SEVERAL PROBLEMS IN OPERATOR THEORY

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1. Introduction

We discuss some problems and conjectures in higher dimensional operator theory. These all have something to do with the basic problem of developing an effective Fredholm theory of d-contractions, and completing the index theorem that was partially established in Theorem B of [Arv02], following up on [Arv00]. My opinion is that good progress on any one of these problems will be a significant advance, if not a breakthrough.

Most of what follows is an exposition of the theory of Dirac operators, Fredholmness, and index from scratch, in a form accessible to anyone with a good basic knowledge of operators on Hilbert spaces. The conjectures and problems will be found in Section 4.

2. Dirac Operators in dimension d

Let $\bar{T} = (T_1, \dots, T_d)$ be a multioperator of complex dimension d, that is to say, a d-tuple of mutually commuting bounded operators acting on a common Hilbert space H. All geometric properties of $\bar{T}$ are reflected in properties of its associated Dirac operator, and we begin by recalling the basic facts about Dirac operators from [Arv02].

The Dirac operator associated with a d-dimensional multioperator $\bar{T}$ is constructed as follows. Let Z be a complex Hilbert space of dimension d, which of course we may take as $\mathbb{C}^d$. The exterior algebra over Z is defined as the direct sum of Hilbert spaces

$$\Lambda Z = \Lambda^0 Z \oplus \Lambda^1 Z \oplus \cdots \oplus \Lambda^d Z,$$

where $\Lambda^k Z$ denotes the kth exterior power of Z. By definition, $\Lambda^0 Z = \mathbb{C}$, and the last summand $\Lambda^d Z$ is also isomorphic to $\mathbb{C}$. ΛZ is a 2^d-dimensional Hilbert space, and $\Lambda^k Z$ is spanned by vectors of the form $z_1 \wedge z_2 \wedge \cdots \wedge z_k$, where the inner product on $\Lambda^k Z$ is determined by

$$\langle z_1 \wedge \cdots \wedge z_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle z_i, w_j \rangle)$$

the right side denoting the determinant of the $k \times k$ matrix of inner products $\langle z_i, w_j \rangle$. For each $z \in Z$, there is a *creation operator* $C(z)$ that maps $\Lambda^k Z$ to $\Lambda^{k+1} Z$, and acts on generators as follows

$$C(z) x_1 \wedge \cdots \wedge x_k = z \wedge x_1 \wedge \cdots \wedge x_k.$$

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C becomes a linear map of Z into $\mathcal{B}(\Lambda Z)$ that gives rise to an irreducible representation of the canonical anticommutation relations; more explicitly, $C(Z)$ is an irreducible set of operators such that for all $z, w \in Z$ one has

$$C(z)C(w) + C(w)C(z) = 0, \quad C(z)C(w)^* + C(w)^*C(z) = \langle z, w\rangle \mathbf{1}.$$

In a less coordinate-free form, one can choose an orthonormal basis $e_1, \dots, e_d$ for Z, set $c_k = C(e_k)$ for $1 \le k \le d$, and one finds that

$$c_i c_j + c_j c_i = 0, \quad c_i c_j^* + c_j^* c_i = \delta_{ij}\mathbf{1}.$$

An elementary discussion of these facts can be found in [Arv01].

Given a d-dimensional multioperator $\bar{T}$ acting on a Hilbert space H, we can form the somewhat larger Hilbert space

$$\tilde{H} = H \otimes \Lambda Z = (H \otimes \Lambda^0 Z) \oplus \cdots \oplus (H \otimes \Lambda^d),$$

and the operator B defined on it by

$$B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d,$$

where $c_k = C(e_k)$ is as above. The fact is that B does not depend on the choice of basis in the sense that an operator B' associated with a different basis $e_1', \dots, e_d'$ for Z must be naturally isomorphic to B (see [Arv02] for more detail). For purposes of these notes, we define the *Dirac operator* of $\bar{T}$ to be the bounded self-adjoint operator $D = B + B^*$. When it is necessary to refer to the underlying multioperator in the notation for Dirac operators we will do that by writing $D_{\bar{T}}$ rather than D.

