GENERATORS OF NONCOMMUTATIVE DYNAMICS

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ABSTRACT. For a fixed C^*-algebra A, we consider all noncommutative dynamical systems that can be generated by A. More precisely, an A-dynamical system is a triple (i, B, α) where α is a $*$-endomorphism of a C^*-algebra B, and $i : A \subseteq B$ is the inclusion of A as a C^*-subalgebra with the property that B is generated by $A \cup \alpha(A) \cup \alpha^2(A) \cup \cdots$. There is a natural hierarchy in the class of A-dynamical systems, and there is a universal one that dominates all others, denoted $(i, \mathcal{P}A, \alpha)$. We establish certain properties of $(i, \mathcal{P}A, \alpha)$ and give applications to some concrete issues of noncommutative dynamics.

For example, we show that every contractive completely positive linear map $\varphi : A \to A$ gives rise to to a unique A-dynamical system (i, B, α) that is "minimal" with respect to φ, and we show that its C^*-algebra B can be embedded in the multiplier algebra of $A \otimes \mathcal{K}$.

1. Generators

The flow of time in quantum theory is represented by a one-parameter group of $*$-automorphisms $\{\alpha_t : t \in \mathbb{R}\}$ of a C^*-algebra B. There is often a C^*-subalgebra $A \subseteq B$ that can be singled out from physical considerations which, together with its time translates, generates B. For example, in nonrelativistic quantum mechanics the flow of time is represented by a one-parameter group of automorphisms of $\mathcal{B}(H)$, and the set of all bounded continuous functions of the configuration observables at time 0 is a commutative C^*-algebra A. The set of all time translates $\alpha_t(A)$ of A generates an irreducible C^*-subalgebra B of $\mathcal{B}(H)$. In particular, for different times $t_1 \neq t_2$, the C^*-algebras $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ do not commute with each other. Indeed, no nontrivial relations appear to exist between $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ when $t_1 \neq t_2$.

In this paper we look closely at this phenomenon, in a simpler but analogous setting. Let A be a C^*-algebra, fixed throught.

Definition 1.1. An A-dynamical system is a triple (i, B, α) consisting of a $*$-endomorphism α acting on a C^*-algebra B and an injective $*$-homomorphism $i : A \to B$, such that B is generated by $i(A) \cup \alpha(i(A)) \cup \alpha^2(i(A)) \cup \cdots$.

We lighten notation by identifying A with its image $i(A)$ in B, thereby replacing i with the inclusion map $i : A \subseteq B$. Thus, an A-dynamical system is a dynamical system (B, α) that contains A as a C^*-subalgebra in

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a specified way, with the property that B is the norm-closed linear span of finite products of the following form

$$B = \overline{\text{span}}\{\alpha^{n_1}(a_1)\alpha^{n_2}(a_2)\cdots\alpha^{n_k}(a_k)\} \tag{1}$$

where $n_1, \dots, n_k \geq 0$, $a_1, \dots, a_k \in A$, $k = 1, 2, \dots$

Our aim is to say something sensible about the class of *all* A-dynamical systems, and to obtain more detailed information about certain of its members. The opening paragraph illustrates the fact that in even the simplest cases, where A is $C(X)$ or even a matrix algebra, the structure of individual A-dynamical systems can be very complex.

There is a natural hierarchy in the class of all A-dynamical systems, defined by $(i_1, B_1, \alpha_1) \geq (i_2, B_2, \alpha_2)$ iff there is a $*$-homomorphism $\theta : B_1 \to B_2$ satisfying $\theta \circ \alpha_1 = \alpha_2 \circ \theta$ and $\theta(a) = a$ for $a \in A$. Since θ fixes A, it follows from (1) that θ must be surjective, $\theta(B_1) = B_2$, hence (i_2, B_2, α_2) is a *quotient* of (i_1, B_1, α_1). Two A-dynamical systems are said to be *equivalent* if there is a map θ as above that is an isomorphism of C^*-algebras. This will be the case iff each of the A-dynamical systems dominates the other. One may also think of the class of all A-dynamical systems as a category, whose objects are A-dynamical systems and whose maps θ are described above.

