subject to no condition other than the triviality of their common nullspace

$$\xi \in H, \quad \pi_t(A)\xi = \{0\}, \quad t \geq 0 \implies \xi = 0.$$

A simple argument establishes the existence of $\mathcal{P}A$ by taking the direct sum of a sufficiently large set of such representation sequences $\bar{\pi}$.

This definition does not exhibit $\mathcal{P}A$ in concrete terms (see §2.4 for that), but it does allow us to define a universal A-dynamical system. The universal property of $\mathcal{P}A$ implies that there is a semigroup of shift endomorphisms $\sigma = \{\sigma_t : t \geq 0\}$ acting on $\mathcal{P}A$ and defined uniquely by $\sigma_t \circ \theta_s = \theta_{t+s}$, $s, t \geq 0$. Using the universal properties of $\mathcal{P}A$, it is quite easy to verify that θ_0 is an injective $*$-homomorphism of A in $\mathcal{P}A$, and we use this map to identify A with $\theta_0(A) \subseteq \mathcal{P}A$. Thus *the triple $(i, \mathcal{P}A, \sigma)$ becomes an A-dynamical system with the property that every other A-dynamical system is subordinate to it.*

2.2. α-Expectations and C^*-Dilations. Suppose now that we are given a semigroup $P = \{P_t : t \geq 0\}$ of completely positive contractions acting on a C^*-algebra A. We are interested in singling out certain A-dynamical systems (ι, B, α) that admit a conditional expectation $E : B \to A$ with the property

$$E(\alpha_t(a)) = P_t(a), \qquad a \in A, \quad t \geq 0.$$

Of course, for many A-dynamical systems (ι, B, α) there will be no such conditional expectation; and even if such an expectation exists, there is no reason to expect it to be uniquely determined by the preceding formula. We now introduce a class of conditional expectations, called α-expectations, that occupy a central position in the dilation theory of CP semigoups.

Let us first review some common terminology. Let $A \subseteq B$ be an inclusion of C^*-algebras. For any subset S of B we write $[S]$ for the norm-closed linear span of S. The subalgebra A is said to be *hereditary* if for $a \in A$ and $b \in B$, one has

$$0 \leq b \leq a \implies b \in A.$$

The hereditary subalgebra of B generated by a subalgebra A is the closed linear span $[ABA]$ of all products axb, $a, b \in A$, $x \in B$, and in general $A \subseteq [ABA]$. A *corner* of B is a hereditary subalgebra of the particular form $A = pBp$ where p is a projection in the multiplier algebra $M(B)$ of B.

We also make essential use of *conditional expectations* $E : B \to A$. A conditional expectation is an idempotent positive linear map with range A, satisfying $E(ax) = aE(x)$ for $a \in A$, $x \in B$. Conditional expectations are completely positive linear maps of norm 1 whenever $A \neq \{0\}$. When $A = pBp$ is a *corner* of B, the map $E(x) = pxp$, defines a conditional expectation of B onto A. On the other hand, many of the conditional expectations encountered here do not have this simple form, even when A has a unit. Indeed, if A is subalgebra of B that is *not* hereditary, then there is no natural conditional expectation $E : B \to A$. For example, the universal dynamical system $(\iota, \mathcal{P}A, \sigma)$ never contains A as a hereditary subalgebra, hence there is no "obvious" conditional expectation $E : \mathcal{P}A \to A$.

Definition 2.2.1. Let (ι, B, α) be an A-dynamical system. An α-expectation is a conditional expectation $E : B \to A$ having the following two properties:

E1. Equivariance: $E \circ \alpha_t = E \circ \alpha_t \circ E$, $t \geq 0$.

E2. The restriction of E to the hereditary subalgebra generated by A is multiplicative, $E(xy) = E(x)E(y)$, $x, y \in [ABA]$.

Note that an *arbitrary* conditional expectation $E : B \to A$ gives rise to a family of linear maps $P = \{P_t : t \geq 0\}$ of A to itself by way of $P_t(a) = E(\alpha_t(a))$, $a \in A$. Each P_t is a completely positive contraction. When E is an α-expectation property E1 implies that P_t is related to α_t by

$$E \circ \alpha_t = P_t \circ E, \qquad t \geq 0, \tag{2.2}$$

and from the relation (2.2) one finds that P must satisfy the semigroup property $P_s \circ P_t = P_{s+t}$.

Property E2 is of course automatic if A is a hereditary subalgebra of B. It is a fundamentally *noncommutative* hypothesis on B. For example, if Y is a compact Hausdorff space and $B = C(Y)$, then every unital subalgebra $A \subseteq C(Y)$ generates $C(Y)$ as a hereditary algebra, and the only linear maps $E : C(Y) \to A$ satisfying E2 are $*$-endomorphisms of $C(Y)$.

Definition 2.2.2. Let $P = \{P_t : t \geq 0\}$ be a semigroup of completely positive contractions acting on a C^*-algebra A. By a C^*-dilation of (A, P) we mean an A-dynamical system (ι, B, α) with the additional property that there is an α-expectation $E : B \to A$ satisfying

$$P_t(a) = E(\alpha_t(a)), \qquad a \in A, \quad t \geq 0.$$

We will see in the following section that an α-expectation $E : B \to A$ is uniquely determined by the family of completely positive maps $a \in A \mapsto E(\alpha_t(a))$, $t \geq 0$;

thus, the α-expectation associated with a C^*-dilation of a given CP semigroup P is *uniquely determined* by P.

The following result implies that C^*-dilations always exist; and in fact, the universal A-dynamical system serves as a simultaneous C^*-dilation of every semigroup of completely positive contractions acting on A.

THEOREM 2.2.3. *For every semigroup of completely positive contractions $P = \{P_t : t \geq 0\}$ acting on A, there is a unique σ-expectation $E : \mathcal{P}A \to A$ satisfying*

$$P_t(a) = E(\sigma_t(a)), \qquad a \in A, \quad t \geq 0. \tag{2.3}$$

Both assertions are nontrivial; we discuss uniqueness in the following section, existence is discussed in §2.4.

2.3. Moment Polynomials. The theory of C^*-dilations rests on properties of certain noncommutative polynomials that are defined recursively as follows.

PROPOSITION 2.3.1. *Let A be an algebra over a field $\mathbb{F}$. For every family of linear maps $\{P_t : t \geq 0\}$ of A to itself satisfying the semigroup property $P_{s+t} = P_s \circ P_t$ and $P_0 = \mathrm{id}$, there is a unique family of multilinear mappings from A to itself, indexed by the k-tuples of nonnegative real numbers, $k = 1, 2, \dots,$ where for a fixed k-tuple $\bar{t} = (t_1, \dots, t_k)$*

$$a_1, \dots, a_k \in A \mapsto [\bar{t}; a_1, \dots, a_k] \in A$$

is a k-linear mapping, all of which satisfy

MP1. $P_s([\bar{t}; a_1, \dots, a_k]) = [t_1 + s, t_2 + s, \dots, t_k + s; a_1, \dots, a_k]$.

MP2. *Given a k-tuple for which $t_\ell = 0$ for some ℓ between 1 and k,*

$$[\bar{t}; a_1, \dots, a_k] = [t_1, \dots, t_{\ell-1}; a_1, \dots, a_{\ell-1}]a_\ell[t_{\ell+1}, \dots, t_k; a_{\ell+1}, \dots, a_k].$$

REMARK 2.3.2. The proofs of both existence and uniqueness are straightforward arguments using induction on the number k of variables. Note that in the second axiom MP2, we make the natural conventions when ℓ has one of the extreme values 1, k. For example, if $\ell = 1$, then MP2 should be interpreted as

$$[0, t_2, \dots, t_k; a_1, \dots, a_k] = a_1[t_2, \dots, t_k; a_2, \dots, a_k].$$

In particular, in the linear case $k = 1$, MP2 makes the assertion

$$[0; a] = a, \qquad a \in A;$$

and after applying axiom MP1 one obtains

$$[t; a] = P_t(a), \qquad a \in A, \quad t \geq 0.$$

One may calculate any particular moment polynomial explicitly, but the computations quickly become a tedious exercise in the arrangement of parentheses. For example,

$$[2, 6, 3, 4; a, b, c, d] = P_2(aP_1(P_3(b)cP_1(d))),$$
$$[6, 4, 2, 3; a, b, c, d] = P_2(P_2(P_2(a)b)cP_1(d)).$$

Finally, we remark that when A is a C^*-algebra and the linear maps satisfy $P_t(a)^* = P_t(a^*)$, $a \in A$, $t \geq 0$, then the associated moment polynomials obey the following symmetry

$$[t_1, \dots, t_k; a_1, \dots, a_k]^* = [t_k, \dots, t_1; a_k^*, \dots, a_1^*]. \tag{2.4}$$

Indeed, one finds that the sequence of polynomials $[[\cdot;\cdot]]$ defined by

$$[[t_1,\dots,t_k;a_1,\dots,a_k]] = [t_k,\dots,t_1;a_k^*,\dots,a_1^*]^*$$

also satisfies axioms MP1 and MP2, and hence must coincide with the moment polynomials of $\{P_t\}$ by the uniqueness assertion of Proposition 2.3.1.

These polynomials are important because they are the expectation values of certain A-dynamical systems.

THEOREM 2.3.3. *Let $P = \{P_t : t \geq 0\}$ be a semigroup of completely positive maps on A satisfying $\|P_t\| \leq 1$, $t \geq 0$, with associated moment polynomials $[t_1,\dots,t_k;a_1,\dots,a_k]$.*

Let (i, B, α) be an A-dynamical system and let $E : B \to A$ be an α-expectation with the property $E(\alpha_t(a)) = P_t(a)$, $a \in A$, $t \geq 0$. Then

$$E(\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)\cdots\alpha_{t_k}(a_k)) = [t_1,\dots,t_k;a_1,\dots,a_k]. \tag{2.5}$$

for every $k = 1, 2, \dots$, $t_k \geq 0$, $a_k \in A$. In particular, there is at most one α-expectation $E : B \to A$ satisfying $E(\alpha_t(a)) = P_t(a)$, $a \in A$, $t \geq 0$.

PROOF. One applies the uniqueness of moment polynomials as follows. Properties E1 and E2 of Definition 2.2.1 imply that the sequence of polynomials $[[\cdot;\cdot]]$ defined by

$$[[t_1,\dots,t_k;a_1,\dots,a_k]] = E(\alpha_{t_1}(a_1)\cdots\alpha_{t_k}(a_k))$$

must satisfy the two axioms MP1 and MP2. For example, notice that E2 implies

$$E(xay) = E(x)aE(y) \qquad x, y \in B, \quad a \in A, \tag{2.6}$$

since for an approximate unit e_n for A we can write $E(xay)$ as follows:

$$\begin{aligned}\lim_{n\to\infty} e_n E(xae_n y)e_n &= \lim_{n\to\infty} E(e_n xae_n ye_n) = \lim_{n\to\infty} E(e_n xa)E(e_n ye_n)\\ &= \lim_{n\to\infty} e_n E(x)ae_n E(y)e_n = E(x)aE(y).\end{aligned}$$

Property (2.6) implies that the polynomials $[[\cdot;\cdot]]$ satisfy MP2. Moreover, from (2.2) we find that

$$\begin{aligned}P_s([[t_1,\dots,t_n;a_1,\dots,a_n]]) = P_s(E(\alpha_{t_1}(a_1))\cdots\alpha_{t_n}(a_n))) =\\ E(\alpha_s(\alpha_{t_1}(a_1)\cdots\alpha_{t_n}(a_n))) = E(\alpha_{t_1+s}(a_1)\cdots\alpha_{t_n+s}(a_n)),\end{aligned}$$

hence MP1 is satisfied as well. Thus (2.5) follows from the uniqueness assertion of Proposition 2.3.1. The uniqueness of the α-expectation associated with P is now apparent from formulas (2.5) and (2.1). $\square$

2.4. Construction of C^*-dilations. The proof of the existence assertion of Theorem 2.2.3 is based on a construction that exhibits $\mathcal{P}A$ as the enveloping C^*-algebra of a Banach $*$-algebra $\ell^1(\Sigma)$, in such a way that the desired α-expectation appears as a completely positive map on $\ell^1(\Sigma)$. We now describe this construction of $\mathcal{P}A$ in some detail, but we do not prove that the α-expectation is completely positive here. Full details can be found in [**Arv03**].

Let S be the set of finite sequences $\bar{t} = (t_1, t_2, \dots, t_k)$ of nonnegative real numbers t_i, $k = 1, 2, \dots$ which have distinct neighbors,

$$t_1 \neq t_2, t_2 \neq t_3, \dots, t_{k-1} \neq t_k.$$

Multiplication and involution are defined in S as follows. The product of two elements $\bar{s} = (s_1, \dots, s_k), \bar{t} = (t_1, \dots, t_\ell) \in S$ is defined by conditional concatenation

$$\bar{s} \cdot \bar{t} = \begin{cases} (s_1, \dots, s_k, t_1, \dots, t_\ell), & \text{if } s_k \neq t_1, \\ (s_1, \dots, s_k, t_2, \dots, t_\ell), & \text{if } s_k = t_1, \end{cases}$$

where we make the natural conventions when $\bar{t} = (t)$ is of length 1, namely $\bar{s} \cdot (t) = (s_1, \dots, s_k, t)$ if $s_k \neq t$, and $\bar{s} \cdot (t) = \bar{s}$ if $s_k = t$. The involution in S is defined by reversing the order of components

$$(s_1, \dots, s_k)^* = (s_k, \dots, s_1).$$

One finds that S is an associative $*$-semigroup.

Fixing a C^*-algebra A, we attach a Banach space Σ_ν to every k-tuple $\tau = (t_1, \dots, t_k) \in S$ as follows

$$\Sigma_\tau = \underbrace{A \hat{\otimes} \cdots \hat{\otimes} A}_{k \text{ times}},$$

the k-fold projective tensor product of copies of the Banach space A. We assemble the Σ_τ into a family of Banach spaces over S, $p : \Sigma \to S$, by way of $\Sigma = \{(\tau, \xi) : \tau \in S, \xi \in E_\tau\}$, $p(\tau, \xi) = \tau$.

We introduce a multiplication in Σ as follows. Fix $\lambda = (\lambda_1, \dots, \lambda_k)$ and $\mu = (\mu_1, \dots, \mu_\ell)$ in S and choose $\xi \in \Sigma_\mu$, $\eta \in \Sigma_\nu$. If $\lambda_k \neq \mu_1$ then $\xi \cdot \eta$ is defined as the tensor product $\xi \otimes \eta \in \Sigma_{\mu \cdot \nu}$. If $\lambda_k = \mu_1$ then we must tensor over A and make the obvious identifications. More explicitly, in this case there is a natural map of the tensor product $\Sigma_\mu \otimes_A \Sigma_\nu$ onto $\Sigma_{\mu \cdot \nu}$ by making identifications of elementary tensors as follows:

$$(a_1 \otimes \cdots \otimes a_k) \otimes_A (b_1 \otimes \cdots \otimes b_\ell) \sim a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k b_1 \otimes b_2 \otimes \cdots \otimes b_\ell.$$

With this convention $\xi \cdot \eta$ is defined by

$$\xi \cdot \eta = \xi \otimes_A \eta \in \Sigma_{\mu \cdot \nu}.$$

This defines an associative multiplication in the family of Banach spaces Σ. There is also a natural involution in Σ, defined on each Σ_μ, $\mu = (s_1, \dots, s_k)$ as the unique antilinear isometry to Σ_{μ^*} satisfying

$$((s_1, \dots, s_k), a_1 \otimes \cdots \otimes a_k)^* = ((s_k, \dots, s_1), a_k^* \otimes \cdots \otimes a_1^*).$$

This defines an isometric antilinear mapping of the Banach space Σ_μ onto Σ_{μ^*}, for each $\mu \in S$, and thus the structure Σ becomes an involutive $*$-semigroup in which each fiber Σ_μ is a Banach space.

Let $\ell^1(\Sigma)$ be the Banach $*$-algebra of summable sections. The norm and involution are the natural ones $\|f\| = \sum_{\mu \in \Sigma} \|f(\mu)\|$, $f^*(\mu) = f(\mu^*)^*$. Noting that $\Sigma_\lambda \cdot \Sigma_\mu \subseteq \Sigma_{\lambda \cdot \mu}$, the multiplication in $\ell^1(\Sigma)$ is defined by convolution

$$f * g(\nu) = \sum_{\lambda \cdot \mu = \nu} f(\lambda) \cdot g(\mu),$$

and one easily verifies that $\ell^1(\Sigma)$ is a Banach $*$-algebra.

For $\mu = (s_1, \dots, s_k) \in S$ and $a_1, \dots, a_k \in A$ we define the function

$$\delta_\mu \cdot a_1 \otimes \cdots \otimes a_k \in \ell^1(\Sigma)$$

to be zero except at μ, and at μ it has the value $a_1\otimes\cdots\otimes a_k \in \Sigma_\mu$. These elementary functions have $\ell^1(\Sigma)$ as their closed linear span. Finally, there is a natural family of $*$-homomorphisms $\theta_t : A \to \ell^1(\Sigma)$, $t \geq 0$, defined by

$$\theta_t(a) = \delta_{(t)} \cdot a, \qquad a \in A, \quad t \geq 0,$$

and these maps are related to the generating sections by

$$\delta_{(t_1,\dots,t_k)} \cdot a_1 \otimes \cdots \otimes a_k = \theta_{t_1}(a_1)\theta_{t_2}(a_2)\cdots\theta_{t_k}(a_k).$$

The algebra $\ell^1(\Sigma)$ fails to have a unit, but it has the same representation theory as $\mathcal{P}A$ in the following sense. Given a family of representations $\pi_t : A \to \mathcal{B}(H)$, $t \geq 0$, fix $\nu = (t_1,\dots,t_k) \in S$. There is a unique bounded linear operator $L_\nu : \Sigma_\nu \to \mathcal{B}(H)$ of norm 1 that is defined by its action on elementary tensors as follows

$$L_\nu(a_1 \otimes \cdots \otimes a_k) = \pi_{t_1}(a_1)\cdots\pi_{t_k}(a_k).$$

Thus there is a bounded linear map $\tilde\pi : \ell^1(\Sigma) \to \mathcal{B}(H)$ defined by

$$\tilde\pi(f) = \sum_{\mu\in S} L_\mu(f(\mu)), \qquad f \in \ell^1(\Sigma).$$

One finds that $\tilde\pi$ is a $*$-representation of $\ell^1(\Sigma)$ with $\|\tilde\pi\| = 1$. This representation satisfies $\tilde\pi \circ \theta_t = \pi_t$, $t \geq 0$. Conversely, every bounded $*$-representation $\tilde\pi$ of $\ell^1(\Sigma)$ on a Hilbert space H is associated with a family of representations π_t, $t \geq 0$, of A on H by way of $\pi_t = \tilde\pi \circ \theta_t$.