We point out that this definition of Dirac operator is less comprehensive than the one in [Arv02], since we have not been careful to state the precise relations that exist between B (and therefore D) and the the anticommuting operators $\mathbf{1}_H \otimes c_1, \dots, \mathbf{1}_H \otimes c_d$. This is discussed in Section 2 of [Arv02]. But for our purposes here, the shorter definition given above will suffice. At the same time, one should keep in mind that with a more comprehensive definition of Dirac operator, the theory of d-dimensional multioperators becomes identical with the theory of d-dimensional Dirac operators in the sense that every abstract Dirac operator is associated with a multioperator, and two multioperators $\bar{S}$ and $\bar{T}$ are unitarily equivalent iff their Dirac operators $D_{\bar{S}}$ and $D_{\bar{T}}$ are isomorphic (see Theorem A of [Arv02]).

3. Fredholm Multioperators and index

A multioperator $\bar{T} = (T_1, \dots, T_d)$ is said to be Fredholm if its Dirac operator D is a Fredholm operator. Since D is self-adjoint, this simply means that the range of D is closed and its kernel is finite-dimensional. The index of a self-adjoint Fredholm operator is of course zero; but in this case there is additional structure available that allows one to make a more subtle definition of index that will be essential in what follows. We now discuss these issues.

Suppose that the operators $T_1, \dots, T_d$ act on a Hilbert space H, so that D acts on $\tilde{H} = H \otimes \Lambda Z$. There is a natural $\mathbb{Z}_2$-grading of $\tilde{H}$ that makes

D into an operator of odd degree, and which is defined as follows. Consider the two subspaces of $\tilde{H}$ defined by

$$\tilde{H}_+ = \sum_{k \text{ even}} H \otimes \Lambda^k Z, \qquad \tilde{H}_- = \sum_{k \text{ odd}} H \otimes \Lambda^k Z.$$

These spaces are mutually orthogonal and sum to $\tilde{H}$, hence they provide $\tilde{H}$ with a $\mathbb{Z}_2$-grading. Elements of $\tilde{H}_+$ (resp. $\tilde{H}_-$) are called even vectors (resp. odd vectors). Since both B and its adjoint map even vectors to odd ones and odd ones to even ones, it follows that D does the same, and in particular the restriction D_+ of D to $\tilde{H}_+$ defines a bounded operator from $\tilde{H}_+$ to $\tilde{H}_-$. The orthogonal decomposition $\tilde{H} = \tilde{H}_+ \oplus \tilde{H}_-$ gives rise to a 2×2 matrix representation

$$D = \begin{pmatrix} 0 & D_+^* \\ D_+ & 0 \end{pmatrix}.$$

It is an instructive exercise to show when D is a Fredholm operator, the range of D_+ is a closed subspace of $\tilde{H}_-$ of finite codimension, so that D_+ *can be regarded as a Fredholm operator from* $\tilde{H}_+$ *to* $\tilde{H}_-$. We can now define the *index* of D as follows

$$\operatorname{ind} D = \dim\ker D_+ - \dim\operatorname{coker} D_+ = \dim(\tilde{H}_+ \cap \ker D) - \dim(H_-/DH_+).$$

The index is an important integer invariant for Fredholm multioperators. By analogy with the work of Atiyah and Singer, one might hope that the computation of the index of Fredholm multioperators will lead to important relations between the geometric and analytic properties of multioperators, and perhaps even feed back into basic issues of algebraic geometry. Thus, one would like to have sharp tools for a) determining when a given Dirac operator is Fredholm and b) computing the index in terms of concrete geometric properties of its underlying multioperator.

Some progress has been made in the direction of b), and we will describe that below. However, the problem a) of proving that natural examples of multioperators are Fredholm remains largely open. It is the latter problem that we want to discuss here in more detail.

Here is a simple and useful sufficient condition.