There is a largest equivalence class in this hierarchy, whose representatives are called *universal* A-dynamical systems. We exhibit one as follows. Consider the free product of an infinite sequence of copies of A,

$$\mathcal{P}A = A * A * \cdots .$$

Thus, we have a sequence of $*$-homomorhisms $\theta_0, \theta_1, \dots$ of A into the C^*-algebra $\mathcal{P}A$ such $\mathcal{P}A$ is generated by $\theta_0(A) \cup \theta_1(A) \cup \cdots$ and such that the following universal property is satisfied: for every sequence $\pi_0, \pi_1, \dots$ of $*$-homomorphisms of A into some other C^*-algebra B, there is a unique $*$-homomorphism $\rho : \mathcal{P}A \to B$ such that $\pi_k = \rho \circ \theta_k$, $k = 0, 1, \dots$. Nondegenerate representations of $\mathcal{P}A$ correspond to sequences $\bar\pi = (\pi_0, \pi_1, \dots)$ of representations $\pi_k : A \to \mathcal{B}(H)$ of A on a common Hilbert space H, subject to no condition other than the triviality of their common nullspace

$$\xi \in H, \quad \pi_k(A)\xi = \{0\}, \quad k = 0, 1, \cdots \implies \xi = 0.$$

A simple argument establishes the existence of $\mathcal{P}A$ by taking the direct sum of a sufficiently large set of such representation sequences $\bar\pi$.

This definition does not exhibit $\mathcal{P}A$ in concrete terms (see §3 for that), but it does allow us to define a universal A-dynamical system. The universal property of $\mathcal{P}A$ implies that there is a shift endomorphism $\sigma : \mathcal{P}A \to \mathcal{P}A$ defined uniquely by $\sigma \circ \theta_k = \theta_{k+1}$, $k = 0, 1, \dots$. It is quite easy to verify that θ_0 is an injective $*$-homomorphism of A in $\mathcal{P}A$, and we use this map to identify A with $\theta_0(A) \subseteq \mathcal{P}A$. Thus *the triple* $(i, \mathcal{P}A, \sigma)$ *becomes an* A-*dynamical system with the property that every other* A-*dynamical system is subordinate to it.*

Before introducing α-expectations, we review some common terminology **[Ped79]**. Let $A \subseteq B$ be an inclusion of C^*-algebras. For any subset S of B

we write $[S]$ for the norm-closed linear span of S. The subalgebra A is said to be *essential* if the two-sided ideal $[BAB]$ it generates is an essential ideal

$$x \in B, \quad xBAB = \{0\} \implies x = 0.$$

It is called *hereditary* if for $a \in A$ and $b \in B$, one has

$$0 \leq b \leq a \implies b \in A.$$

The hereditary subalgebra of B generated by a subalgebra A is the closed linear span $[ABA]$ of all products axb, $a, b \in A$, $x \in B$, and in general $A \subseteq [ABA]$. A *corner* of B is a hereditary subalgebra of the particular form $A = pBp$ where p is a projection in the multiplier algebra $M(B)$ of B.

We also make essential use of *conditional expectations* $E : B \to A$. A conditional expectation is an idempotent positive linear map with range A, satisfying $E(ax) = aE(x)$ for $a \in A$, $x \in B$. When $A = pBp$ is a *corner* of B, the map $E(x) = pxp$, defines a conditional expectation of B onto A. On the other hand, many of the conditional expectations encountered here do not have this simple form, even when A has a unit. Indeed, if A is subalgebra of B that is *not* hereditary, then there is no natural conditional expectation $E : B \to A$. In general, conditional expectations are completely positive linear maps with $\|E\| = 1$.

Definition 1.2. Let (i, B, α) be an A-dynamical system. An α-expectation is a conditional expectation $E : B \to A$ having the following two properties:

- E1. Equivariance: $E \circ \alpha = E \circ \alpha \circ E$.
- E2. The restriction of E to the hereditary subalgebra generated by A is multiplicative, $E(xy) = E(x)E(y)$, $x, y \in [ABA]$.

Note that an *arbitrary* conditional expectation $E : B \to A$ gives rise to a linear map $\varphi : A \to A$ by way of $\varphi(a) = E(\alpha(a))$, $a \in A$. Such a φ is a completely positive map satisfying $\|\varphi\| \leq 1$. Axiom E1 makes the assertion

$$E \circ \alpha = \varphi \circ E. \tag{2}$$

where $\varphi = E \circ \alpha \restriction_A$ is the linear map of A associated with E.

Property E2 is of course automatic if A is a hereditary subalgebra of B. It is a fundamentally *noncommutative* hypothesis on B. For example, if Y is a compact Hausdorff space and $B = C(Y)$, then every unital subalgebra $A \subseteq C(Y)$ generates $C(Y)$ as a hereditary algebra. Thus the only linear maps $E : C(Y) \to A$ satisfying E2 are $*$-endomorphisms of $C(Y)$. The key property of the universal A-dynamical system $(i, \mathcal{P}A, \sigma)$ follows.

Theorem 1.3. *For every completely positive contraction $\varphi : A \to A$, there is a unique σ-expectation $E : \mathcal{P}A \to A$ satisfying*

$$\varphi(a) = E(\sigma(a)), \qquad a \in A. \tag{3}$$

Both assertions are nontrivial. We prove uniqueness in the following section, see Theorem 2.3. Existence is taken up in §3, see Theorem 3.2.