The results of the preceding discussion are summarized as follows:

Proposition 2.4.1. *The enveloping C^*-algebra $C^*(\ell^1(\Sigma))$, together with the family of homomorphisms $\tilde\theta_t : A \to C^*(\ell^1(\Sigma))$, $t \geq 0$, defined by promoting the homomorphisms $\theta_t : A \to \ell^1(\Sigma)$, has the same universal property as the infinite free product $\mathcal{P}A = *_{t\geq 0}a$, and is therefore isomorphic to $\mathcal{P}A$.*

Notice that there is a natural semigroup of $*$-endomorphisms of $\ell^1(\Sigma)$ defined by

$$\sigma_t : \delta_{(s_1,\dots,s_k)} \cdot \xi \mapsto \delta_{(s_1+t,\dots,s_k+t)} \cdot \xi, \qquad (s_1,\dots,s_k) \in \Sigma, \quad \xi \in \Sigma_\nu$$

and it promotes to the natural shift semigroup of $\mathcal{P}A = C^*(\ell^1(\Sigma))$. The inclusion of A in $\ell^1(\Sigma)$ is given by the map $\theta_0(a) = \delta_{(0)}a \in \ell^1(\Sigma)$, and it too promotes to the natural inclusion of A in $\mathcal{P}A$.

Finally, we fix a semigroup of completely positive contractions $P_t : A \to A$, $t \geq 0$, and consider the associated moment polynomials of Proposition 2.3.1. Since $\|P_t\| \leq 1$ for every $t \geq 0$, a straightforward inductive argument based on the two properties MP1 and MP2 shows that

$$\|[t_1,\dots,t_n; a_1,\dots,a_n]\| \leq \|a_1\|\cdots\|a_n\|, \qquad t_k \geq 0, \quad a_k \in A,$$

hence there is a unique bounded linear map $E_0 : \ell^1(\Sigma) \to A$ satisfying

$$E_0(\delta_{(t_1,\dots,t_k)} \cdot a_1 \otimes \cdots \otimes a_k) = [t_1,\dots,t_k; a_1,\dots,a_k],$$

for $(t_1,\dots,t_k) \in S$, $a_1,\dots,a_k \in A$, $k = 1,2,\dots$, and in fact $\|E_0\| \leq 1$. Using the axioms MP1 and MP2, one finds that the map E_0 preserves the adjoint (see Equation (2.4)), satisfies the conditional expectation property $E_0(af) = aE_0(f)$ for $a \in A$, $f \in \ell^1(\Sigma)$, that the restriction of E_0 to the “hereditary” $*$-subalgebra of $\ell^1(\Sigma)$ spanned by $\theta_0(A)\ell^1(\Sigma)\theta_0(A)$ is multiplicative, and that it is related to ϕ by $E_0 \circ \sigma = \phi \circ E_0$ and $E_0(\sigma(a)) = \phi(a)$, $a \in A$. Thus, E_0 satisfies the axioms of Definition 2.2.1, suitably interpreted for the Banach $*$-algebra $\ell^1(\Sigma)$.

In view of the basic fact that a bounded completely positive linear map of a Banach $*$-algebra to A promotes naturally to a completely positive map of its enveloping C^*-algebra to A, the critical property of E_0 reduces to:

THEOREM 2.4.2. *For every $n \geq 1$, $a_1, \dots, a_n \in A$, and $f_1, \dots, f_n \in \ell^1(\Sigma)$, we have*

$$\sum_{i,j=1}^{n} a_j^* E_0(f_j^* f_i) a_i \geq 0.$$

Consequently, E_0 extends uniquely through the completion map $\ell^1(\Sigma) \to \mathcal{P}A$ to a completely positive map $E_\phi : \mathcal{P}A \to A$ that becomes a σ-expectation satisfying Equation (2.3).

COROLLARY 2.4.3. *Every semigroup of completely positive contractions acting on a C^*-algebra has a C^*-dilation.*

2.5. Existence of W^*-dilations. We now show how to use the results of the preceding section to obtain dilations appropriate for the category of von Neumann algebras. The dynamical issues discussed in Lecture 1 involved pairs of E_0-semigroups, but now it is appropriate to broaden the context. While we are primarily interested in the case of E_0-semigroups acting on type I_∞ factors, the basic concepts of dilation theory are best formulated in greater generality, and the general formulation has certain advantages.

DEFINITION 2.5.1 (E-semigroups). An E-semigroup is a semigroup $\{\alpha_t : t \geq 0\}$ of normal $*$-endomorphisms of a von Neumann algebra M that obeys the natural continuity requirement in its time variable; for every normal linear functional ρ on M, the functions $t \mapsto \rho(\alpha_t(x))$, $x \in M$, should be continuous.

Let (M, α) be a pair consisting of a von Neumann algebra M with separable predual and an E-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on it. The operators $\alpha_t(\mathbf{1})$ form a decreasing family of projections in M in general, and if one has $\alpha_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, then α is called an E_0-semigroup, *even when M is not a type I factor.* In order to deal effectively with dilation-theoretic issues, we must allow for more general von Neumann algebras M and the possibility of non-unital semigroups. The general issues discussed in this section do not depend on spatial aspects of M, and for the most part we will not have to realize M in any concrete representation as a subalgebra of $\mathcal{B}(H)$.

A *corner* of M is a von Neumann subalgebra of the particular form $N = pMp$, where p is a projection in M. The corner is said to be *full* if the central carrier of p is $\mathbf{1}$, and in that case pMp is a factor iff M is a factor of the same type.

Given an arbitrary projection $p \in M$, one can ask if there is a semigroup of completely positive maps $P = \{P_t : t \geq 0\}$ that acts on the corner pMp and is related to α as follows

$$P_t(pxp) = p\alpha_t(x)p, \qquad t \geq 0, \quad x \in M. \tag{2.7}$$

Such maps P_t need not exist in general; for example, taking $x = \mathbf{1} - p$, one finds that a necessary condition for P_t to exist is that p should satisfy $p\alpha_t(\mathbf{1} - p)p = 0$. Equivalently, a projection $p \in M$ is said to be *coinvariant* under α if

$$\alpha_t(\mathbf{1} - p) \leq \mathbf{1} - p, \qquad t \geq 0. \tag{2.8}$$

Remark 2.5.2 (Increasing Projections and E_0-semigroups). A projection $p \in M$ is said to be *increasing* if it has the property

$$\alpha_t(p) \geq p, \qquad t \geq 0. \tag{2.9}$$

Notice that in general, an increasing projection must be coinvariant. Indeed, since $\alpha_t(\mathbf{1}) \leq \mathbf{1}$, we will have

$$\alpha_t(\mathbf{1} - p) = \alpha_t(\mathbf{1}) - \alpha_t(p) \leq \mathbf{1} - p$$

whenever p is an increasing projection. The converse is not necessarily true. But in the special case where α is an E_0-semigroup, $\alpha_t(\mathbf{1}-p) = \mathbf{1}-\alpha_t(p)$; we conclude that *a projection is coinvariant under an E_0-semigroup iff it is an increasing projection.*

Now for any projection $p \in M$, one can define a family of linear maps $P = \{P_t : t \geq 0\}$ on $N = pMp$ by compressing each map α_t as follows

$$P_t(a) = p\alpha_t(a)p, \qquad a \in pMp, \quad t \geq 0. \tag{2.10}$$

Obviously, each P_t is a normal completely positive linear map of pMp into itself satisfying $\|P_t\| \leq 1$ for every $t \geq 0$. More significantly, one easily establishes:

Proposition 2.5.3. *Let p be a coinvariant projection for α and consider the family of maps $P = \{P_t : t \geq 0\}$ of pMp defined by (2.10). P is a continuous semigroup of completely positive contractions, satisfying (2.7). If, in addition, α is an E_0-semigroup, then we have $P_t(p) = p$, $t \geq 0$.*

Dilation theory in the category of von Neumann algebras concerns the properties of completely positive semigroups that can be obtained from E-semigroups in this particular way. By a CP semigroup we mean a pair (N, P) where $P = \{P_t : t \geq 0\}$ is a semigroup of normal completely positive linear maps acting on a von Neumann algebra N which satisfies $\|P_t\| \leq 1$ for every $t \geq 0$.

Definition 2.5.4 (Dilation and Compression). A triple (M, α, p) consisting of an E-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on a von Neumann algebra M, together with a distinguished coinvariant projection $p \in M$, is called a dilation triple. Let $N = pMp$ be the corner of M associated with p and let $P = \{P_t : t \geq 0\}$ be the semigroup acting on N as follows

$$P_t(a) = p\alpha_t(a)p, \qquad t \geq 0, \quad a \in N. \tag{2.11}$$

The CP semigroup (N, P) called a compression of (M, α, p), and (M, α, p) is called a dilation of (N, P).

We emphasize that the notion of a compression to a subalgebra has meaning only when (a) the subalgebra is a corner pMp of M and (b) the projection p satisfies (2.8). There are several equivalent notions of *minimality* that are associated with this dilation theory [**Arv03**]; but in this lecture we confine attention to the question of existence. However, we point out that an arbitrary dilation can always be reduced to a minimal one, and that a *necessary* condition for (M, α, p) to be a minimal dilation of (N, P) is that the central carrier of p should be $\mathbf{1}$. Thus, in the context of minimal dilations, if N is a factor then M must be a factor of the same type.

Starting with a CP semigoup (N, P), in order to find a dilation (M, α, p) of (N, P) one has to find a way of embedding N as a corner pMp of a larger von Neumann algebra M, on which there is a specified action of an E-semigroup α that is related to P as above. Notice that Corollary 2.4.3 provides the following infrastructure. If we view N as a unital C^*-algebra and P as a semigroup of contractive completely positive maps on N, forgetting the continuity of P_t in its time variable, then we can assert that the pair (N, P) has a C^*-dilation (ι, B, α). Certainly, B is not a von Neumann algebra and α is not an E-semigroup; thus (ι, B, α) cannot serve as a W^*-dilation of (N, P). However, it is possible to make use of the α-expectation $E : B \to N$ to find another dilation of (N, P) that is subordinate to (ι, B, α) and has all the desired properties. The results are summarized as follows.

THEOREM 2.5.5 (Existence of W^*-dilations). *Let $\{P_t : t \geq 0\}$ be a contractive CP-semigroup acting on a von Neumann algebra N with separable predual. Then (N, P) has a dilation (M, α, p).*

IDEA OF PROOF. Considering $P = \{P_t : t \geq 0\}$ as a semigroup of completely positive contractions acting on the unital C^*-algebra N, we see from Corollary 2.4.3 that P has a C^*-dilation (ι, B, α). We may obviously assume that $N \subseteq \mathcal{B}(H)$ acts concretely and nondegenerately on some separable Hilbert space H. We will construct a representation π of B on a Hilbert space $K \supseteq H$ with the property that each α_t can be extended to a normal $*$-endomorphism of the weak closure M of $\pi(B)$, and this will provide the required dilation of (N, P). The representation π is obtained as follows.

Let $E : B \to N$ be the α-expectation associated with (ι, B, α). Since we may view E as a completely positive map of B to $\mathcal{B}(H)$, it has a minimal Stinespring decomposition $E(x) = V^*\pi(x)V$, $x \in B$, where π is a representation of B on another Hilbert space K and $V : H \to K$ is a bounded linear map such that VH has K as its closed linear span.

Let M be the von Neumann algebra $\pi(B)''$. The remainder of the proof amounts to showing first, that there is a unique E-semigroup $\tilde{\alpha}$ acting on M that satisfies $\tilde{\alpha}_t(\pi(x)) = \pi(\alpha_t(x))$, $t \geq 0$, $x \in B$, second, that $p = VV^*$ is a projection in M whose corner pMp can be naturally identified with N, and that after this identification is made, $(M, \tilde{\alpha}, p)$ becomes a dilation triple for (N, P). $\square$

Historical Remarks. Several approaches to dilation theory for semigroups of completely positive maps have been proposed over the years, including work of Evans and Lewis [**EL77**], Accardi et al [**AL82**], Kümmerer [**Küm85**], Sauvageot [**Sau86**], and many others. Our attention was drawn to these developments by work of Bhat [**Bha99**], building on work of Bhat and Parthasarathy [**BP94**] for noncommutative Markov processes, in which the first dilation theory for CP semigroups acting on $\mathcal{B}(H)$ emerged that was effective for our work on E_0-semigroups [**Arv97b**], [**Arv00**]. SeLegue [**SeL97**] showed how to apply multi-operator dilation theory to obtain Bhat's dilation result for CP semigroups acting on $\mathcal{B}(H)$, and he calculated the expectation values of the n-point functions of such dilations. Recently, Bhat and Skeide [**BS00**] have initiated an approach to the subject that is based on Hilbert modules over C^*-algebras and von Neumann algebras.

3. The Role of Product Systems.

The fundamental problem in the theory of E_0-semigroups is to find a complete set of computable invariants for cocycle conjugacy. There is some optimism that such a classification is possible, but we are far from achieving that goal. We have not yet seen all possible cocycle conjugacy classes of E_0-semigroups, and we do not have an effective and computable set of invariants for the ones we have seen.

The numerical index is an example of a cocycle conjugacy invariant. However, while it serves to classify type I E_0-semigroups up to cocycle conjugacy, it is far from being a complete invariant for examples of type II and it is degenerate for examples of type III. The gauge group introduced below provides a more subtle cocycle conjugacy invariant, but it has not been calculated except in type I cases.

In this lecture we describe how the classification problem can be reduced to the problem of classifying certain simpler objects (product systems) up to natural isomorphism, and we discuss the role of product systems in other dynamical issues related to the theory of E_0-semigroups. This reformulaton has already lead to significant progress in several directions. We discuss these aspects in this lecture, and complete the discussion in Lecture 4.

3.1. Product Systems and Cocycle Conjugacy. It is appropriate to discuss concrete product systems within the context of E-semigroups acting on $\mathcal{B}(H)$. Given an E-semigoup $\alpha = \{\alpha_t : t \geq 0\}$ and $t > 0$, consider the linear space of operators

$$\mathcal{E}_\alpha(t) = \{T \in \mathcal{B}(H) : \alpha_t(A)T = TA, \quad A \in \mathcal{B}(H)\}. \tag{3.1}$$

We assemble the various $\mathcal{E}_\alpha(t)$ into a family of vector spaces $p : \mathcal{E}_\alpha \to (0,\infty)$ over the interval $(0,\infty)$ in the natural way

$$\mathcal{E}_\alpha = \{(t,T) : t > 0, \quad T \in \mathcal{E}_\alpha(t)\} \subseteq (0,\infty) \times \mathcal{B}(H), \tag{3.2}$$

where p is the projection $p(t,T) = t$.

The family $\mathcal{E}_\alpha$ has three important properties. First, the operator norm on each particular space $\mathcal{E}_\alpha(t)$ is actually a Hilbert space norm. To see how the inner product is defined, choose two elements $S, T \in \mathcal{E}_\alpha(t)$ and an arbitrary operator $A \in \mathcal{B}(H)$. Writing

$$T^*SA = T^*\alpha_t(A)S = (\alpha_t(A^*)T)^*S = (TA^*)^*S = AT^*S,$$

we find that T^*S must be a scalar multiple of the identity, and the value of that scalar defines an inner product on $\mathcal{E}_\alpha(t)$:

$$T^*S = \langle S, T\rangle \cdot \mathbf{1}. \tag{3.3}$$

The operator norm on $\mathcal{E}_\alpha(t)$ coincides with the norm defined by this inner product, since for $T \in \mathcal{E}_\alpha(t)$ we have

$$\|T\|^2 = \|T^*T\| = \|\langle T, T\rangle \cdot \mathbf{1}\| = \langle T, T\rangle.$$

In particular, $\langle T, T\rangle = 0$ iff $T = 0$, and we conclude that each fiber $\mathcal{E}_\alpha(t)$ becomes a Hilbert space relative to the inner product defined by (3.3).

Second, one may verify directly that $\mathcal{E}_\alpha(s)\mathcal{E}_\alpha(t) \subseteq \mathcal{E}_\alpha(s+t)$, so that the family $\mathcal{E}_\alpha$ can be made into an associative semigroup by defining multiplication as follows:

$$(s, S) \cdot (t, T) = (s+t, ST), \tag{3.4}$$

and this multiplication makes p into a homomorphism of the multiplicative structure of $\mathcal{E}_\alpha$ onto the additive semigroup of positive reals.

Third, this multiplication *acts like tensoring* in the sense that for every $s, t > 0$ there is a unique unitary operator $W_{s,t} : \mathcal{E}_\alpha(s) \otimes \mathcal{E}_\alpha(t) \to \mathcal{E}_\alpha(s+t)$ satisfying $W_{s,t}(S \otimes T) = ST$ for all $S \in \mathcal{E}_\alpha(s)$, $T \in \mathcal{E}_\alpha(t)$.

The weak operator topology on $\mathcal{B}(H)$ generates a σ-algebra of subsets of $\mathcal{B}(H)$, whose elements we refer to as Borel sets. This makes $B(H)$ into a standard Borel space because H is separable. We now describe an appropriate context for the structure $\mathcal{E}_\alpha$.

Definition 3.1.1 (Concrete Product System). A concrete product system is a Borel subset $\mathcal{E}$ of the cartesian product of Borel spaces $(0, \infty) \times \mathcal{B}(H)$ which has the following properties. Let $p : \mathcal{E} \to (0, \infty)$ be the natural projection $p(t, T) = t$. We require that p should be surjective, and in addition:

(i) For every $t > 0$, the set of operators $\mathcal{E}(t) = p^{-1}(t)$ is a norm-closed linear subspace of $\mathcal{B}(H)$ with the property that B^*A is a scalar for every $A, B \in \mathcal{E}(t)$.
(ii) For every $s, t > 0$, $\mathcal{E}(s+t)$ is the norm-closed linear span of the set of products $\mathcal{E}(s)\mathcal{E}(t)$.
(iii) As a measurable family of Hilbert spaces, $\mathcal{E}$ is isomorphic to the trivial family $(0, \infty) \times H_0$, where H_0 is a separable infinite dimensional Hilbert space.