Proposition 3.1. *Let* $\bar{T} = (T_1, \dots, T_d)$ *be a multioperator with the following two properties:*

(i) $\bar{T}$ *is essentially normal in that all self-commutators* $T_k T_j^* - T_j^* T_k$ *are compact,* $1 \le j, k \le d$.

(ii) $T_1 T_1^* + \cdots + T_d T_d^*$ *is a Fredholm operator.*

Then the Dirac operator of $\bar{T}$ *is Fredholm.*

Proof. Since $D = D^*$, it suffices to show that D^2 is a Fredholm operator. To that end, consider $B = T_1 \otimes c_1 + \cdots + T_d \otimes c_d$. Since T_j commutes with T_k and c_j anticommutes with c_k, a straightforward computation shows that

$B^2 = 0$. Hence

$$D^2 = (B + B^*)^2 = B^*B + BB^* = \sum_{k,j=1}^{d} T_k^* T_j \otimes c_k^* c_j + \sum_{k,j=1}^{d} T_j T_k^* \otimes c_j c_k^*.$$

Using the anticommutation relations $c_j c_k^* = \delta_{jk}\mathbf{1} - c_k^* c_i$, we can write the second term on the right as

$$A \otimes \mathbf{1} - \sum_{k,j=1}^{d} T_j T_k^* \otimes c_k^* c_j,$$

where $A = T_1 T_1^* + \cdots + T_d T_d^*$, so that

$$D^2 = A \otimes \mathbf{1} + \sum_{k,j=1}^{d} (T_k^* T_j - T_j T_k^*) \otimes c_k^* c_j.$$

Since $A \otimes \mathbf{1}$ is a Fredholm operator by (ii) and each summand in the second term is compact by (i), it follows that D^2 is invertible modulo compacts. □

Remark 3.2 (Commutativity vs. Noncommuativity). It has become fashionable these days to look in the most noncommutative directions for results that generalize previously known commutative ones. The reader will note, however, that we have used commutativity in an essential way in the preceding proof to conclude that $B^2 = 0$. In turn, the fact that $B^2 = 0$ is critical for developing the homology and cohomology of multioperators; indeed, B is nothing other than the coboundary operator for the Koszul complex of $\bar{T}$. If one gives up commutativity of the operators $T_1, \dots, T_d$ then one must also give up the fact that $H \otimes \Lambda Z$ is a complex.

4. Finite Rank d-contractions

Let d be a positive integer. A d-contraction is a multioperator $\bar{T} = (T_1, \dots, T_d)$ acting on a Hilbert space H that defines a row contraction in the sense that

$$T_1 T_1^* + \cdots + T_d T_d^* \leq \mathbf{1}.$$

The *rank* of $\bar{T}$ is the rank of the defect operator $\mathbf{1} - T_1 T_1^* - \cdots - T_d T_d^*$, and $\bar{T}$ is said to be *pure* if the completely positive operator map defined by

$$\phi(X) = T_1 X T_1^* + \cdots + T_d X T_d^*, \qquad X \in \mathcal{B}(H)$$

satisfies $\phi^n(\mathbf{1}) \downarrow 0$ as $n \to \infty$.

A principal theme of [Arv98], [Arv00], and [Arv02] has been to understand the nature of pure finite rank d-contractions. In particular, we have focused on the following three related questions. When is such a $\bar{T}$ Fredholm? How does one compute the index of $D_{\bar{T}}$? How is the index related to concrete geometric properties of $\bar{T}$? The papers [Arv00] and [Arv02] give various partial answers to the second two of these three questions. However, they do not address the first, and that is the question we want to discuss here: Which pure finite rank d-contractions are Fredholm? While it is conceivable that they *all* are, approaching the question in that generality appears today to be well out of reach. Instead, we fix attention on certain sub-questions which seem accessible, which have a genuine connection with algebraic geometry, and for which there is evidence that the answers are positive.

In order to discuss these issues, we first point out that finite rank pure d-contractions can all be realized as compressions of finite-multiplicity d-shifts to quotients (see [Arv98]). This is a significant reduction, since the properties of such compressions can be approached in very concrete terms. We now describe these examples of d-contractions in more detail.