Property (iii) requires some elaboration. The appropriate notion of isomorphism here is the one that belongs with the abstract theory of the following section, namely:

Definition 3.1.2. Two concrete product systems $\mathcal{E} \subseteq (0, \infty) \times \mathcal{B}(H)$ and $\mathcal{F} \subseteq (0, \infty) \times \mathcal{B}(K)$ are said to be *isomorphic* if there is an isomorphism of Borel spaces $\theta : \mathcal{E} \to \mathcal{F}$, which restricts to a unitary operator from $\mathcal{E}(t)$ to $\mathcal{F}(t)$ for every $t > 0$, and which satisfies $\theta(xy) = \theta(x)\theta(y)$, $x, y \in \mathcal{E}$.

There are several topologies that can used to topologize a given concrete product systems $\mathcal{E}$. However, a basic result in the theory of Borel structures implies that all of these topologies generate the same Borel structure on $\mathcal{E}$ because it is a standard Borel space. Thus any "topological" isomorphism of product systems must be an isomorphism in the sense of Definition 3.1.2. This feature of product systems allows for considerable flexibility.

Item (iii) makes the assertion that *there is an isomorphism of Borel spaces* $\theta : \mathcal{E} \to (0, \infty) \times H_0$ *with the property that the restriction of* θ *to each fiber* $\mathcal{E}(t)$ *is a unitary operator with range* H_0. The fact that the concrete product system $\mathcal{E}_\alpha$ associated with an E-semigroup α satisfies (iii) is nontrivial, and we refer the reader to [**Arv03**] for the proof, as well as for other characterizations of this property.

Finally, we point out that every concrete product system $\mathcal{E} \subseteq (0, \infty) \times \mathcal{B}(H)$ arises in the above way from a *unique* E-semigroup α acting on $\mathcal{B}(H)$, in such a way that the correspondence $\mathcal{E} \leftrightarrow \alpha$ is a bijection. More precisely:

Proposition 3.1.3. *Let* $\mathcal{E} \subseteq (0, \infty) \times \mathcal{B}(H)$ *be a concrete product system. There is a unique* E*-semigroup* $\alpha = \{\alpha_t : t \geq 0\}$ *acting on* $\mathcal{B}(H)$ *whose endomorphisms satisfy the following two conditions for every* $t > 0$

(i) $\alpha_t(A)T = TA$*, for every* $T \in \mathcal{E}(t)$*,* $A \in \mathcal{B}(H)$*.*

(ii) $\alpha_t(\mathbf{1})$ *is the projection on* $[\mathcal{E}(t)H]$.

Moreover, $\mathcal{E} = \mathcal{E}_\alpha$ *is the concrete product system associated with* α.

In order to define α_t for $t > 0$, one chooses an orthonormal basis $T_1(t), T_2(t), \dots$ for $\mathcal{E}(t)$ and sets $\alpha_t(A) = T_1(t)AT_1(t)^* + T_2(t)AT_2(t)^* + \cdots$. In particular, we conclude that *E_0-semigroups correspond bijectively with concrete product systems $p : \mathcal{E} \to (0,\infty)$ with the property that $[\mathcal{E}(t)H] = H$ for every $t > 0$.*

It is significant that the fibers $\mathcal{E}(t)$ of a concrete product system are all infinite dimensional except in degenerate cases. More precisely, let $V = \{V_t : t \geq 0\}$ be a strongly continuous semigroup of isometries acting on a Hilbert space H and let α be the associated E-semigroup

$$\alpha_t(A) = V_tAV_t^*, \qquad A \in \mathcal{B}(H), \quad t \geq 0. \tag{3.5}$$

One may verify by a direct calculation that $\mathcal{E}_\alpha(t) = \mathbb{C} \cdot V_t$ for every $t > 0$, and in particular the spaces $\mathcal{E}_\alpha(t)$ are all one-dimensional. It is less obvious that the converse is true. Indeed, for any concrete product system $p : \mathcal{E} \to (0,\infty)$, the fiber spaces $\mathcal{E}(t) = p^{-1}(t)$ are either all infinite dimensional or else they are all one-dimensional, and in the latter case the underlying E-semigroup must have the form (3.5).

Let us now examine the connection between cocycle conjugacy and product systems. Let α be an E_0-semigroup acting on $\mathcal{B}(H)$, let $U = \{U_t : t \geq 0\}$ be an α-cocycle, i.e., a strongly continuous family of unitary operators in H which satisfy

$$U_{a+t} = U_s\alpha_s(U_t), \qquad s, t \geq 0, \tag{3.6}$$

and let $\beta_t(A) = U_t\alpha_t(A)U_t^*$ be the corresponding cocycle perturbation of α. We can define a map $\theta : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ as follows:

$$\theta(t, T) = (t, U_tT), \qquad T \in \mathcal{E}_\alpha(t), \quad t > 0.$$

Indeed, one sees that $U_t\mathcal{E}_\alpha(t) \subseteq \mathcal{E}_\beta(t)$ for every $t > 0$ because

$$\beta_t(A)U_tT = U_t\alpha_t(A)T = U_tTA, \qquad A \in \mathcal{B}(H),$$

and in fact $U_t\mathcal{E}_\alpha(t) = \mathcal{E}_\beta(t)$. Thus, θ defines an isomorphism of measurable families of Hilbert spaces that is unitary on fibers. θ also preserves multiplication by virtue of the cocycle equation (3.6) and the fact that $\alpha_s(U_t)S = SU_t$:

$$\begin{aligned}\theta((s, S)(t, T)) &= \theta(s + t, ST) = (s + t, U_{s+t}ST) = (s + t, U_s\alpha_s(U_t)ST)\\ &= (s + t, U_sSU_tT) = (s, U_sS)(t, U_tT) = \theta(s, S)\theta(t, T).\end{aligned}$$

It can be seen by a more involved argument that this argument is reversible in the sense that every other E_0-semigroup β that acts on $\mathcal{B}(H)$ whose product system $\mathcal{E}_\beta$ is isomorphic to $\mathcal{E}_\alpha$ must actually be a cocycle perturbation of α (see [**Arv03**]). With these results in hand, one easily deduces:

THEOREM 3.1.4. *Two E_0-semigroups α and β acting, respectively, on $\mathcal{B}(H)$ and $\mathcal{B}(K)$, are cocycle conjugate iff their product systems are isomoprhic.*

Theorem 3.1.4 implies that the cocycle conjugacy class of an E_0-semigroup is completely determined by the *structure* of its product system. Thus the classification of E_0-semigroups up to cocycle conjugacy should begin with a systematic development of a general theory of continuous tensor products of Hilbert spaces that is appropriate for E_0-semigroups. We turn now to this more general discussion.

3.2. Abstract Product Systems. An effective theory should have at least the following three properties. First, it should be possible to associate a continuous tensor product of Hilbert spaces with every E_0-semigroup. Second, the structure of the continuous tensor product associated with an E_0-semigroup should be an invariant for cocycle conjugacy, and optimally a complete invariant. Third, every continuous tensor product in this category of objects should be associated with an E_0-semigroup.

Given a theory with these three properties, the isomorphism classes of such structures will then provide a full description of the cocycle conjugacy classes of E_0-semigroups; hence the problem of classifying E_0-semigroups to cocycle conjugacy is reduced to the problem of classifying these considerably simpler objects, their symmetries, and their related structures.

We now describe an axiomatic approach to continuous tensor products of Hilbert spaces that satisfies these requirements. We will find that the first two requirements follow from the results described in the preceding section; the proof that the third property is satisfied requires additional C^*-algebraic tools, and will be taken up in Lecture 4.

This program has already led to significant progress in the classification of E_0-semigroups up to cocycle conjugacy, in two directions. The classification of type I E_0-semigroups was completely settled through an analysis of the structure of their product systems [**Arv89a**], and the description of decomposable product systems in [**Arv97a**]. In [**Pow99**], Powers showed that for every $n = 1, 2, \dots$ there is a type II E_0-semigroups whose numerical index is n; hence there are infinitely many type II E_0-semigroups that are mutually non cocycle conjugate. In a previous paper [**Pow87**], he showed that type III E_0-semigroups exist, but the methods of [**Pow87**] do not lend themselves to differentiating between cocycle conjugacy classes in the family of examples exhibited there. More recently, Tsirelson showed [**Tsi00a**] that there is a continuum of type II product systems that are mutually non-isomorphic, and that there is a product system that is not anti-isomorphic to itself. In a subsequent paper [**Tsi00b**], he constructed a one parameter family of type III product systems that are mutually non isomorphic; we describe these product systems in the following section. Given that every product system can be associated with an E_0-semigroup (see Theorem 4.5.2 below), Tsirelson's results on product systems have immediate implications for the problem of classifying E_0-semigroups up to cocycle conjugacy.

Heuristically, a *product system* is a measurable family of Hilbert spaces $E = \{E(t) : t > 0\}$ which behaves as if each $E(t)$ were a continuous tensor product

$$E(t) = \bigotimes_{0<s<t} H_s, \qquad H_s = H \tag{3.7}$$

of copies of a single separable Hilbert space H. While this heuristic picture is often useful, one must be careful not to push it too far. Indeed, we will see that this picture is basically correct for the simplest examples of product systems, but that there are other examples with the remarkable property that the "germ" H fails to exist. We first illustrate the essentials of the structure of product systems in the discrete case, where the positive real line is replaced with the discrete set $\mathbb{N} = \{1, 2, \dots\}$ of positive integers. Then we will indicate how to change the axioms to pass from $\mathbb{N}$ to $\mathbb{R}^+$.

Let H be a separable Hilbert space. For every $n = 1, 2 \dots$ let $E(n)$ be the full tensor product of n copies of H:

$$E(n) = \underbrace{H \otimes H \otimes \cdots \otimes H}_{n \text{ times}}.$$

We may organize these spaces into a family of Hilbert spaces $p : E \to \mathbb{N}$ over $\mathbb{N}$ by setting

$$E = \{(t, \xi) : t \in \mathbb{N}, \quad \xi \in E(t)\},$$

with projection $p(t, \xi) = t$. We introduce an associative multiplication on the structure E by making use of the tensor product

$$(s, \xi) \cdot (t, \eta) = (s + t, \xi \otimes \eta),$$

$\xi \in E(s)$, $\eta \in E(t)$. This multiplication is bilinear on fibers, and has the two additional properties

$$E(s + t) = \overline{\mathrm{span}}\ E(s) \cdot E(t), \qquad s, t \in \mathbb{N} \tag{3.8}$$

$$< ux, vy > =< u, v >< x, y >, \quad u, v \in E(s), \quad x, y \in E(t). \tag{3.9}$$

Notice that the Hilbert space associated with the sections of $p : E \to \mathbb{N}$ is the direct sum

$$\sum_{t \in \mathbb{N}} E(t) = \sum_{n=1}^{\infty} H^{\otimes n},$$

namely the full Fock space over the one-particle space H.

A *unit* is a section $n \in \mathbb{N} \mapsto u_n \in E(n)$ satisfying

$$u_{m+n} = u_m u_n,$$

and which is not the zero section. The most general unit has the form

$$u_n = \underbrace{x \otimes x \otimes \cdots \otimes x}_{n \text{ times}}$$

$n \geq 1$, where x is a nonzero element of the one-particle space H. A vector $u \in E(n)$ is called *decomposable* if for every $k = 1, 2, \dots, n - 1$ there are vectors $v_k \in E(k)$, $w_k \in E(n - k)$ such that

$$u = v_k w_k.$$

The most general decomposable vector in $E(n)$ is an elementary tensor of the form

$$u = x_1 \otimes x_2 \otimes \cdots \otimes x_n,$$

where $x_k \in H$ for $k = 1, 2, \dots, n$.

A product system is a similar structure, except that it is associated with the space of positive reals rather than $\mathbb{N}$.

Definition 3.2.1. A product system is a family of separable Hilbert spaces $p : E \to (0, \infty)$ over the semi-infinite interval $(0, \infty)$, with fiber Hilbert spaces $E(t) = p^{-1}(t)$, endowed with an associative multiplication that restrcts to a bilinear map on fibers

$$(x, y) \in E(s) \times E(t) \mapsto xy \in E(s + t), \qquad s, t > 0,$$

which acts like tensoring in the sense that properties (3.8) and (3.9) are satisfied. In addition, E should be endowed with the structure of a standard Borel space that is compatible with projection onto $(0, \infty)$, multiplication, the vector space operations and the inner product, and which has the further property that there should be a separable Hilbert space H such that

$$E \cong (0, \infty) \times H, \tag{3.10}$$

where $\cong$ denotes an isomorphism of measurable families of Hilbert spaces.

Remark 3.2.2. For example, measurability of the inner product means that if one considers the subset $\Delta = \{(x, y) \in E \times E : p(x) = p(y)\}$ of the standard Borel space $E \times E$, then Δ is a Borel subset because $p : E \to (0, \infty)$ is a Borel measurable function, and measurability of the inner product means that the complex-valued function defined on Δ by $(x, y) \mapsto \langle x, y\rangle$ should be Borel-measurable.

The requirement (3.10) is nontrivial, and is the counterpart for this category of local triviality of Hermitian vector bundles. It is equivalent to the existence of a sequence of measurable sections $t \in (0, \infty) \mapsto e_n(t) \in E(t)$ with the property that $\{e_1(t), e_2(t), \dots\}$ is an orthonormal basis for $E(t)$, for every $t > 0$.

Definition 3.2.3. By an isomorphism of product systems we mean an isomorphism of Borel spaces $\theta : E \to F$, such that $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in E$, whose restriction to each fiber space is a unitary operator $\theta_t : E(t) \to F(t)$, $t > 0$.

One can show easily that a concrete product system in the sense of Definition 3.1.1 is a product system in the the more abstract sense of Definition 3.2.1. We have also seen that every E_0-semigroup α gives rise to a concrete product system $\mathcal{E}_\alpha$, and Theorem 3.1.4 asserts that α and β are cocycle conjugate iff their product systems $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are isomorphic. Moreover, we will see in Lecture 4 that every abstract product system is isomorphic to the product system $\mathcal{E}_\alpha$ associated with some E_0-semigroup α. Thus, *the problem of classifying product systems up to isomorphism becomes a central problem in noncommutative dynamics.*

One might expect that it should be possible to write down a comprehensive list of (continuous) product systems as we have done above for their discrete analogues. In the discrete case there is, up to isomorphism, exactly one "product system" for every integer $d = 1, 2, \dots, \aleph_0$, and of course d is the dimension of the one-particle space $E(1)$. In the continuous case, however, nothing like that is true. While there is a family of "natural" examples parameterized by the values $d = 1, 2, \dots, \aleph_0$, there are many others as well.

3.3. Examples of Product Systems. We begin this section by describing the simplest examples of product systems and we discuss their role in the classification of type I E_0-semigroups. We then describe Tsirelson-Vershik product systems, without technical details.

3.3.1. *Exponential Product Systems.* Recall first the basic features of the symmetric Fock space e^H over a one-particle space H. It is defined as the direct sum of Hilbert spaces

$$e^H = \sum_{n=0}^{\infty} H^n$$

where H^n denotes the n-fold symmetric tensor product of copies of H when $n \geq 1$, and $H^0 = \mathbb{C}$. There is a natural exponential map $f \in H \mapsto \exp(f) \in e^H$ defined by

$$\exp(f) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}} f^{\otimes n}.$$

e^H is the closed linear span of the set of exponentials $\exp(f)$, $f \in H$, and one has

$$\langle \exp(f), \exp(g)\rangle = e^{\langle f,g\rangle}, \qquad f, g \in L^2((0,\infty); K).$$

This construction is functorial in that for every unitary operator $U : H_1 \to H_2$ there is a natural second quantization $\Gamma(U) : e^{H_1} \to e^{H_2}$ defined by

$$\Gamma(U) = V_0 \oplus V_1 \oplus V_2 \oplus \cdots$$

where, for $n \geq 1$, $V_n : H_1^n \to H_2^n$ is the n-fold tensor product of copies of U, and where V_0 is the identity map of $\mathbb{C}$. Equivalently, $\Gamma(U)$ is defined as the unique unitary operator satisfying $\Gamma(U) : \exp(f) \to \exp(Uf)$, $f \in H$. Another fundamental property of this construction is that for any two Hilbert spaces H_1, H_2, $e^{H_1 \oplus H_2}$ is naturally identified with the tensor product $e^{H_1} \otimes e^{H_2}$; indeed, there is a unique unitary operator $W : e^{H_1\oplus H_2} \to e^{H_1} \otimes e^{H_2}$ that satisfies

$$W : \exp(f \oplus g) \mapsto \exp(f) \otimes \exp(g), \qquad f \in H_1, \quad g \in H_2.$$

The simplest product systems are the ones associated with the CAR/CCR flows, and are described as follows. Let N be a positive integer or $\infty = \aleph_0$ and let K be a Hilbert space of dimension N. We form the symmetric Fock space $e^{L^2((0,\infty);K))}$ over the one-particle space $L^2((0,\infty); K)$ consisting of all square-integrable K-valued functions $\xi : (0,\infty) \to K$. For every $t > 0$ let $E(t)$ be the closed subspace

$$E(t) = e^{L^2((0,t);K)} \subseteq e^{L^2((0,\infty);K)},$$

$L^2((0,t); K)$ denoting the subspace of $L^2((0,\infty); K)$ consisting of functions vanishing almost everywhere outside $(0,t)$, and consider the family of Hilbert spaces $p : E_N \to (0,\infty)$ defined by

$$E_N = \{(t, \xi) : t > 0, \quad \xi \in E(t)\}, \qquad p(t, \xi) = t.$$

The shift semigroup $S = \{S_t : t \geq 0\}$ acting on $L^2((0,\infty); K)$ gives rise to a semigroup of isometries by way of second quantization

$$U_t = \Gamma(S_t), \qquad t \geq 0,$$

and we use $\{U_t : t \geq 0\}$ to introduct a multiplication in E_N as follows. Noting that U_s maps $e^{L^2((0,t);K)}$ onto $e^{L^2((s,s+t);K)}$, and noting the natural identification

$$e^{L^2((0,s+t);K)} = e^{L^2((0,s);K)} \otimes e^{L^2((s,s+t);K)},$$

we can define multiplication in E_N as follows: for $f \in e^{L^2((0,s);K)}$ and $g \in e^{L^2((0,t);K)}$ we set

$$(s, f) \cdot (t, g) = (s + t, f \otimes U_s g).$$

E_N is a closed subset of the Polish space

$$E_N \subseteq (0,\infty) \times e^{L^2((0,\infty);K)},$$

the topology being the obvious one arising from the usual metric on $(0,\infty)$ and the Hilbert space norm of $e^{L^2((0,\infty);K)}$, hence E_N is a standard Borel space. It is not hard to verify directly that E_N is a product system.