Let Z be a d-dimensional Hilbert space. The symmetric Fock space over Z will be denoted H^2_d, since it can be identified with the completion of the algebra of polynomials $\mathbb{C}[z_1, \dots, z_d]$ with respect to a natural inner product. Choosing an orthonormal basis $e_1, \dots, e_d$ for Z, one can form a row contraction $\bar{S} = (S_1, \dots, S_d)$ of operators on H^2_d, where S_k denotes symmetric tensoring by e_k; equivalently, S_k can be interpreted as the operator that multiplies polynomials by the kth coordinate variable z_k. Let r be a positive integer and let $r \cdot H^2_d$ be the direct sum of r copies of H^2_d. There is a natural way to define an action of $S_1, \dots, S_d$ on $r \cdot H^2_d$ simply by increasing the multiplicity by a factor of r. We refer to that universal d-tuple as the *d-shift of multiplicity r*.

Given any closed invariant subspace $M \subseteq r \cdot H^2_d$ for the latter, one can form a quotient Hilbert space $H = (r \cdot H^2_d)/M$ and a d-contraction $\bar{T}$ on H by compressing the action of the d-shift or multiplicity r to the quotient $\mathbb{C}[z_1, \dots, z_d]$-module $(r \cdot H^2_d)/M$. It is not hard to see that such a multioperator $\bar{T}$ is a pure d-contraction of rank at most r; indeed, the rank will be r except in the presence of simple degeneracies. We are interested in determining whether such multioperators $\bar{T}$ are Fredholm, for various classes of invariant subspaces M.

The simplest versions of this general problem connect with algebraic geometry, and are described as follows. Let r be a positive integer. We may consider r-tuples of polynomials as elements of $r \cdot H^2_d$, and in an obvious sense such elements are vector-valued polynomials. Choose a finite set $\mathbf{p}_1, \dots, \mathbf{p}_s$ of vector-valued polynomials in $r \cdot \mathbb{C}[z_1, \dots, z_d]$, such that each $\mathbf{p}_i$ is homogeneous of some degree n_i - i.e., each of the r components of $\mathbf{p}_i$ is a homogeneous polynomial of the same degree n_i. Let

$$M = [f_1\mathbf{p}_1 + \cdots + f_s\mathbf{p}_s : f_1, \dots, f_s \in \mathbb{C}[z_1, \dots, z_d]]$$

be the invariant subspace of $r \cdot H^2_d$ generated by $\mathbf{p}_1, \dots, \mathbf{p}_s$.

Conjecture 1. Let $H = (r \cdot H^2_d)/M$ be the quotient Hilbert space, and let $\bar{T} = (T_1, \dots, T_d)$ be the d-contraction defined on H by compressing the multiplicity r d-shift. Then the self-commutators $T_jT_k^* - T_k^*T_j$, $1 \leq j, k \leq d$, belong to the Schatten-von Neumann class $\mathcal{L}^p$ for every $p > d$.

Remark 4.1 (Consequences of Conjecture 1). Assuming that Conjecture 1 is true, it follows that the Dirac operators of such multioperators $\bar{T}$ are all

Fredholm. Indeed, since $T_1T_1^* + \cdots + T_dT_d^* = \mathbf{1} - F$ where the defect operator F is positive of rank at most r, and since all self-commutators are compact, Proposition 3.1 applies in a straightforward manner.

In turn, the fact that $D_{\bar T}$ is Fredholm can be applied to strengthen the index theorem proved in [Arv02] as follows. In [Arv00] we introduced a curvature invariant $K(\bar T)$ that can be associated with any pure finite-rank d-contraction $\bar T$, and showed that it occupies a position analogous to the average Gaussian curvature of a compact oriented Riemannian manifold. Thanks in part to work of Greene, Richter and Sundberg [GRS02], it is now known that $K(\bar T)$ is always an integer. But what does that integer represent? It had been shown earlier in [Arv00] that $K(\bar T)$ frequently coincides with the Euler characteristic of a certain finitely generated algebraic module associated with $\bar T$, but it was also known that there are examples for which $K(\bar T)$ is *not* the Euler characteristic of that module.