The CCR flow of rank N is an E_0-semigroup α that acts on $\mathcal{B}(e^{L^2((0,\infty);K)})$. It is defined most explicitly in terms of the the natural representation of the canonical commutation relations on $e^{L^2((0,\infty);K)}$, and its concrete product system $\mathcal{E}_\alpha$ is isomorphic to E_N: the following result exhibits that isomorphism $E_N \sim \mathcal{E}_\alpha$. Since an E_0-semigroup is uniquely defined by its product system, it will not be necessary to reiterate the definition of α in terms of the CCRs here.

Proposition 3.3.1. *For every $t > 0$, $f \in L^2((0,t);K)$, there is a bounded operator T_f on the symmetric Fock space $e^{L^2((0,\infty);K)}$, defined uniquely on the spanning set of vectors $\{\exp(g) : g \in L^2((0,\infty);K)\}$ by*

$$T_f(\exp(g)) = f \otimes \exp(S_t g), \quad t \geq 0, \quad g \in L^2((0,\infty);K). \tag{3.11}$$

The mapping $\theta : E_N \to (0,\infty) \times \mathcal{B}(e^{L^2((0,\infty);K)})$ defined by

$$\theta : (t,f) \mapsto (t, T_f) \tag{3.12}$$

is an isomorphism of E_N onto a concrete product system $\mathcal{E}$ acting on the Hilbert space $e^{L^2((0,\infty);K)}$. $\mathcal{E}$ is the product system of the CCR flow of rank N.

Definition 3.3.2. For every $N = 1, 2, \ldots, \infty$, E_N is called the exponential product system of rank N.

Remark 3.3.3 (Type of a Product System). Let $p : E \to (0,\infty)$ be a product system and choose $t > 0$. A vector $u \in E(t)$ is said to be *decomposable* if for every $0 < s < t$ there are vectors $v \in E(s)$, $w \in E(t-s)$ such that $u = vw$. The set of all decomposable vectors in $E(t)$ spans a closed subspace $D(t) \subseteq E(t)$, which can be the trivial subspace $D(t) = \{0\}$, but in general one have

$$D(s+t) = \overline{\operatorname{span}} D(s)D(t), \qquad s,t > 0.$$

Moreover, if $D(t_0) \neq \{0\}$ for some particular $t_0 > 0$ then $D(t) \neq \{0\}$ for every $t > 0$. The product system E is said to be of type I if $D(t)$ spans $E(t)$ for some (and therefore every) positive t, of type II if it is not of type I but $D(t) \neq \{0\}$ for some (and therefore every) postive t, and of type III if $D(t) = \{0\}$ for some (and therefore every) $t > 0$. An E_0-semigroup α is said to be of type I, II, or III according as its product system is of that type.

In the section 3.4 below we will describe a reformulation of the index invariant of E_0-semigroups into a dimension function defined on the category of product systems. The basic results on the classification of type I product systems and type I E_0-semigroups are summarized as follows (see [**Arv03**]): Every type I product system E is isomorphic to an exponential product system E_N, where $N = \dim E$. Every type I E_0-semigroup α is conjugate to a cocycle perturbation of the CAR/CCR flow of rank N, where $N = \operatorname{index}(\alpha)$.

3.3.2. *Tsirelson-Vershik Product Systems.* Tsirelson and Vershik [**VT98**] constructed a family of continuous tensor product of Hilbert spaces that could not be described in terms of the classical symmetric Fock space construction. Responding to a question of the author, Tsirelson [**Tsi00b**] adapted those ideas so as to generate a one parameter family of product systems; moreover, by very ingenious arguments he was able to show that these product systems are not only of type III, but they are mutually non-isomorphic. In this section we describe these product systems in enough detail so that their basic features are exposed, avoiding most technicalities. The reader is referred to Tsirelson's contribution to these proceedings for more detail. Our exposition differs somewhat from the original; for example, we use complex-valued Gaussian random variables rather than real-valued ones, and our treatment of quasiorthogonality is formulated somewhat differently. But the resulting constructions are fundamentally the same as Tsirelson's.

We first describe the correlation functions of Tsirelson and Vershik. For every real number $\theta > 1$ we fix, once and for all, a continous real-valued function C_θ, defined on the punctured line $\mathbb{R} \setminus \{0\}$, which vanishes outside some small interval $(-\epsilon, +\epsilon)\setminus\{0\}$ with $\epsilon < 1$ (ϵ may depend on θ), and which has the following properties

(i) The restriction of C_θ to the positive real line is nonnegative, continous, decreasing, and convex.
(ii) $C_\theta(-t) = C_\theta(t)$, for every $t \in \mathbb{R} \setminus \{0\}$.
(iii) For some positive number $0 < \delta < \epsilon$ we have

$$C_\theta(t) = \frac{1}{|t| \cdot |\log |t||^\theta}, \qquad 0 < |t| < \delta.$$

Obviously, the conditions (i)–(iii) can be achieved with ϵ as small as we please. Notice the singularity of C_θ near the origin; and this is an essential feature. Thus, such functions C_θ "approximate" the delta function. The singularity is mild enough that these functions belong to $L^1(\mathbb{R})$, and therefore they define bounded convolution operators $f \mapsto C_\theta * f$ on the Hilbert space $L^2(\mathbb{R})$.

Heuristically, one thinks of C_θ as the correlation function

$$C_\theta(t-s) = E(X_s X_t), \qquad s, t \in \mathbb{R},$$

associated with a stationary Gaussian random process $\{X_t : t \in \mathbb{R}\}$. However, because of the singularity (iii) at $t = 0$, this formula cannot be achieved with a classical process, but only with a stationary random distribution [**GV64**]. This random distribution shares certain properties with white noise, and it is suggestive to think of it as "off-white" noise or, as Tsirelson puts it, slightly colored noise.

The second key property of these functions is that they are positive definite in the sense that the sesquilinear form they define on $L^2(\mathbb{R})$ by way of

$$\langle f, g\rangle = \int_{\mathbb{R}} (C_\theta * f)(x)\bar{g}(x)\, dx \tag{3.13}$$

is a semidefinite inner product on $L^2(\mathbb{R})$. Indeed, any real function $C_\theta \in L^1(\mathbb{R})$ that satisfies properties (i) and (ii) above can be seen to be positive definite, and therefore the form defined by (3.13) will satisfy $\langle f, f\rangle \geq 0$ for all $f \in L^2(\mathbb{R})$ (see Chapter 14 of [**Arv03**] for more detail). The completion of $L^2(\mathbb{R})$ in the inner product $\langle \cdot, \cdot\rangle$ is a Hilbert space that will be denoted H_θ.

Third, notice that the inner product of (3.13) is invariant under the action of the one-parameter group of translation operators acting on $L^2(\mathbb{R})$ in the sense that

$$\langle T_t f, T_t g\rangle = \langle f, g\rangle, \qquad f, g \in L^2(\mathbb{R}). \tag{3.14}$$

Thus there is a unique one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ in $\mathcal{B}(H_\theta)$ that extends that action of the translation operators to H_θ.

For a bounded interval $I = (a, b) \subseteq \mathbb{R}$ we write $L^2(I)$ for the subspace of $L^2(\mathbb{R})$ consisting of functions that vanish almost everywhere on the complement of I. Finally, notice that if $I = (a, b)$ and $J = (c, d)$ are two bounded intervals whose separation is greater than 2ϵ, then $L^2(I)$ and $L^2(J)$ are orthogonal in H_θ:

$$\langle f, g\rangle = 0, \qquad f \in L^2(I), \quad g \in L^2(J).$$

3.3.3. *Quasiorthogonal Subspaces.* If the separation between I and J is smaller than 2ϵ then this is no longer true. But for arbitrary disjoint intervals there is a generalized sense in which it is approximately true. Let H be a Hilbert space, let $M_1, \dots, M_n$ be a finite set of closed subspaces of H, and let $M_1 \oplus \cdots \oplus M_n$

be the direct sum of Hilbert spaces. There is a unique bounded linear map $L : M_1 \oplus \cdots \oplus M_n \to H$ satisfying

$$L(\xi_1, \dots, \xi_n) = \xi_1 + \cdots + \xi_n, \qquad \xi \in M, \quad \eta \in N.$$

In general, L is a bounded linear map of $M_1 \oplus \cdots \oplus M_n$ onto the algebraic sum $M_1 + \cdots + M_n \subseteq H$.

Definition 3.3.4. A finite set of subspaces $M_1, \dots, M_n$ is said to be quasiorthogonal if the linear map L is injective, and in addition $\mathbf{1} - L^*L$ is a Hilbert Schmidt operator on $M_1 \oplus \cdots \oplus M_n$.

Remark 3.3.5 (Equivalence Operators). The hypothesis that $\mathbf{1} - L^*L$ is compact implies that the range of L is closed. It follows that the algebraic linear span $M_1 + \cdots + M_n$ is a closed subspace of H that is linearly isomorphic to the orthogonal direct sum of the various M_k. More generally, let $L : H_1 \to H_2$ be a bounded operator from one Hilbert space to another. We will say that L is an *equivalence operator* if it is one-to-one, has dense range, and $\mathbf{1} - L^*L$ is a Hilbert Schmidt operator on H_1. This forces the range of L to be closed, hence L is an invertible operator. One verifies easily that the both the inverse and the adjoint of an equivalence operator are equivalence operators, and that the class of equivalence operators is closed under composition. In particular, the set of equivalence operators in $\mathcal{B}(H)$ is a subgroup of the general linear group of H.

The first important property of these subspaces of H_θ is based on ideas originating in [**VT98**]; a detailed proof appears in [**Tsi00b**].

Theorem 3.3.6. *Let $I_1, \dots, I_n$ be a pairwise disjoint sequence of bounded intervals in $\mathbb{R}$ and let $H_\theta(I_k)$ be the closure of $L^2(I_k)$ in H_θ. Then $H_\theta(I_1), \dots, H_\theta(I_n)$ is quasiorthogonal set of subspaces of H_θ.*

3.3.4. *Equivalence Operators and Gaussian Measures.* The second key element of this construction of product systems is the following result that describes Gaussian measures that are mutually absolutely continuous with respect to a given one, and which is in some sense a refinement of Kakutani's characterization of mutual absolute continuity for infinite product measures [**Kak48**]. The result is due to Feldman [**Fel58**], Hájek [**Háj58**], and Segal [**Seg58**].

Let $(\Omega, \mathcal{B}, P)$ be a probability space, which we may assume is modelled on a standard Borel space $(\Omega, \mathcal{B})$. By a Gaussian random variable we mean a complex-valued function in $L^2(\Omega, \mathcal{B}, P)$ having the form $z = x + iy$ where x and y are independent real Gaussian random variables with mean zero and equal variances. A Gaussian space is a complex linear subspace $G \subseteq L^2(\Omega, \mathcal{B}, P)$ consisting of Gaussian random variables. Finally, with any subspace S of $L^2(\Omega, \mathcal{B}, P)$ there is an associated sub σ-algebra $\mathcal{B}_S$ of $\mathcal{B}$, namely the σ-algebra generated by a sequence of functions in S that has S as its closed linear span. Up to sets of measure zero, $\mathcal{B}_S$ does not depend on the choice of spanning sequence.

Suppose we are given a Gaussian space $G \subseteq L^2(\Omega, \mathcal{B}, P)$ of random variables, along with a second probability measure Q on $(\Omega, \mathcal{B})$ that is mutually absolutely continuous with P and which has the further property that every random variable in G is also Gaussian when viewed as a random variable relative to Q. We may consider the two inner products defined on G by

$$\langle z_1, z_2 \rangle_P = \int_\Omega z_1 \bar{z}_2 \, dP, \qquad \langle z_1, z_2 \rangle_Q = \int_\Omega z_1 \bar{z}_2 \, dQ.$$

Let us write G_P for the Hilbert space structure of G relative to the inner product $\langle\cdot,\cdot\rangle_P$ and G_Q for the inner product space associated with $\langle\cdot,\cdot\rangle_Q$. Since $Q \sim P$, it follows that G is also a closed subspace of $L^2(\Omega, \mathcal{B}, Q)$, and an application of the closed graph theorem implies that the identity map of G defines an invertible operator from G_P to G_Q. In particular, the two Hilbert norms on G are equivalent, hence there is a unique positive invertible operator $B \in \mathcal{B}(G_P)$ such that

$$\langle z_1, z_2\rangle_Q = \langle Bz_1, z_2\rangle_P, \qquad z_k \in G. \tag{3.15}$$

The possibilities for B are characterized as follows.

Theorem 3.3.7 (Feldman, Hájek, Segal). *Any operator satisfying (3.15) has the form $B = \mathbf{1} + C$ where C is a Hilbert Schmidt operator in $\mathcal{B}(G_P)$. Conversely, if B is a positive invertible operator in $\mathcal{B}(G_P)$ such that $\mathbf{1} - B$ is Hilbert Schmidt, then there is a probability measure Q on $(\Omega, \mathcal{B})$, mutually absolutely continuous with P, such that G is also a Gaussian space relative to Q and for which*

$$\langle Bz_1, z_2\rangle_P = \int_\Omega z_1 \bar{z}_2 \, dQ, \qquad z_k \in G.$$

This formula determines Q uniquely on the σ-algebra $\mathcal{B}_G$ associated with G.

Suppose now that M, N are two subspaces of a Gaussian space $G \subseteq L^2(\Omega, \mathcal{B}, P)$, with associated triples $(\Omega, \mathcal{B}_M, P_M)$, $(\Omega, \mathcal{B}_N, P_N)$. If M and N happen to be orthogonal, then it is a fundamental property of Gaussian random variables that

$$P(A \cap B) = P(A)P(B), \qquad A \in \mathcal{B}_M, \quad B \in \mathcal{B}_N$$

and this property allows one to identify the L^2 space associated with the sum $M+N$ with the tensor product of L^2 spaces

$$L^2(\Omega, \mathcal{B}_{M+N}, P_{M+N}) \cong L^2(\Omega, \mathcal{B}_M, P_M) \otimes L^2(\Omega, \mathcal{B}_N, P_N). \tag{3.16}$$

Indeed, this identification associates a product of functions of the form F_1F_2 with F_1 $\mathcal{B}_M$-measurable and F_2 $\mathcal{B}_N$-measurable, with the tensor product $F_1 \otimes F_2$.

If M and N are merely quasiorthogonal, then we have the following substitute, which implies that they are independent with respect to an equivalent Gaussian measure.

Theorem 3.3.8. *Let M, N be a quasiorthogonal pair of subspaces of a Gaussian space $G \subseteq L^2(\Omega, \mathcal{B}, P)$. There is a unique probability measure Q on the Borel space $(\Omega, \mathcal{B}_{M+N})$ satisfying the three conditions*

(i) $Q \sim P_{M+N}$,
(ii) *$M + N$ is a Gaussian subspace of $L^2(\Omega, \mathcal{B}_{M+N}, Q)$.*
(iii) *$(\xi, \eta) \in M \oplus N \mapsto \xi + \eta \in M + N \subseteq L^2(\Omega, \mathcal{B}_{M+N}, Q)$ is an isometry.*

M and N are probabilistically independent Gaussian subspaces relative to Q, and moreover the restrictions of Q to the sub σ-algebras $\mathcal{B}_M$ and $\mathcal{B}_N$ agree with P_M and P_N respectively.

Sketch of Proof. Let $L : M \oplus N \to M + N \subseteq L^2(\Omega, \mathcal{B}, P)$ be the natural linear map $L(w_1, w_2) = w_1 + w_2$. Since M and N are quasiorthogonal, L is an equivalence operator, hence $L^{-1*}L^{-1} = \mathbf{1}+C$ where C is a Hilbert Schmidt operator on $M + N$. Theorem 3.3.7 implies that there is a unique probability measure Q on $(\Omega, \mathcal{B}_{M+N})$ that satisfies (i), (ii), and obeys

$$\int_\Omega z_1 \bar{z}_2 \, dQ = \langle L^{-1}z_1, L^{-1}z_2\rangle_{M\oplus N}. \qquad z_1, z_2 \in M + N,$$

and the assertion (iii) follows from this formula. The last sentence follows from basic properties of Gaussian spaces and the uniqueness assertion of Theorem 3.3.7. □

REMARK 3.3.9. Theorem 3.3.8 gives a precise sense in which certain pairs of Gaussian subspaces of $L^2(\Omega, \mathcal{B}, P)$ can be "straightened" by replacing P with another Gaussian measure that is equivalent to it. The same thing can be done for any finite set of n quasiorthogonal subspaces of a Gaussian space, the proof being a straightforward variation of the one above. We require only the case $n = 2$ for this discussion.

Because of these remarks we can make an identification

$$L^2(\Omega, \mathcal{B}_{M+N}, Q_{M+N}) \cong L^2(\Omega, \mathcal{B}_M, Q_M) \otimes L^2(\Omega, \mathcal{B}_N, Q_N).$$

In order to make use of this in the construction to follow it will be necessary to make such identifications in a more invariant way, in terms of generalizations of measure spaces called measure classes. We now describe that procedure.

3.3.5. *The L^2 Space of a Measure Class.* By a *measure class* we mean a triple $(X, \mathcal{B}, \mathcal{M})$ consisting of a Borel space $(X, \mathcal{B})$ together with a nonempty set $\mathcal{M}$ of finite positive measures on $(X, \mathcal{B})$ with the property

$$\mu \in \mathcal{M}, \qquad \nu \sim \mu \implies \nu \in \mathcal{M},$$

$\mu \sim \nu$ denoting mutual absolute continuity. Given two positive finite measures μ, ν on $(X, \mathcal{B})$ there is a notion of the geometric mean $\sqrt{\mu\nu}$ due to Kakutani: $\sqrt{\mu\nu}$ is characterized as the largest positive measure σ with the property

$$|\int_X f\bar{g}\, d\sigma|^2 \leq \int_X |f|^2\, d\mu \int_X |g|^2\, d\nu,$$

for all bounded measurable functions f, g. It can be defined in more concrete terms as the measure $\sqrt{uv}(\mu + \nu)$ where u, v are the Radon-Nikodym derivatives

$$u = \frac{d\mu}{d(\mu+\nu)}, \qquad v = \frac{d\nu}{d(\mu+\nu)}.$$

The map $\mu, \nu \mapsto \sqrt{\mu\nu}$ has the following property: for every set of finite positive measures $\mu_1, \dots, \mu_n$ on $(X, \mathcal{B})$ and every set $f_1, \dots, f_n$ of bounded measurable functions on X we have

$$\sum_{j,k=1}^n \int_X f_j \bar{f}_k\, d\sqrt{\mu_j \mu_k} \geq 0,$$

(see Chapter 14 of [**Arv03**] for more detail).

Fixing a measure class $(X, \mathcal{B}, \mathcal{M})$, we form the complex vector space V of all formal finite sums

$$f_1\sqrt{\mu_1} + \cdots + f_n\sqrt{\mu_n}$$

where $\mu_1, \dots, \mu_n \in \mathcal{M}$ and $f_1, \dots, f_n$ are bounded measurable functions. The preceding inequalities imply that we can define a positive semidefinite inner product $\langle \cdot, \cdot \rangle$ on V uniquely by setting

$$\langle f\sqrt{\mu}, g\sqrt{\nu} \rangle = \int_X f\bar{g}\, d\sqrt{\mu\nu},$$

where $\mu, \nu \in \mathcal{M}$ and f, g are bounded functions. After dividing out by elements of norm zero and completing, we obtain a Hilbert space $L^2(X, \mathcal{B}, \mathcal{M})$.

There is a natural "square root" map $\mu \in \mathcal{M} \mapsto \sqrt{\mu}$ of $\mathcal{M}$ into $L^2(X, \mathcal{B}, \mathcal{M})$ and these square roots can be shown to span $L^2(X, \mathcal{B}, \mathcal{M})$. There is also a natural

$*$-representation π of the C^*-algebra $B(X)$ of all bounded Borel functions on X on $L^2(X, \mathcal{B}, \mathcal{M})$, defined uniquely by requiring $\pi(f)g\sqrt{\mu} = fg\sqrt{\mu}$. If $\mathcal{M} = [\mu_0]$ consists of all finite measures ν that are mutually absolutely continuous with respect to a fixed finite positive measure μ_0, then it is possible to identify $L^2(X, \mathcal{B}, \mathcal{M})$ with $L^2(X, \mathcal{B}, \mu_0)$. Considering such formal expressions as elements of $L^2(X, \mathcal{B}, \mathcal{M})$, one finds that for any strictly positive bounded Borel function f and a measure $\mu \in \mathcal{M}$, one has $\sqrt{f^2\mu} = f\sqrt{\mu}$ where $f^2\mu$ denotes the obvious measure in $\mathcal{M}$.

More significantly, this assignment of a Hilbert space to a measure class has the following property: Given two Borel spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ and a measure class $\mathcal{P}$ on the cartesian product of Borel spaces $(X \times Y, \mathcal{A} \times \mathcal{B})$ which is generated as a measure class by a finite product measure $\mu \times \nu$, where $\mathcal{A} = [\mu]$ and $\mathcal{B} = [\nu]$, then $L^2(X \times Y, \mathcal{A} \times \mathcal{B}, \mathcal{P})$ decomposes into a tensor product of Hilbert spaces in a way analogous to (3.16) above: there is a unique unitary operator

$$W : L^2(X, \mathcal{A}, [\mu]) \otimes L^2(Y, \mathcal{B}, [\nu]) \to L^2(X \times Y, \mathcal{A} \times \mathcal{B}, \mathcal{P})$$

that satisfies

$$W(f\sqrt{\mu} \otimes g\sqrt{\nu}) = f(x)g(y)\sqrt{\mu \times \nu}, \qquad (x, y) \in X \times Y \tag{3.17}$$

for all bounded Borel functions $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$.

The key feature of this identification is that while the unitary operator W of (3.17) *appears* to depend on the particular choice of μ and ν, it actually does not. Indeed, if we choose other measures μ_1 on X and ν_1 on Y such that $\mathcal{A} = [\mu_1]$ and $\mathcal{B} = [\nu_1]$, then $\mathcal{P} = [\mu_1 \times \nu_1]$ and one can verify that the operator W of (3.17) also satisfies

$$W(f\sqrt{\mu_1} \otimes g\sqrt{\nu_1}) = f(x)g(y)\sqrt{\mu_1 \times \nu_1}, \qquad (x, y) \in X \times Y.$$

This follows from the Radon-Nikodym theorem, since one can check that for every measure $\lambda \in \mathcal{M}$ and every bounded nonnegative function u, the two formal expressions $\sqrt{u\lambda}$ and $\sqrt{u(x)}\sqrt{\lambda}$ define the same element of $L^2(X, \mathcal{A}, \mathcal{M})$.

With these preparations, we can now define the Tsirelson-Vershik product systems. Fix $\theta > 1$, choose a correlation function C_θ satisfying conditions (i), (ii), (iii) above, and let H_θ be the completion of $L^2(\mathbb{R})$ in the inner product (3.13). There is a standard construction whereby we can realize the Hilbert space H_θ as a Gaussian space $H_\theta \subseteq L^2(\Omega, \mathcal{B}, P)$. For every interval $I = (a, b) \subseteq \mathbb{R}$ let $H_\theta(I)$ be the closure of the set $L^2(I)$ of functions in $L^2(\mathbb{R})$ that vanish almost everywhere outside I, let $\mathcal{B}_I$ be the corresponding σ subalgebra of $\mathcal{B}$ and let $\mathcal{P}_I$ be the set of all measures on $\mathcal{B}_I$ that are mutually absolutely continuous with the restriction of P to $\mathcal{B}_I$.

Each $(\Omega, \mathcal{B}_I, \mathcal{P}_I)$ is a measure class, and we can form its canonical Hilbert space $L^2(\Omega, \mathcal{B}_I, \mathcal{P}_I)$. For every $t > 0$ $E_\theta(t)$ is defined as the Hilbert space

$$E_\theta(t) = L^2(\Omega, \mathcal{B}_{(0,t)}, \mathcal{P}_{(0,t)}),$$

and $p : E_\theta \to (0, \infty)$ is the assembled family of Hilbert spaces

$$E_\theta = \{(t, \xi) : t > 0, \xi \in E_\theta(t)\}.$$

There is a natural Borel structure on E_θ, and it is a nontrivial result of [**Tsi00b**] that this Borel structure is standard.

Multiplication is defined in E_θ as follows. Given $s, t > 0$, $\xi \in E_\theta(s)$ and $\eta \in E_\theta(t)$ we define the product $\xi \cdot \eta \in E_\theta(s + t)$ as follows. The translation operator $T_s \in \mathcal{B}(L^2(\mathbb{R}))$ restricts to a unitary map of $L^2(0,t)$ onto $L^2(s,s+t)$, and thus its closure defines a unitary operator

$$U_{(s,t)} : H_\theta(0,t) \to H_\theta(s,s+t).$$

It follows from the properties of Gaussian random variables that there is a unique unitary operator

$$\tilde{U}_{(s,t)} : L^2(\Omega, \mathcal{B}_{(0,t)}, \mathcal{P}_{(0,t)}) \to L^2(\Omega, \mathcal{B}_{(s,s+t)}, \mathcal{P}_{(s,s+t)});$$

indeed, $\tilde{U}_{(s,t)}$ is implemented by an isomorphism of measure classes. Now up to sets of measure zero, $(0,s+t) = (0,s) \cup (s,s+t)$ decomposes into a disjoint union of such intervals. However, the restriction of P to $\mathcal{B}_{(0,s+t)} \sim \mathcal{B}_{(0,s)} \times \mathcal{B}_{(s,s+t)}$ does not decompose into a product measure. Nevertheless, since the subspaces $H_\theta(0,s)$ and $H_\theta(s,s+t)$ are quasiorthogonal, Theorems 3.3.6 and 3.3.8 imply that the *measure class* $\mathcal{P}_{(0,s+t)}$ decomposes into a product of measure classes $\mathcal{P}_{(0,s)} \times \mathcal{P}_{(s,s+t)}$. Therefore we have a natural way of making the identification

$$L^2(\Omega, \mathcal{B}_{(0,s+t)}, \mathcal{P}_{(0,s+t)}) \sim L^2(\Omega, \mathcal{B}_{(0,s)}, \mathcal{P}_{(0,s)}) \otimes L^2(\Omega, \mathcal{B}_{(s,s+t)}, \mathcal{P}_{(s,s+t)})$$

using a unitary operator W of the form (3.17). At this point the product $\xi \cdot \eta$ can be defined as the image of

$$\xi \cdot \eta = \xi \otimes \tilde{U}_{(s,t)}\eta$$

under the latter unitary operator. This is perhaps not the best definition of the multiplication in E_θ, but it is the quickest. There are several technical details that must be carefully checked to establish that E_θ is a product system, see [**Tsi00b**].

The main two results of [**Tsi00b**] are the following:

THEOREM 3.3.10 (Tsirelson). *For every $\theta > 1$, the product system E_θ is of type III. E_θ and $E_{\theta'}$ are isomophic only if $\theta = \theta'$.*

3.4. Dimension Function. We have defined the numerical index of an E_0-semigroup α in terms of certain structures that are associated with its concrete product system $\mathcal{E}_\alpha$. We now describe briefly how all of those considerations carry over to the setting of abstract product systems. In that context, it appears more appropriate to think of this numerical invariant as a logarithmic dimension function.

DEFINITION 3.4.1. Let E be a product system. A unit of E is a measurable cross section $t \in (0,\infty) \mapsto u(t) \in E(t)$ that is multiplicative

$$u(s+t) = u(s)u(t), \qquad s,t > 0, \tag{3.18}$$

and is not the trivial section $u \equiv 0$.

The set of units of E is denoted $\mathcal{U}_E$. The following result provides a covariance function for product systems that have units; we refer the reader to [**Arv03**] for the proof.

PROPOSITION 3.4.2. *Let E be a product system and let $u, v \in \mathcal{U}_E$. Then there is a unique complex number $c_E(u,v)$ satisfying*

$$\langle u(t), v(t)\rangle = e^{tc_E(u,v)}, \qquad t > 0.$$

The function $c_E : \mathcal{U}_E \times \mathcal{U}_E \to \mathbb{C}$ is conditionally positive definite.

DEFINITION 3.4.3. Let E be a product system for which $\mathcal{U}_E \neq \emptyset$. The function $c_E : \mathcal{U}_E \times \mathcal{U}_E \to \mathbb{C}$ is called the covariance function of E.

The covariance function is conditionally positive definite, thus there is a natural way to use it to construct a Hilbert space $H(\mathcal{U}_E, c_E)$. It is known that $H(\mathcal{U}_E, c_E)$ is separable whenever $\mathcal{U}_E \neq \emptyset$, see [**Arv03**].

Definition 3.4.4. Let E be a product system. The dimension $\dim E$ of E is defined as the dimension of the Hilbert space $H(\mathcal{U}_E, c_E)$ if E has units, and is defined as $\dim E = 2^{\aleph_0}$ otherwise.

Suppose now that we are given an E_0-semigroup α acting on $\mathcal{B}(H)$, with concrete product system $\mathcal{E}_\alpha$. Paraphrasing the definition of units given in Section 1.4, one has semigroups $\{U_t : t > 0\}$ that are strongly continuous sections $t \in (0, \infty) \mapsto U_t \in \mathcal{E}_\alpha(t)$ that also tend strongly to $\mathbf{1}$ as $t \to 0+$. On the other hand, Definition 3.4.1 above merely requires measurable sections of $\mathcal{E}_\alpha$ defined on the open interval $(0, \infty)$ that form a nonzero semigroup. It can be shown that the two definitions coincide. Moreover, once this identification is made, it becomes obvious that the covariance function defined by Proposition 3.4.2 is identical with the covariance function defined in Section 1.4. Thus the two Hilbert spaces are the same, and in particular:

Proposition 3.4.5. *For every E_0-semigroup α, we have*

$$\operatorname{index}(\alpha) = \dim \mathcal{E}_\alpha.$$

Remark 3.4.6 (Index of the CCR Flows). In order to calculate the dimension of a product system, one has to calculate its set of units, its covariance function, and the dimension of the associated Hilbert space. In order to illustrate the procedure, we show how to compute the dimension of the exponential product systems E_N and the index of the CCR flows.

Letting K to be an N-dimensional Hilbert space, we have

$$E_N(t) = e^{L^2((0,t);K)} \subseteq e^{L^2((0,\infty);K)}, \qquad t > 0,$$

and the multiplication of functions $f \in E(s)$ and $g \in E(t)$ is defined by

$$f \cdot g = f \otimes U_s g \in E(s+t)$$

where $U_\lambda = \Gamma(S_\lambda)$ is the second-quantized shift semigroup of multiplicity N, and where the tensor product is interpreted in the sense of (3.11).

Writing $\chi_{(0,t)} \otimes \zeta$ for the function in $L^2((0,t); K)$ that has the constant value $\zeta \in K$ on the interval $0 < x \leq t$ and is zero elsewhere, we find that

$$\exp(\chi_{(0,s)} \otimes \zeta) \cdot \exp(\chi_{(0,t)} \otimes \zeta) = \exp(\chi_{(0,s+t)} \otimes \zeta),$$

hence $u(t) = \exp(\chi_{(0,t)} \otimes \zeta)$, $t > 0$, defines a unit of E_N. Moreover, using Proposition 3.3.1 it is possible to show that the most general unit of E_N is given by

$$u^{(a,\zeta)}(t) = e^{ta} \exp(\chi_{(0,t)} \otimes \zeta), \qquad t > 0, \tag{3.19}$$

where a is a complex number and ζ is a vector in K. From the formula

$$\langle \exp(\chi_{(0,t)} \otimes \zeta), \exp(\chi_{(0,t)} \otimes \omega)\rangle_{E(t)} = e^{t\langle \zeta, \omega\rangle_K}, \qquad t > 0,$$

we find that the covariance function of E_N is

$$c_{E_N}(u^{(a,\zeta)}, u^{(b,\omega)}) = a + \bar{b} + \langle \zeta, \omega \rangle_K. \tag{3.20}$$

A straightforward calculation based on (3.20) shows that the Hilbert space $H(\mathcal{U}_{E_N}, c_{E_N})$ is naturally identified with K, and therefore $\dim(E_N) = N$. From Propositions 3.4.5 and 3.3.1, we deduce the following result, which implies, for example, that the CCR flow of rank 2 cannot be realized as a cocycle perturbation of the CCR flow of rank 1.

COROLLARY 3.4.7. *Let α be the CCR flow of rank $N = 1, 2, \dots, \infty$. Then*

$$\mathrm{index}(\alpha) = N.$$

3.5. The Classifying Structure Σ. The fundamental problem in this subject is the classification of E_0-semigroups up to cocycle conjugacy. The results of Section 3.1 imply that the problem reduces to the problem of classifying product systems up to isomorhism, and therefore *one should approach the classification problem for E_0-semigroups by examining the structure of product systems on their own terms.*

We now make this more precise by introducing a classifying structure Σ for E_0-semigroups. The elements of Σ are isomorphism classes of product systems. The formation of tensor products of product systems gives rise to a commutative "addition" in Σ. There is also a natural involution in Σ which makes Σ into an involutive abelian semigroup with a zero element, and we discuss the significance of this involution for dynamics. We also describe how Σ can be naturally identified with the set of all cocycle conjugacy classes of E_0-semigroups. One may conclude that the problem of classifying E_0-semigroups up to cocycle conjugacy reduces to that of determining the structure of the involutive semigroup Σ and discovering computable invariants for its elements.

In order to fully carry out the discussion of this section, we must depart from the logical development by making use of a key result that has not yet been discussed, namely that for every product system E there is an E_0-semigroup whose concrete product system is isomorphic to E. We will discuss that result in Lecture 4.

The *trivial product system* is the trivial family of one-dimensional Hilbert spaces

$$Z = (0, \infty) \times \mathbb{C},$$

where $\mathbb{C}$ has its usual inner product $\langle z, w \rangle = z\bar{w}$, where multiplication is defined by

$$(s, z)(t, w) = (s + t, zw), \qquad s, t > 0, \quad z, w \in \mathbb{C},$$

and where the Borel structure on $Z = (0, \infty) \times \mathbb{C}$ is the obvious one. It is significant that Z is the only "line bundle" in the category of product systems, as asserted by the following result.

THEOREM 3.5.1. *Let E be a product system such that $E(t)$ is one dimensional for every $t > 0$. Then E is isomorphic to the trivial product system $Z = (0, \infty) \times \mathbb{C}$.*

Opposite of a Product System. We will see momentarily that anti-isomorphisms of product systems play a significant role in noncommutative dynamics. By an *anti-isomorphism* of product systems $\theta : E \to F$ we mean a Borel isomorphism that restricts to a unitary operator on each fiber $\theta_t : E(t) \to F(t)$, $t > 0$, such that $\theta(xy) = \theta(y)\theta(x)$ for $x, y \in E$.

There is a natural involution $E \to E^{\mathrm{op}}$ in the category of product systems, defined as follows. For every product system E, E^{op} is defined as the same measurable family of Hilbert spaces $p : E \to (0, \infty)$, but the multiplication in E^{op} is reversed; for $x, y \in E$, the product $x \cdot y$ in E^{op} is defined as $yx \in E$. E^{op} is called the *opposite* product system of E. If we consider the identity map of E as a map of E to E^{op}, then it becomes an anti-isomorphism of product systems. Thus, a *product system is anti-isomorphic to E iff it is isomorphic to E^{op}.*

The tensor product. There is also a natural notion of tensor product in this category. Given product systems E, F, and $t > 0$, we can form the Hilbert space $E(t) \otimes F(t)$, and the associated family of Hilbert spaces

$$E \otimes F = \{(t, x) : t > 0,\ x \in E(t) \otimes F(t)\}. \tag{3.21}$$

Multiplication is defined in $E \otimes F$ in the natural way. In more detail, given elementary tensors $x \otimes y \in E(s) \otimes F(s)$ and $x' \otimes y' \in E(t) \otimes F(t)$, the map

$$(x \otimes y, x' \otimes y') \mapsto xx' \otimes yy' \in E(s + t) \otimes F(s + t)$$

extends uniquely to a bounded bilinear map of $(E(s) \otimes F(s)) \times (E(t) \otimes F(t))$ into $E(s + t) \otimes F(s + t)$, which in turn can be associated with a unitary operator

$$(E(s) \otimes F(s)) \otimes (E(t) \otimes F(t)) \to E(s + t) \otimes F(s + t)$$

as required for the multiplication of a product system. The Borel structure of $E \otimes F$ has a natural definition that we omit.