In [Arv02] a new approach was initiated, in which one sought to relate the value of $K(\bar T)$ to the index of a Dirac operator. The results apply to *graded* d-contractions - essentially, those associated with Conjecture 1. The key formula is Theorem B of [Arv02], which makes the following assertion about the d-contractions $\bar T$ of Conjecture 1 and their associated Dirac operators D: Both $\ker D_+$ and $\ker D_+^*$ are finite dimensional, and

$$(-1)^d K(\bar T) = \dim\ker D_+ - \dim\ker D_+^*. \tag{4.1}$$

Unlike the earlier result that related $K(\bar T)$ to an Euler characteristic, *there are no known exceptions to this formula*, even for nongraded d-contractions.

However, note that Theorem B makes no assertion about whether or not the range of D_+ is closed, and if the range of D_+ is not closed then the right side of (4.1) is unstable. But if Conjecture 1 is true, then the range of D_+ must be closed and the index formula (4.1) appears as a conventional assertion about the index of a Fredholm operator. Such a strengthening of Theorem B can now be seen as a full counterpart of the Gauss-Bonnet-Chern formula in its modern dress as an index theorem [GM91].

There are other consequences as well, relating to the stability of the curvature invariant under compact perturbations and homotopy, that we will not describe here. See Section 4 of [Arv02].

Remark 4.2 (Evidence for the truth of Conjecture 1). All concrete examples, for which the truth of Conjecture 1 can be decided, support it. A variety of such examples is described in [Arv02]. In this remark we describe another family of examples that also support Conjecture 1. These results are unpublished, and we give no details beyond pointing out the existence of such examples. Though this class of examples has little geometric interest (their associated algebraic varieties are trivial), the conclusion is significant from the point of view of operator theory. A *monomial* in $\mathbb{C}[z_1, \dots, z_d]$ is a homogeneous polynomial of the particular form

$$p(z_1, \dots, z_d) = z_1^{k_1} z_2^{k_2} \cdots z_d^{k_d}$$

where $k_1, \dots, k_d$ are nonnegative integers. A vector-valued monomial is an element $\mathbf{p}$ of $r\cdot\mathbb{C}[z_1, \dots, z_k]$ of the form $(p_1, p_2, \dots, p_r)$ where each component p_i is a scalar multiple of the *same* monomial $z_1^{k_1} \cdots z_d^{k_d}$. The positive result that we want to report is that Conjecture 1 is true provided that each of the generators $\mathbf{p}_1, \dots, \mathbf{p}_s$ of M is a vector-valued monomial.

Moving on to more general conjectures, we propose:

Conjecture 2. Let $\mathbf{p}_1, \dots, \mathbf{p}_s$ be a finite set of vector polynomials in $r \cdot \mathbb{C}[z_1, \dots, z_d]$, not necessarily homogeneous, let M be the closed invariant subspace of $r \cdot H^2_d$ that they generate, and let $\bar{T}$ be the compression of the multiplicity r shift to the quotient Hilbert space $(r \cdot H^2_d)/M$. Then the self-commutators $T_j T_k^* - T_k^* T_j$ are all compact.

Assuming that this conjecture is true, the argument given above shows that all such compressed d-tuples $\bar{T}$ are Fredholm. Such examples are not "graded" as were the ones discussed in the context of Conjecture 1, and we do not know if the index formula (4.1) holds for ungraded Hilbert modules. But there is enough evidence to lead us to:

Conjecture 3. Assuming that Conjecture 2 is true, then the index formula (4.1) holds in that setting as well.

Finally, let us skip directly to the most general question, about which we have the least evidence. It is conceivable (perhaps even likely) that for some invariant subspaces $M \subseteq H^2_d$, the compression of the d-shift to H^2_d/M does *not* have compact self-commutators. While this would be unfortunate, the index theory can be salvaged if the answer to the following question is yes.

Problem 4. Is the Dirac operator of every pure finite-rank d-contraction a Fredholm operator?

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