It is a nontrivial property of the dimension function that it obeys the following logarithmic addition formula

$$\dim(E \otimes F) = \dim E + \dim F$$

in all cases. The proof of this formula amounts to establishing the fact that a unit w of a tensor product $E \otimes F$ of product systems *must* decompose into a tensor product of units

$$w_t = u_t \otimes v_t, \qquad t > 0,$$

where $u \in \mathcal{U}_E$ and $v \in \mathcal{U}_f$ (see [**Arv03**] for more detail). In view of the relation between index and dimension (Proposition 3.4.5), and because of Proposition 3.5.2 to follow, the preceding formula leads to the addition formula for the index of E_0-semigroups

$$\operatorname{index}(\alpha \otimes \beta) = \operatorname{index}(\alpha) + \operatorname{index}(\beta).$$

Now let α and β be two E_0-semigroups acting, respectively, on $\mathcal{B}(H)$ and $\mathcal{B}(K)$, and let $\mathcal{E}_\alpha$, $\mathcal{E}_\beta$ be their concrete product systems. For every $t > 0$ consider the operator space

$$\mathcal{E}_\alpha(t) \otimes \mathcal{E}_\beta(t) = \overline{\operatorname{span}}^{\|\cdot\|}\{A \otimes B : A \in \mathcal{E}_\alpha(t), B \in \mathcal{E}_\beta(t)\},$$

the closure being relative to the *operator* norm. The total family of spaces

$$\mathcal{E}_\alpha \otimes \mathcal{E}_\beta = \{(t, C) : t > 0, \quad C \in \mathcal{E}_\alpha(t) \otimes \mathcal{E}_\beta(t)\}$$

is called the *spatial tensor product* of the concrete product systems $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$. The tensor product of product systems corresponds to the tensor product of E_0-semigroups as follows:

Proposition 3.5.2. *For any two E_0-semigroups α, β, the product system of $\alpha \otimes \beta$ is the spatial tensor product $\mathcal{E}_\alpha \otimes \mathcal{E}_\beta$. Moreover, the spatial tensor product $\mathcal{E}_\alpha \otimes \mathcal{E}_\beta$ is naturally isomorphic to the tensor product of (3.21).*

For every product system E, let $[E]$ denote the class of all product systems that are isomorphic to E. The set Σ of all such equivalence classes can be made into an abelian semgroup by defining the sum of classes as follows

$$[E] + [F] = [E \otimes F],$$

and Σ admits a natural involution

$$[E]^* = [E^{\mathrm{op}}],$$

satisfying $(\xi+\eta)^* = \xi^* + \eta^*$, $\xi, \eta \in \Sigma$. It is a straightforward exercise to verify that $E \cong E \otimes Z \cong Z \otimes E$ for every product system E, hence the class of the trivial product system $[Z]$ functions as a zero element for Σ.

The role of Σ in the classification problem for E_0-semigroups is spelled out as follows (where we assume the result of Theorem 4.5.2 below).

THEOREM 3.5.3. *For every E_0-semigroup α, let $[\mathcal{E}_\alpha]$ be the representative of its product system in Σ. This association defines a bijection of the set of cocycle conjugacy classes of E_0-semigroups onto Σ, and one has*

$$[\mathcal{E}_{\alpha\otimes\beta}] = [\mathcal{E}_\alpha] + [\mathcal{E}_\beta].$$

3.6. Role of the Involution in Dynamics. The involution of Σ is of fundamental importance for dynamics, as we now describe.

Suppose that we are given two E_0-semigroups α, β acting respectively on $\mathcal{B}(H)$ and $\mathcal{B}(K)$. We seek conditions on the pair α, β which imply that there is a one-parameter group of unitary operators $W = \{W_t : t \in \mathbb{R}\}$ acting on the tensor product $H \otimes K$ whose associated automorphism group $\gamma_t(C) = W_t C W_t^*$ satisfies

$$\gamma_t(A \otimes \mathbf{1}_K) = \alpha_t(A) \otimes \mathbf{1}_K, \quad \text{for } t \geq 0 \tag{3.22}$$

$$\gamma_t(\mathbf{1}_H \otimes B) = \mathbf{1}_H \otimes \beta_{-t}(B), \quad \text{for } t \leq 0. \tag{3.23}$$

When such a group exists, α and β are said to be *paired*. This relation was introduced by Powers and Robinson in [**PR89**] as an intermediate step in their definition of another index. We will not pursue the Powers-Robinson index here, but we do want to emphasize the importance of the pairing concept for dynamics.

Let us first recall the context of Lecture 1. Considering the von Neumann algebra $\mathcal{M} = \mathcal{B}(H) \otimes \mathbf{1}_K$ as a type I subfactor of $\mathcal{B}(H \otimes K)$, with commutant $\mathcal{M}' = \mathbf{1}_H \otimes \mathcal{B}(K)$, we are given a pair of E_0-semigroups α, β acting, respectively, on $\mathcal{M}$ and $\mathcal{M}'$, and we are asking if there is a one-parameter group of automorphisms γ of $\mathcal{B}(H \otimes K)$ that satisfies the two conditions of (1.8).

The following result implies that a necessry and sufficient condition for the existence of an automorphism γ satisfying (3.22) and (3.23) is that the product systems of α and β should satisfy $[\mathcal{E}_\beta] = [\mathcal{E}_\alpha]^*$.

THEOREM 3.6.1. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a type I factor and let α and β be E_0-semigroups acting, respectively, on $\mathcal{M}$ and $\mathcal{M}'$. The following are equivalent.*

(i) *There is a one-parameter automorphism group $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ acting on $\mathcal{B}(H)$ that satisfies (3.22) and (3.23).*
(ii) *The product systems $\mathcal{E}_\alpha$ and $\mathcal{E}_\beta$ are anti-isomorphic.*

More explicitly, if $U = \{U_t : t \in \mathbb{R}\}$ is a strongly continuous one parameter unitary group on H whose automorphism group $\gamma_t(A) = U_t A U_t^*$ implements α and β as in (i), then for every $t > 0$ we have $U_t^* \mathcal{E}_\alpha(t) = \mathcal{E}_\beta(t)$, and the map $\theta : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ defined by

$$\theta((t, T)) = (t, U_t^* T), \qquad t > 0, \quad T \in \mathcal{E}_\alpha(t) \tag{3.24}$$

is an anti-isomorphism of product systems.

Conversely, every anti-isomorphism $\theta : \mathcal{E}_\alpha \to \mathcal{E}_\beta$ has the form (3.24) for a unique family $\{U_t : t > 0\}$ of unitary operators. This family is a strongly continuous semigroup tending strongly to $\mathbf{1}$ as $t \to 0+$, and its extension to a one-parameter unitary group in $\mathcal{B}(H)$ gives rise to an automorphism group γ satisfying (i) as above.

Sketch of Proof. In order to communicate the flavor of the argument, we prove the implication (i) $\implies$ (ii), referring the reader to [**Arv03**] for more detail.

Recalling that every one-parameter group γ of automorphisms of $\mathcal{B}(H)$ is implemented by a strongly continuous one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ by way of $\gamma_t(A) = U_t A U_t^*$ for $t \in \mathbb{R}$, $A \in \mathcal{B}(H)$, it suffices to prove that any such group γ that satisfies (i) must give rise to a map θ as in (3.24) that defines an anti-isomorphism of product systems.

Note first that $U_t^* \mathcal{E}_\alpha(t) \subseteq \mathcal{M}'$ for every $t > 0$. Indeed, if $A \in \mathcal{M}$ then for every $T \in \mathcal{E}_\alpha(t)$ we have

$$AU_t^*T = U_t^*\gamma_t(A)T = U_t^*\alpha_t(A)T = U_t^*TA.$$

We claim that $U_t^*\mathcal{E}_\alpha(t) = \mathcal{E}_\beta(t)$. For the inclusion $\subseteq$, choose $T \in \mathcal{E}_\alpha(t)$. The preceding paragraph implies that $U_t^*T \in \mathcal{M}'$, so it remains to show that $\beta_t(B)U_t^*T = U_t^*TB$ for every $B \in M'$. For that, write

$$\beta_t(B)U_t^*T = \gamma_{-t}(B)U_t^*T = U_t^*BU_tU_t^*T = U_t^*BT = U_t^*TB,$$

the last equality because $T \in \mathcal{M}$ commutes with $B \in \mathcal{M}'$.

For the inclusion $\mathcal{E}_\beta(t) \subseteq U_t^*\mathcal{E}_\alpha(t)$, choose $S \in \mathcal{E}_\beta(t)$ and set $T = U_tS$. Note that $T \in \mathcal{M}'' = \mathcal{M}$ because for every $B \in \mathcal{M}'$ we have

$$BT = BU_tS = U_t\gamma_{-t}(B)S = U_t\beta_t(B)S = U_tSB = TB.$$

Moreover, $T = U_tS \in \mathcal{M}$ actually belongs to $\mathcal{E}_\alpha(t)$, since for $A \in \mathcal{M}$

$$\alpha_t(A)T = \gamma_t(A)U_tS = U_tAS = U_tSA = TA,$$

hence $S = U_t^*T \in U^*\mathcal{E}_\alpha(t)$.

Thus for every $t > 0$ we can define a map $\theta_t : \mathcal{E}_\alpha(t) \to \mathcal{E}_\beta(t)$ by $\theta_t(T) = U_t^*T$. By assembling these maps we get a bijective Borel-measurable function

$$\theta : (t, T) \in \mathcal{E}_\alpha \to (t, U_t^*T) \in \mathcal{E}_\beta$$

that is a linear isomorphism on each fiber. Each θ_t is actually unitary, since for $T_1, T_2 \in \mathcal{E}_\alpha(t)$ we have

$$\langle \theta_t(T_1), \theta_t(T_2)\rangle \mathbf{1} = \theta_t(T_2)^*\theta_t(T_1) = (U_t^*T_2)^*(U_t^*T_1) = T_2^*T_1 = \langle T_1, T_2\rangle \mathbf{1}.$$

Finally, since for $S \in \mathcal{E}_\alpha(s)$ and $T \in \mathcal{E}_\alpha(t)$ we have

$$\theta_{s+t}(ST) = U_{s+t}^*ST = U_t^*(U_s^*S)T = U_t^*\theta_s(S)T = U_t^*T\theta_s(S) = \theta(T)\theta(S),$$

it follows that θ is an anti-isomorphism of product systems. $\square$

We now indicate how a key result from Lecture 2 is deduced from Theorem 3.6.1.

Proof of Theorem 1.5.1. For every $n = 1, 2, \dots, \infty$, let E_n be the exponential product system of dimension n. We point out that each E_n is anti-isomorphic to itself. That follows, for example, from the classification results for type I product systems, since the product system opposite to E_n is a decomposable product system of the same dimension n, and therefore isomorphic to E_n. Alternately, one can simply write down an explicit anti-automorphism of E_n using the concrete description of it given in Proposition 3.3.1.

One concludes from these remarks that E_m is anti-isomorphic to E_n iff $m = n$. It follows that the product systems of two cocycle perturbations α, β of CAR/CCR flows are anti-isomorphic iff α and β have the same numerical index. Thus, Theorem

1.5.1 is now seen as the special case of Theorem 3.6.1 for cocycle perturbations of CAR/CCR flows, in the setting in which $\mathcal{M} = \mathcal{B}(H)\otimes\mathbf{1}_K$ and $\mathcal{M}' = \mathbf{1}_H\otimes\mathcal{B}(K)$. $\square$

3.7. Gauge Group. Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(H)$. A gauge cocycle is a cocycle $U = \{U_t : t \geq 0\}$ for α with the property that the corresponding perturbation of α is is the trivial one:

$$U_t\alpha_t(A)U_t^* = \alpha_t(A), \qquad t \geq 0, \quad A \in \mathcal{B}(H).$$

One sees that the pointwise product $UV = \{U_tV_t : t \geq 0\}$ of two gauge cocycles U, V is a gauge cocycle, as is $\{U_t^* : t \geq 0\}$. Thus the gauge cocycles form a group $G(\alpha)$ under pointwise multiplication, called the *gauge group* of α. There is also a natural topology on $G(\alpha)$ with respect to which it is a Polish topological group, but it will not be necessary to deal with topological issues here. The gauge group reflects the "internal symmetries" of α as we will see presently.

It is quite easy to see that if $\beta = \{\beta_t : t \geq 0\}$ is another E_0-semigroup that is cocycle conjugate to α, then $G(\alpha)$ and $G(\beta)$ are isomorphic. Thus the gauge group provides a rather subtle cocycle conjugacy invariant for E_0-semigroups. The purpose of this section is to point out the role of the gauge group in dynamical issues, to clarify its status as the group of internal symmetries, and to exhibit the structure of the gauge groups of type I E_0-semigroups in very explicit terms.

Let us first examine the role of the gauge group in dynamics. Given two E_0-semigroups α, β acting respectively on $\mathcal{B}(H)$, $\mathcal{B}(K)$, we have seen that a necessary and sufficient for there to exist a one-paramter group of automorphisms $\gamma = \{\gamma_t : t \in \mathbb{R}\}$ satisfying (3.22) and (3.23) is that the product systems of α and β should by anti-isomorphic. Assuming that this is the case, one would then like to know how to parameterize the set of all such groups γ in concrete terms. The following result, a consequence of Theorem 3.6.1, shows that the set of all such automorphism groups is parameterized naturally by the elements of the gauge group $G(\alpha)$.

Theorem 3.7.1. *Let α and β be two E_0-semigroups acting, respectively, on $\mathcal{B}(H)$ and $\mathcal{B}(K)$, and assume there is a one-parameter group $\gamma^0 = \{\gamma_t^0 : t \in \mathbb{R}\}$ of automorphisms of $\mathcal{B}(H \otimes K)$ that satisfies (3.22) and (3.23). For every other one-parameter group of automorphisms γ of $\mathcal{B}(H)$, the following are equivalent:*

(i) *γ satisfies (3.22) and (3.23).*
(ii) *There is a gauge cocycle $U = \{U_t : t \geq 0\}$ in $G(\alpha)$ such that the actions of γ and γ^0 on $\mathcal{B}(H)$ are related as follows*

$$\gamma_t(X) = U_t\gamma_t^0(X)U_t^*, \qquad X \in \mathcal{B}(H), \quad t \geq 0. \tag{3.25}$$

Moreover, for every gauge cocycle $U \in G(\alpha)$, formula (3.25) defines a semigroup $\{\gamma_t : t \geq 0\}$ of automophisms of $\mathcal{B}(H)$ whose unique extension to a one-parameter automorphism group of $\mathcal{B}(H)$ satisfies (3.22) and (3.23).

While Theorem 3.7.1 pins down the lack of uniqueness that accompanies the automorphism groups γ that solve (3.22) and (3.23), it provides no insight into the structure or even the cardinality of gauge group, and does not lead to an explicit parameterization of the these automorphism groups. We now show how to identify the gauge group in a more concrete way that reveals the role of elements of $G(\alpha)$ as internal symmetries.

For every gauge cocycle $U = \{U_t : t \geq 0\} \in G(\alpha)$ let $\theta^U : \mathcal{E}_\alpha \to (0, \infty) \times \mathcal{B}(H)$ be the map defined by

$$\theta^U(t, T) = (t, U_t T), \qquad t > 0, \quad T \in \mathcal{E}_\alpha(t).$$

Note first that $\theta^U(\mathcal{E}_\alpha) \subseteq \mathcal{E}_\alpha$. Indeed, for every $t > 0$ and $A \in \mathcal{B}(H)$ we have

$$\alpha_t(A) U_t T = U_t \alpha_t(A) T = U_t T A,$$

since U_t commutes with $\alpha_t(A)$. Indeed, θ^U is a bijection of $\mathcal{E}_\alpha$ onto itself that is unitary on fibers. It is multiplicative because

$$\begin{aligned} \theta^U(s, S)\theta^U(t, T) &= (s, U_s S)(t, U_t T) = (s + t, U_s S U_t T) \\ &= (s + t, U_s \alpha_s(U_t) S T) = (s + t, U_{s+t} S T) \end{aligned}$$

since U is an α-cocycle. Obviously, $\theta^U \circ \theta^V = \theta^{UV}$, and in fact we have an isomorphism of groups:

THEOREM 3.7.2. *The mapping $U \in G(\alpha) \mapsto \theta^U$ defines an isomorphism of the gauge group $G(\alpha)$ onto the group* $\operatorname{aut} \mathcal{E}_\alpha$ *of all automorphisms of the product system of α.*

Thus, in order to calculate gauge groups one should attempt to compute the automorphism groups of product systems. However, little is known about type III product systems, and even less is known about their automorphism groups. The type II case is somewhat less mysterious, but still poorly understood. A type II product system E of dimension $n = 1, 2, \ldots, \infty$ must contain a type I part E_I of dimension n as a subsystem, but as yet we lack a clear understanding of how to assemble E out of its type I part E_I and perhaps other data.

On the other hand, for type I product systems these calculations can be carried out in very explicit terms. We now describe this result from [**Arv89a**], passing lightly over topological issues. For every separable Hilbert space H, we define a group G_H as follows. As a set, G_H is the cartesian product

$$G_H = \mathbb{R} \times H \times \mathcal{U}(H), \tag{3.26}$$

$\mathcal{U}(H)$ being the unitary group of H. The multiplication in G_H is defined by

$$(\lambda, \xi, U)(\mu, \eta, V) = (\lambda + \mu + \omega(\xi, U\eta), \xi + U\eta, UV),$$

ω denoting the symplectic form on $H \times H$ given by the imaginary part of the inner product $\omega(\xi, \eta) = \Im\langle \xi, \eta \rangle$. The identity of G_H is $(0, 0, \mathbf{1})$, and inverses are given by the formula

$$(\lambda, \xi, U)^{-1} = (-\lambda + \omega(\xi, U\xi), U^{-1}\xi, U^{-1}).$$

There is a natural topology on G_H which makes it into a Polish topological group; it becomes a Lie group when H is finite dimensional.

REMARK 3.7.3 (G_H and the Canonical Commutation Relations). The map $\lambda \in \mathbb{R} \mapsto (\lambda, 0, \mathbf{1})$ is an isomorphism of $\mathbb{R}$ onto the center of G_H, and the map $\pi(\lambda, \xi, U) = U$ defines a surjective homomorphism of G_H onto $\mathcal{U}(H)$ whose kernel is $K = \mathbb{R} \times H \times \{\mathbf{1}\}$. Noting the multiplication rule in K,

$$(\lambda, \xi, \mathbf{1})(\mu, \eta, \mathbf{1}) = (\lambda + \mu + \omega(\xi, \eta), \xi + \eta, \mathbf{1}), \tag{3.27}$$

we see that when H is finite dimensional, K is isomorphic to the universal covering group of the Heisenberg group of an appropriate dimension [**How80**]. The split exact sequence of groups

$$1 \longrightarrow K \longrightarrow G_H \longrightarrow \mathcal{U}(H) \longrightarrow 1$$

exhibits G_H as a semi direct product of K with the unitary group $\mathcal{U}(H)$.

Perhaps one can see the relation to the canonical commutation relations in more concrete terms by looking at the unitary representations of G_H. The most general strongly continuous unitary representaation of G_H on a Hilbert space L is obtained as follows. Let (U, W, Γ) be a triple consisting of (i) a strongly continuous one-parameter unitary group $U = \{U_t : t \in \mathbb{R}\}$ acting on L, (ii) a strongly continuous mapping

$$\xi \in H \mapsto W(\xi) \in \mathcal{U}(L)$$

satisfying the commutation relations

$$W(\xi)W(\eta) = U(\omega(\xi, \eta))W(\xi + \eta), \tag{3.28}$$

and (iii) a strongly continuous representation Γ of $\mathcal{U}(H)$ on L, all of which satisfy the compatibility relations

(iv) $U(\lambda)$ commutes with $U(\mathbb{R}) \cup W(H) \cup \Gamma(\mathcal{U}(H))$ for every $\lambda \in \mathbb{R}$,
(v) $\Gamma(U)W(\xi)\Gamma(U)^{-1} = W(U\xi), \quad \xi \in H, U \in \mathcal{U}(H)$.

Given such a triple (U, W, Γ), one readily verifies that

$$R(\lambda, \xi, V) = U(\lambda)W(\xi)\Gamma(V) \tag{3.29}$$

defines a strongly continuous unitary representation $R : G_H \to \mathcal{B}(L)$. Conversely, every unitary representation R of G_H decomposes uniquely into a product as in (3.29).

Notice that if R is an *irreducible* representation, then necessarily $U(\lambda)$ must be a scalar for every $\lambda \in \mathbb{R}$, hence there is a real constant c such that $U(\lambda) = e^{ic\lambda}\mathbf{1}$, $\lambda \in \mathbb{R}$. In this case, (3.28) reduces to

$$W(\xi)W(\eta) = e^{ic\omega(\xi,\eta)}W(\xi + \eta), \qquad \xi, \eta \in H.$$

Thus, W is simply a Weyl system for the symplectic form on $H \times H$ given by $c \cdot \omega(\xi, \eta)$, with the additional property that it is equivariant under the action of the unitary group of H in the sense of (v) above. We conclude that G_H *is the appropriate group for the analysis of representations of the canonical commutation relations that include rotational symmetry.*

THEOREM 3.7.4. *For every $N = 1, 2, \dots, \infty$, the gauge group of the CAR/CCR flow α of index N is isomorphic to the group G_H of (3.26), where H is a Hilbert space of dimension N.*

We have not indicated a specific isomorphism of groups $\theta : G(\alpha) \to G_H$ in the discussion above, but it is not hard to do so (see [**Arv03**]). Once this is done, Theorem 3.7.4 provides an explicit parameterization of the set of anti-isomorphisms of Theorem 3.6.1, and therefore a parameterization of the set of all one-parameter automorphism groups γ that satisfy (3.22) and (3.23) in cases where both α and β are cocycle perturbations of CAR/CCR flows of the same index N.

4. Spectrum of an E_0-Semigroup.

Every product system E is associated with an intrinsic Hilbert space $L^2(E)$. There is also a natural way of representing E as a concrete product system $\mathcal{E}$ acting on $L^2(E)$, and $\mathcal{E}$ is isomorphic to E. In turn, $\mathcal{E}$ is associated with a semigroup of endomorphisms $\alpha = \{\alpha_t : t \geq 0\}$ of $\mathcal{B}(L^2(E))$ by Proposition 3.1.3. However, this E-semigroup is not an E_0-semigroup because the projections $\alpha_t(\mathbf{1})$ decrease to 0 as $t \to \infty$.

This lecture focuses on the problem of constructing an E_0-semigroup whose product system is isomorphic to E by exploiting properties of a C^*-algebra naturally associated with E, called the spectral C^*-algebra of E. We describe the basic theory of spectral C^*-algebras, emphasizing their role in the representation theory of product systems, and we discuss the issue of simplicity. The results were originally obtained in the series [**Arv90a**], [**Arv89b**], [**Arv90b**]; and have been substantially rewritten in [**Arv03**].

4.1. The C^*-algebra of a Product System. We now describe the fundamental properties of the regular representation and antirepresentation of a product system E, we define the spectral C^*-algebra $C^*(E)$, and we discuss the most basic properties of these structures.

With every product system E there is a naturally associated Banach algebra $L^1(E)$ of integrable sections $t \in (0, \infty) \mapsto f(t) \in E(t)$. The norm on $L^1(E)$ is

$$\|f\|_1 = \int_0^\infty \|f(t)\| \, dt,$$

and multiplication is defined by convolution

$$f * g(t) = \int_0^t f(s)g(t-s)\, ds, \qquad f, g \in L^1(E), \quad t > 0.$$

Notice that $f(s)g(t-s) \in E(t)$ for every s satisfying $0 < s < t$, so that $f * g$ is a well-defined section; one verifies the inequality $\|f * g\| \leq \|f\|\|g\|$ by a familiar application of the Fubini theorem and the fact that for $v \in E(s)$, $w \in E(t-s)$ one has $\|vw\| = \|v\|\|w\|$.

A *representation* of a product system E is a Borel measurable operator valued map $\phi : E \to \mathcal{B}(H)$ satisfying

(i) $\phi(y)^*\phi(x) = \langle x, y\rangle \mathbf{1}$, for every $x, y \in E(t)$, $t > 0$, and
(ii) $\phi(z)\phi(w) = \phi(zw)$, for arbitrary $z, w \in E$.

Condition (i) implies that the restriction of ϕ to each fiber $E(t)$ is a linear map. Anti-representations of E are defined similarly, with (ii) replaced by its opposite $\phi(z)\phi(w) = \phi(wz)$. Given a representation ϕ of E on a Hilbert space H, then for every integrable section $f \in L^1(E)$ the operator function $t \in (0, \infty) \mapsto \phi(f(t)) \in \mathcal{B}(H)$ is measurable with respect to the weak opeator topology, and a standard application of the Riesz Lemma allows one to form the operator integral

$$\int_0^\infty \phi(f(t))\, dt, \tag{4.1}$$

which is an operator of norm at most $\|f\|_1$. We abuse notation slightly by writing this operator as $\phi(f)$; thus

$$\phi(f) = \int_0^\infty \phi(f(t))\, dt, \qquad f \in L^1(E).$$

One verifies that $\phi(f)\phi(g) = \phi(f * g)$, and in fact ϕ is a contractive representation of the Banach algebra $L^1(E)$. In a similar way, an anti-representation of E on H can be integrated to give a contractive *anti*representation of $L^1(E)$ on H.

We now exhibit a representation and an anti-representation of E on the intrinsic L^2 space of E. Consider the Hilbert space

$$L^2(E) = \int_{(0,\infty)}^{\oplus} E(t)\, dt$$

of square-integrable sections $\xi : t \in (0, \infty) \mapsto \xi(t) \in E(t)$, $t > 0$, with inner product

$$\langle \xi, \eta \rangle = \int_0^\infty \langle \xi(t), \eta(t) \rangle\, dt.$$

For each $v \in E(t)$, $t > 0$ and $\xi \in L^2(E)$ let $v\xi$ be the following function in $L^2(E)$

$$v\xi(x) = \begin{cases} v\xi(x - t), & x > t, \\ 0, & 0 < x \leq t. \end{cases}$$

One verifies easily that the map $\ell : E \to \mathcal{B}(L^2(E))$ defined by $\ell_v \xi = v\xi$ is a representation of E. As with any representation of E, this can be integrated to a representation $\ell : L^1(E) \to \mathcal{B}(L^2(E))$,

$$\ell_f \xi = \int_0^\infty f(t)\xi\, dt,$$

and of course when $\xi \in L^1(E) \cap L^2(E)$, $\ell_f \xi$ is seen to be convolution

$$\ell_f \xi\,(t) = f * \xi\,(t) = \int_0^t f(s)\xi(t - s)\, ds.$$

Similarly, for every $\xi \in L^2(E)$ we can define $\xi v \in L^2(E)$ by

$$\xi v(x) = \begin{cases} \xi(x - t)v, & x > t, \\ 0, & 0 < x \leq t. \end{cases}$$

This defines an antirepresentation of E on $L^2(E)$. The corresponding antirepresentation of $L^1(E)$ on $L^2(E)$ is written $f \in L^1(E) \mapsto r_f \in \mathcal{B}(L^2(E))$,

$$r_f \xi = \int_0^\infty \xi\, f(t)\, dt, \qquad f \in L^1(E), \quad \xi \in L^2(E);$$

and when $\xi \in L^1(E) \cap L^2(E)$ we have $r_f \xi = \xi * f$.

Remark 4.1.1 (Left and Right Semigroups). There are *two* concrete product systems that act naturally on $\mathcal{L}^2(E)$, one associated with the regular representation and the other associated with the regular antirepresentation

$$\mathcal{E}_\ell(t) = \{\ell_v : v \in E(t)\}, \quad \mathcal{E}_r(t) = \{r_v : v \in E(t)\}, \qquad t > 0.$$

Correspondingly, Proposition 3.1.3 implies that there are two semigroups of endomorphisms α, β associated with these concrete product systems. One obtains a more explicit expression for α_t and β_t by choosing an arbitrary orthonormal basis $\{e_1(t), e_2(t), \dots\}$ for $E(t)$, letting $U_1(t), U_2(t), \dots, V_1(t), V_2(t), \dots$ be the sequences of isometries $U_n(t) = \ell_{e_n(t)}$, $V_n(t) = r_{e_n(t)}$, and writing

$$\alpha_t(A) = \sum_{n=1}^\infty U_n(t) A U_n(t)^*, \quad \beta_t(A) = \sum_{n=1}^\infty V_n(t) A V_n(t)^*, \quad A \in \mathcal{B}(L^2(E)).$$

Since $U_m(s)$ commutes with $V_n(t)$ for every m, n and every $s, t > 0$, we have $\alpha_s \circ \beta_t = \beta_t \circ \alpha_s$.

For every $t > 0$, let P_t be the projection onto the subspace $L^2((t, \infty); E)$ of $L^2(E)$ consisting of all square summable sections that vanish almost everywhere on $(0, t]$. Then we have

$$\alpha_t(\mathbf{1}) = \beta_t(\mathbf{1}) = P_t, \qquad t > 0. \tag{4.2}$$

Since the intersection $\cap_t L^2((t, \infty); E)$ is the trivial subspace $\{0\}$, it follows that both $\alpha_t(\mathbf{1})$ and $\beta_t(\mathbf{1})$ decrease to 0 as $t \to \infty$. In particular, *neither of the semigroups* α, β *is an* E_0*-semigroup.* Nevertheless these semigroups, and especially β, play a central role in the analysis of the spectral C^*-algebra $C^*(E)$ introduced below.

Before introducing the spectral C^*-algebra of E, we point out that for every pair of functions $f, g \in L^1(E) \cap L^2(E)$, the product $\ell_f^* \ell_g$ decomposes into a sum

$$\ell_f^* \ell_g = \ell_h + \ell_k^* \tag{4.3}$$

where h and k belong to $L^1(E) \cap L^2(E)$. One verifies this by a direct computation, and while it is possible give explicit formulas for h and k, we shall not do so here.

Remark 4.1.2 (On Morita Equivalence). Consider the algebra

$$\mathcal{A} = \{\ell_f : f \in L^1(E)\} \subseteq \mathcal{B}(L^2(E))$$

of left convolution operators. There are two C^*-algebras of operators on $L^2(E)$ that one might associate with $\mathcal{A}$. Perhaps the most natural one is $C^*(\mathcal{A})$, the C^*-algebra spanned by finite products of elements of $\mathcal{A}$ and their adjoints. However, $C^*(\mathcal{A})$ is somewhat more cumbersome than a certain subalgebra of it that we now describe, and it is the latter subalgebra that we choose to work with. The purpose of these remarks is to point out that, since the two C^*-algebras are strongly Morita equivalent, they become isomorphic after tensoring with the compact operators. In particular, they have the same representation theory, the same lattice of closed two-sided ideals, the same K-theory, etc.

We write $[\mathcal{S}]$ for the norm-closed linear span of a set $\mathcal{S} \subseteq \mathcal{B}(L^2(E))$ of operators. Formula (4.3) implies that $\mathcal{A}^*\mathcal{A}$ is contained in $[\mathcal{A} + \mathcal{A}^*]$, and it follows that $[\mathcal{A}\mathcal{A}^*]$ is a C^*-algebra. Actually, $[\mathcal{A}\mathcal{A}^*]$ is a full hereditary subalgebra of $C^*(\mathcal{A})$. To see that, note that

$$\mathcal{R} = [\mathcal{A} + \mathcal{A}\mathcal{A}^*]$$

is a right ideal in $C^*(\mathcal{A})$ because $\mathcal{A}^*\mathcal{A} \subseteq [\mathcal{A} + \mathcal{A}^*]$, and it has the properties

$$[\mathcal{R}^*\mathcal{R}] = C^*(\mathcal{A}), \qquad [\mathcal{R}\mathcal{R}^*] = [\mathcal{A}\mathcal{A}^*].$$

Thus $[\mathcal{A}\mathcal{A}^*]$ is a full hereditary subalgebra of $C^*(\mathcal{A})$; in particular, it is strongly Morita equivalent to $C^*(\mathcal{A})$.

Definition 4.1.3. The spectral C^*-algebra of E is defined as the norm-closed linear span of the set of operators $\mathcal{A}\mathcal{A}^* = \{\ell_f \ell_g^* : f, g \in L^1(E)\}$, and is denoted $C^*(E)$.

It is significant that representations of $C^*(E)$ can always be obtained by integrating representations of the simpler structure E.

Theorem 4.1.4. *For every nondegenerate representation* π *of* $C^*(E)$ *on a Hilbert space* H *there is a representation* $\phi : E \to \mathcal{B}(H)$ *such that*

$$\pi(\ell_f \ell_g^*) = \phi(f)\phi(g)^*, \qquad f, g \in L^1(E), \tag{4.4}$$

where for $f \in L^1(G)$, $\phi(f)$ is defined as the integral (4.1).

COROLLARY 4.1.5. *For every nondegenerate representation $\pi : C^*(E) \to \mathcal{B}(H)$ there is a concrete product system $\mathcal{E} \subseteq (0,\infty) \times \mathcal{B}(H)$ that is naturally associated with π, and which is isomorphic to E.*

We emphasize that it is not at all apparent at this point that every concrete product system $\mathcal{E}$, that is isomorphic to E, can be obtained from a representation of $C^*(E)$ as in Corollary 4.1.5. Fortunately, this is true; it is a consequence of results on amenability in [**Arv03**]. Because of these amenability results, it is possible to realize spectral C^*-algebras in the concrete form given in Definition 4.1.3, rather than in the more intangible way they were originally defined in [**Arv90a**] using universal properties.

We now discuss a useful formula that expresses operators of the form $\ell_f \ell_g^*$ in terms of the rank-one operator $f \otimes \bar{g}$, as an absolutely convergent weak integral

$$\ell_f \ell_g^* = \int_0^\infty \beta_t(f \otimes \bar{g})\, dt \tag{4.5}$$

when $f, g \in L^1(E) \cap L^2(E)$. Notice that the left side involves *left* convolution operators, while the integral on the right involves the semigroup of endomorphisms β associated with *right* multiplication operators. Formula (4.5) is a consequence of the following more precise assertion:

PROPOSITION 4.1.6. *Consider the natural action β_* of the E-semigroup β on the Banach space $\mathcal{L}^1$ of all trace class operators on $L^2(E)$, defined for $t \geq 0$ by*

$$\operatorname{trace}(\beta_{*t}(A)B) = \operatorname{trace}(A\beta_t(B)), \qquad A \in \mathcal{L}^1, \quad B \in \mathcal{B}(L^2(E)).$$

Then for every $A \in \mathcal{L}^1$ and every pair of functions $f, g \in L^1(E) \cap L^2(E)$,

$$\int_0^\infty |\langle \beta_{t*}(A)f, g\rangle|\, dt \leq \operatorname{trace}|A| \cdot \|f\|_1 \|g\|_1,$$

and moreover

$$\operatorname{trace}(A\ell_f \ell_g^*) = \int_0^\infty \langle \beta_{*t}(A)f, g\rangle\, dt = \int_0^\infty \operatorname{trace}(A\beta_t(f \otimes \bar{g}))\, dt.$$

REMARK 4.1.7 ($C^*(E)$ versus $C^*(E^{\mathrm{op}})$). We have encountered the opposite product system E^{op} in Lecture 3, and we now relate it to the current discussion. The regular antirepresentation of E on $L^1(E)$ gives rise to a representation of E^{op} on $L^2(E)$; and in fact we can identify $L^2(E^{\mathrm{op}})$ with $L^2(E)$ in such a way that the regular representation of E^{op} on $L^2(E^{\mathrm{op}})$ is unitarily equivalent to the representation of E^{op} on $L^2(E)$ associated with right multiplications.

Thus, we can identify the spectral C^*-algebra of E^{op} with the following C^*-algebra associated with right convolution operators on $L^2(E)$

$$C^*(E^{\mathrm{op}}) = \overline{\operatorname{span}}\{r_f r_g^* : f, g \in L^1(E)\}.$$

Since this realizes both $C^*(E)$ and $C^*(E^{\mathrm{op}})$ on the same Hilbert space, we can look for concrete relations between them, and the most basic ones follow.

THEOREM 4.1.8. *Let E be a nontrivial product system and let $\mathcal{K}$ be the algebra of all compact operators on $L^2(E)$. Then $C^*(E)$ and $C^*(E^{\mathrm{op}})$ are irreducible C^*-algebras with the following properties:*

(i) $C^*(E) \cap \mathcal{K} = C^*(E^{op}) \cap \mathcal{K} = \{0\}$.

(ii) *For every* $A \in C^*(E)$, $B \in \mathcal{C}^*(E^{op})$ *and* $v \in E$,

$$[A, r_v] \in \mathcal{K}, \quad [B, \ell_v] \in \mathcal{K},$$

$[X, Y]$ *denoting the commutator bracket* $XY - YX$.

(iii) $[C^*(E), C^*(E^{op})] \subseteq \mathcal{K}$.

4.2. Infinitesimal Description of $C^*(E)$**.** Let E be a nontrivial product system, fixed throughout this section. Corollary 4.1.5 implies that, in order to study E_0-semigroups α such that $\mathcal{E}_\alpha \cong E$, one should look closely at the representatation theory of the spectral C^*-algebra $C^*(E)$. Since every representation of a C^*-algebra is a direct sum of cyclic representations and since cyclic representations are associated with positive linear functionals, we are led to examine the state space of $C^*(E)$ where by definition, a *state* of a C^*-algebra is a positive linear functional of norm 1. In the following sections we describe an analysis of the state space of $C^*(E)$, culminating in Theorems 4.4.2 and 4.4.6. The results are applied to establish the existence of E_0-semigroups whose product systems have arbitrary structure in Theorem 4.5.2 below.

In order to obtain an effective description of the state space of $C^*(E)$, we require an alternate description of this C^*-algebra in terms of the infinitesimal generator of the semigroup $\beta = \{\beta_t : t \geq 0\}$ of endomorphisms of $\mathcal{B}(L^2(E))$ associated with the *right* antirepresentation of E on $L^2(E)$, see Remark 4.1.1. In this section we examine the properties of this generator, its inverse, and their relation to the spectral C^*-algebra.

Let δ be the generator of β. Thus, the domain of δ is the set $\mathcal{D}$ of all operators $A \in \mathcal{B}(L^2(E))$ with the property that the limit

$$\delta(A) = \lim_{t \to 0+} t^{-1}(A - \beta_t(A)) \tag{4.6}$$

exists in the strong operator topology of $\mathcal{B}(L^2(E))$. This definition of $\delta(A)$ misses the usual definition up to a sign, but the definition above will be more convenient for our purposes here.

Since β is a semigroup of endomorhisms, $\mathcal{D}$ is a $*$-algebra of operators and δ is a derivation from $\mathcal{D}$ to $\mathcal{B}(H)$

$$\delta(AB) = \delta(A)B + A\delta(B), \qquad A, B \in \mathcal{D}. \tag{4.7}$$

Moreover, since the individual maps β_t are normal, $\mathcal{D}$ is strongly dense in $\mathcal{B}(H)$.

For every interval $I \subseteq (0, \infty)$ there is a corresponding subspace $L^2(I; E) \subseteq L^2(E)$, consisting of all square summable sections that vanish almost everywhere off I, and we write P_I for the projection onto $L^2(I; E)$. An operator $A \in \mathcal{B}(L^2(E))$ is said to be *supported in* I if $A = P_I A = AP_I$. An operator is said to have *bounded support* if there is a $t > 0$ such that it is supported in $(0, t]$. The set

$$\mathcal{B}_0 = \bigcup_{t>0} P_{(0,t]} \mathcal{B}(L^2(E)) P_{(0,t]}$$

of all operators of bounded support is obviously a $*$-subalgebra of $\mathcal{B}(L^2(E))$ which contains the $*$-algebra $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{B}_0$ of all compact operators of bounded support as a self adjoint two-sided ideal, and in particular it is strongly dense.

The following result asserts that $C^*(E)$ can be defined purely in terms of the generator of the semigroup of endomorphisms of $\mathcal{B}(L^2(E))$ associated with *right* multiplication operators. This fact is central to the description of the state space of $C^*(E)$ that we will describe below.

THEOREM 4.2.1. *Let $\mathcal{A}$ be the space of all operators A in the domain of δ such that $\delta(A)$ is a compact operator with bounded support. If $A, B \in \mathcal{A}$ are such that $\delta(A)$ is supported in $(0, a]$ and $\delta(B)$ is supported in $(0, b]$, then $\delta(AB)$ is supported in $(0, a+b]$.*

$\mathcal{A}$ is a $*$-algebra whose norm closure is $C^*(E)$.

4.3. Decreasing Weights. In this section we introduce a family of linear functionals defined on the $*$-algebra $\mathcal{A} \subseteq C^*(E)$ of Theorem 4.2.1, and we characterize those among them that are formally positive.

DEFINITION 4.3.1. A locally normal weight is a linear functional

$$\omega : \mathcal{B}_0 \to \mathbb{C}$$

defined on the local algebra $\mathcal{B}_0 \subseteq \mathcal{B}(L^2(E))$ with the property that for every $t > 0$ the restriction of ω to $P_{(0,t]}\mathcal{B}(L^2(E))P_{(0,t]}$ is a positive normal linear functional.

REMARK 4.3.2. For example, given a normal weight $\tilde{\omega} : \mathcal{B}(L^2(E))^+ \to [0, +\infty]$ with the property that $\tilde{\omega}(P_{(0,t]}) < \infty$ for every $t > 0$, the restriction of $\tilde{\omega}$ to the cone of positive operators in $\mathcal{B}_0$ extends uniquely to a linear functional on $\mathcal{B}_0$ that is a locally normal weight. On the other hand, there are locally normal weights on $\mathcal{B}_0$ that cannot be associated with normal weights of $\mathcal{B}(L^2(E))$ (a class of examples is constructed in Appendix A of [**Arv90b**]). Thus one should consider locally normal weights as somewhat more general than normal weights.

Notice that the semigroup $\beta = \{\beta_t : t \geq 0\}$ associated with the left antirepresentation of E on $L^2(E)$ acts naturally on $\mathcal{B}_0$; indeed, if B is a bounded operator supported in the interval $(0, b]$ then, for every $s \geq 0$, $\beta_s(B)$ is a bounded operator supported in the interval $(s, s+b] \subseteq (0, s+b]$.

DEFINITION 4.3.3. A locally normal weight ω is called decreasing if for every $B \in \mathcal{B}_0$ and $t \geq 0$ we have $\omega(\beta_t(B^*B)) \leq \omega(B^*B)$.

To maintain euphony, we refer to such objects simply as *decreasing weights.* The set of all decreasing weights is a cone of linear functionals on $\mathcal{B}_0$, and it is partially ordered by the relation $\omega_1 \leq \omega_2$ iff $\omega_2 - \omega_1$ is a decreasing weight.

Now let $\mathcal{A} \subseteq C^*(E)$ be the $*$-algebra of Theorem 4.2.1, let δ be the generator of β as defined in (4.6), and let ω be a locally normal weight. Since $\delta(\mathcal{A})$ is contained in the domain of ω we can define a linear functional $d\omega$ on $\mathcal{A}$ as follows

$$d\omega(A) = \omega(\delta(A)), \qquad A \in \mathcal{A}. \tag{4.8}$$

One may interpret $d\omega$ as the derivative of ω in the direction opposite to the flow of the semigroup of endomorphisms β. Typically, both ω and $d\omega$ are unbounded linear functionals on their respective domains $\mathcal{B}_0$ and $\mathcal{A}$. The following result identifies the linear functionals $d\omega$ that are formally positive on $\mathcal{A}$.

THEOREM 4.3.4. *Let ω be a locally normal weight and let $d\omega : \mathcal{A} \to \mathbb{C}$ be the linear functional of (4.8). The following are equivalent.*

(i) $d\omega(A^*A) \geq 0$ *for every* $A \in \mathcal{A}$.
(ii) ω *is decreasing.*

4.4. State Space of $C^*(E)$. Given that $C^*(E)$ is exhibited as the norm closure of the $*$-algebra $\mathcal{A}$ as in Theorem 4.2.1, we now give a concrete description of the cone of positive linear functionals on $C^*(E)$ in terms of the infinitesimal structure of $\mathcal{A}$ (Theorem 4.4.2), and we determine which positive linear functionals on $C^*(E)$ give rise to E_0-semigroups (Theorem 4.4.6).

Remark 4.4.1 (Growth of a Decreasing Weight). Let $\omega : \mathcal{B}_0 \to \mathbb{C}$ be a decreasing weight. For every nondegenerate bounded interval $I \subseteq (0,\infty)$ let $L^2(I;E)$ be the corresponding subspace of $L^2(E)$, consisting of square integrable sections that vanish almost everywhere off I, with projection $P_I : L^2(E) \to L^2(I;E)$. We will be concerned with decreasing weights that satisfy the growth condition

$$\sup_I \frac{\omega(P_I)}{|I|} < +\infty,$$

$|I|$ denoting the length of I, the supremum being taken over all bounded intervals $I \subseteq (0,\infty)$. Using the fact that ω is decreasing, it is not hard to show that the supremum can be restricted to intervals of the form $(0,\epsilon]$ where ϵ is arbitrarily small; consequently

$$\sup_{I\subseteq(0,\infty)} \frac{\omega(P_I)}{|I|} = \limsup_{t\to 0+} \frac{\omega(\mathbf{1}-\beta_t(\mathbf{1}))}{t}. \tag{4.9}$$

The common value of (4.9) is a number in $[0,+\infty]$, called the *growth* of ω.

After these preparations, one has the following description of the state space of $C^*(E)$ in terms of decreasing weights.

Theorem 4.4.2. *Let Ω be the partially ordered cone of all locally normal decreasing weights ω on $\mathcal{B}_0$ of finite growth. For every $\omega \in \Omega$, let $d\omega$ be the linear functional defined on $\mathcal{A}$ by*

$$d\omega(A) = \omega(\delta(A)), \qquad A \in \mathcal{A}.$$

Then $d\omega$ is bounded and extends to a positive linear functional on the norm closure $C^*(E)$ of $\mathcal{A}$. The map $\omega \mapsto d\omega$ defines an affine order isomorphism of Ω onto the cone of all positive linear functionals on $C^*(E)$, and one has

$$\limsup_{t\to 0+} \frac{\omega(\mathbf{1}-\beta_t(\mathbf{1})}{t} \le \|d\omega\| \le 4\cdot \limsup_{t\to 0+} \frac{\omega(\mathbf{1}-\beta_t(\mathbf{1})}{t}. \tag{4.10}$$

Remark 4.4.3 (Essential States). States of $C^*(E)$ give rise to semigroups of endomorphisms of $\mathcal{B}(H)$, and now we need to make that correspondence quite explicit. Let E be a product system, and let ρ be a positive linear functional on $C^*(E)$. We assert that there is a triple (ϕ,ξ,H) consisting of a representation ϕ of E on a Hilbert space H and a vector $\xi \in H$ with the following properties

$$\rho(\ell_f\ell_g^*) = \langle \phi(f)\phi(g)^*\xi,\xi\rangle, \tag{4.11}$$

$$H = \overline{span}\{\phi(f)\phi(g)^*\xi : f,g \in L^1(E)\}, \tag{4.12}$$

where ϕ is the associated representation of the Banach algebra $L^1(E)$

$$\phi(f) = \int_0^\infty \phi(f(t))\,dt, \qquad f \in L^1(E).$$

Indeed, the GNS construction gives rise to a triple (π,ξ,H) consisting of a representation π of $C^*(E)$ on a Hilbert spaces H and a cyclic vector $\xi \in H$, such that $\rho(A) = \langle \pi(A)\xi, \xi\rangle$ for A in $C^*(E)$; Theorem 4.1.4 provides a representation $\phi : E \to \mathcal{B}(H)$ such that $\pi(\ell_f \ell_g^*) = \phi(f)\phi(g)^*$ for $f, g \in L^1(E)$, and (4.11) and (4.12) follow. There is a uniqueness assertion that goes with such "GNS" triples (ϕ, ξ, H) for ρ, but we will not require that.

In particular, given such a triple (ϕ, ξ, H) for ρ, there is an associated E-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ by way of

$$\alpha_t(A) = \sum_{n=1}^{\infty} \phi(e_n(t))A\phi(e_n(t))^*, \qquad t \geq 0, \quad A \in \mathcal{B}(H)$$

where $\{e_1(t), e_2(t), \dots\}$ is an orthonormal basis for $E(t)$. α will be an E_0-semigroup iff $\alpha_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$. Since $\alpha_t(\mathbf{1})$ is the projection onto the subspace of H spanned by the ranges of the operators in $\phi(E(t))$, this will be the case iff

$$[\phi(E(t))H] = H, \qquad t \geq 0. \tag{4.13}$$

Definition 4.4.4. A positive linear functional ρ on $C^*(E)$ is called essential if the representation $\phi : E \to \mathcal{B}(H)$ associated with ρ satisfies (4.13), and therefore gives rise to an E_0-semigroup.

Remark 4.4.5 (Invariant Weights). Let $\omega : \mathcal{B}_0 \to \mathbb{C}$ be a locally normal weight that is invariant under β in the sense that $\omega(\beta_t(B)) = \omega(B)$, $t \geq 0$, $B \in \mathcal{B}_0$. It is obvious that ω is decreasing, and we assert that the growth (4.9) is finite. Indeed, we claim that when $\omega \neq 0$ there is a positive constant c such that

$$\omega(\mathbf{1} - \beta_t(\mathbf{1})) = ct, \qquad t \geq 0. \tag{4.14}$$

To see that, let P be the spectral measure defined on $[0, \infty)$ by the property

$$P([a, b]) = \beta_a(\mathbf{1}) - \beta_b(\mathbf{1}), \qquad 0 \leq a < b < +\infty.$$

Than we can define a positive measure μ on the Borel subsets of $[0, \infty)$ by $\mu(S) = \omega(P(S))$, $S \subseteq [0, \infty)$. μ is finite on compact sets and positive on some nondegenerate intervals because $\omega \neq 0$. Moreover, for every interval $I = [a, b] \subseteq [0, \infty)$ and every $t \geq 0$ we have

$$\mu(I + t) = \omega(P_{I+t}) = \omega(\beta_t(P_I)) = \omega(P_I) = \mu(I),$$

and it follows that $\mu(S+t) = \mu(S)$ for every Borel set S and $t \geq 0$. Such a measure must be a nonzero multiple of Lebesgue measure and (4.14) follows.

The key fact is that the states of $C^*(E)$ that give rise to E_0-semigroups are precisely the derivatives of β-invariant weights.

Theorem 4.4.6. *Let ω be a locally normal weight satisfying*

$$\omega(\beta_t(B)) = \omega(B), \qquad B \in \mathcal{B}_0, \quad t \geq 0.$$

Then ω belongs to Ω, and $d\omega$ is an essential positive linear functional on $C^*(E)$. Conversely, if $\omega \in \Omega$ is such that $d\omega$ is essential, then $\omega \circ \beta_t = \omega$ for every $t \geq 0$.

4.5. Existence of E_0-Semigroups. We now indicate how the results of the preceding discussion are applied to construct essential states of $C^*(E)$.

Proposition 4.5.1. *Let $\beta = \{\beta_t : t \geq 0\}$ be the semigroup of endomorphisms of $\mathcal{B}(L^2(E))$ associated with the antirepresentation of E on $L^2(E)$. There is a* *normal weight ω of $\mathcal{B}(L^2(E))$ with the property that $\omega(\beta_t(B)) = \omega(B)$ for every $t \geq 0$, $B \in \mathcal{B}(L^2(E))^+$, and which satisfies*

$$\omega(\mathbf{1} - \beta_t(\mathbf{1})) = t, \qquad t \geq 0. \tag{4.15}$$

Sketch of Proof. Consider the single endomorphism β_1. Since $\mathbf{1} - \beta_1(\mathbf{1})$ is a nonzero projection, we may choose a normal state ν_0 on $\mathcal{B}(L^2(E))$ such that $\nu_0(\mathbf{1} - \beta_1(\mathbf{1})) = 1$. Let V be any isometry satisfying

$$\beta_1(A)V = VA, \qquad A \in \mathcal{B}(L^2(E)), \tag{4.16}$$

and define a sequence of normal states $\nu_1, \nu_2, \dots$ on $\mathcal{B}(L^2(E))$ by

$$\nu_n(A) = \nu_0(V^{*n}AV^n), \qquad A \in \mathcal{B}(L^2(E)), \quad n = 1, 2, \dots.$$

Since ν_0 annihilates the projection $\beta_1(\mathbf{1})$ we have $\nu_0 \circ \beta_1 = 0$; and for $n \geq 1$ the commutation relation (4.16) implies $\nu_n \circ \beta_1 = \nu_{n-1}$. Hence

$$\nu = \sum_{n=0}^{\infty} \nu_n$$

defines a normal weight of $\mathcal{B}(L^2(E))$ satisfying $\nu \circ \beta_1 = \nu$. We can now define a normal weight ω on $\mathcal{B}(L^2(E))^+$ as follows:

$$\omega(A) = \int_0^1 \nu(\beta_s(A))\, ds, \qquad A \in \mathcal{B}(L^2(E))^+.$$

One finds that ω is invariant under the full semigrouop $\{\beta_t : t \geq 0\}$, and that $\omega(P_I) < \infty$ for every bounded interval $I \subseteq (0, \infty)$. Remark 4.4.5 implies that there is a positive constant c such that $\omega(\mathbf{1} - \beta_t(\mathbf{1})) = ct$ for $t > 0$, and that is obviously sufficient. □

The central result on the existence of E_0-semigroups now follows:

Theorem 4.5.2. *For every product system E, there is an E_0-semigroup α such that $\mathcal{E}_\alpha \cong E$.*

Proof. Proposition 4.5.1 implies that there is a locally normal weight ω on $\mathcal{B}_0$ satisfying

$$\omega \circ \beta_t = \omega, \quad \text{and} \quad \omega(\mathbf{1} - \beta_t(\mathbf{1})) = t, \qquad t \geq 0.$$

In particular, ω is a decreasing weight satisfying the growth requirement for membership in Ω, and Theorem 4.4.2 implies that $d\omega$ is a positive linear functional on $C^*(E)$. It must be essential by Theorem 4.4.6, so that the representation $\phi : E \to \mathcal{B}(H)$ associated with $d\omega$ as in Remark 4.4.3 gives rise to a concrete product system $\mathcal{E}$ that is isomorphic to E and which is the concrete product system of an E_0-semigroup. □

4.6. Simplicity. Let E be a product system. The one-parameter unitary group $\Gamma = \{\Gamma(\lambda) : \lambda \in \mathbb{R}\}$ defined on $L^2(E)$ by

$$\Gamma(\lambda)\xi\,(t) = e^{it\lambda}\xi(t), \qquad t > 0, \quad \xi \in L^2(E)$$

implements a one parameter group of $*$-automorphisms of the spectral C^*-algebra by way of

$$\gamma_\lambda(A) = \Gamma(\lambda)A\Gamma(\lambda)^*, \qquad A \in C^*(E), \quad \lambda \in \mathbb{R}.$$

Experience has led us to believe that $C^*(E)$ has no nontrivial closed two-sided ideals; however, the issue remains unresolved in general. What we do know is summarized in the following two results from [**Arv90a**].

Theorem 4.6.1. *Let E be an arbitrary product system and consider the one parameter group $\gamma = \{\gamma_\lambda : \lambda \in \mathbb{R}\}$ of gauge automorphisms of $C^*(E)$. Then $C^*(E)$ is γ-simple in the sense that the only closed γ-invariant ideals in $C^*(E)$ are the trivial ones $\{0\}$ and $C^*(E)$.*

Theorem 4.6.2. *Let E be a product system that is not of type III. Then $C^*(E)$ is a simple C^*-algebra.*

The following problem remains:

Problem: Is the spectral C^*-algebra of a type III product system simple?

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