2. MOMENT POLYNOMIALS

This theory of generators rests on properties of certain noncommutative polynomials that are defined recursively as follows.

Proposition 2.1. *Let A be an algebra over a field $\mathbb{F}$. For every linear map $\varphi : A \to A$, there is a unique sequence of multilinear mappings from A to itself, indexed by the k-tuples of nonnegative integers, $k = 1, 2, \dots,$ where for a fixed k-tuple $\bar{n} = (n_1, \dots, n_k)$*

$$a_1, \dots, a_k \in A \mapsto [\bar{n}; a_1, \dots, a_k] \in A$$

is a k-linear mapping, all of which satisfy

MP1. $\varphi([\bar{n}; a_1, \dots, a_k]) = [n_1 + 1, n_2 + 1, \dots, n_k + 1; a_1, \dots, a_k]$.
MP2. *Given a k-tuple for which $n_\ell = 0$ for some ℓ between 1 and k,*

$$[\bar{n}; a_1, \dots, a_k] = [n_1, \dots, n_{\ell-1}; a_1, \dots, a_{\ell-1}] a_\ell [n_{\ell+1}, \dots, n_k; a_{\ell+1}, \dots, a_k].$$

Remark 2.2. The proofs of both existence and uniqueness are straightforward arguments using induction on the number k of variables, and we omit them. Note that in the second axiom MP2, we make the natural conventions when ℓ has one of the extreme values $1, k$. For example, if $\ell = 1$, then MP2 should be interpreted as

$$[0, n_2, \dots, n_k; a_1, \dots, a_k] = a_1 [n_2, \dots, n_k; a_2, \dots, a_k].$$

In particular, in the linear case $k = 1$, MP2 makes the assertion

$$[0; a] = a, \qquad a \in A;$$

and after repeated applications of axiom MP1 one obtains

$$[n; a] = \varphi^n(a), \qquad a \in A, \quad n = 0, 1, \dots.$$

One may calculate any particular moment polynomial explicitly, but the computations quickly become a tedious exercise in the arrangement of parentheses. For example,

$$[2, 6, 3, 4; a, b, c, d] = \varphi^2(a\varphi(\varphi^3(b)c\varphi(d))),$$
$$[6, 4, 2, 3; a, b, c, d] = \varphi^2(\varphi^2(\varphi^2(a)b)c\varphi(d)).$$

Finally, we remark that when A is a C^*-algebra and $\varphi : A \to A$ is a linear map satisfying $\varphi(a)^* = \varphi(a^*)$, $a \in A$, then its associated moment polynomials obey the following symmetry

$$[n_1, \dots, n_k; a_1, \dots, a_k]^* = [n_k, \dots, n_1; a_k^*, \dots, a_1^*]. \tag{4}$$

Indeed, one finds that the sequence of polynomials $[[\cdot;\cdot]]$ defined by

$$[[n_1, \dots, n_k; a_1, \dots, a_k]] = [n_k, \dots, n_1; a_k^*, \dots, a_1^*]^*$$

also satisfies axioms MP1 and MP2, and hence must coincide with the moment polynomials of φ by the uniqueness assertion of Proposition 2.1.

These polynomials are important because they are the expectation values of certain A-dynamical systems.

Theorem 2.3. *Let $\varphi : A \to A$ be a completely positive map on A, satisfying $\|\varphi\| \leq 1$, with associated moment polynomials $[n_1, \dots, n_k; a_1, \dots, a_k]$.*

Let (i, B, α) be an A-dynamical system and let $E : B \to A$ be an α-expectation with the property $E(\alpha(a)) = \varphi(a)$, $a \in A$. Then

$$E(\alpha^{n_1}(a_1)\alpha^{n_2}(a_2)\cdots\alpha^{n_k}(a_k)) = [n_1, \dots, n_k; a_1, \dots, a_k]. \tag{5}$$

for every $k = 1, 2, \dots$, $n_k \geq 0$, $a_k \in A$. In particular, there is at most one α-expectation $E : B \to A$ satisfying $E(\alpha(a)) = \varphi(a)$, $a \in A$.

Proof. One applies the uniqueness of moment polynomials as follows. Properties E1 and E2 of Definition 1.2 imply that the sequence of polynomials $[[\cdot;\cdot]]$ defined by

$$[[n_1, \dots, n_k; a_1, \dots, a_k]] = E(\alpha^{n_1}(a_1)\cdots\alpha^{n_k}(a_k))$$

must satisfy the two axioms MP1 and MP2. Notice here that E2 implies

$$E(xay) = E(x)aE(y) \qquad x, y \in B, \quad a \in A, \tag{6}$$

since for an approximate unit e_n for A we can write $E(xay)$ as follows:

$$\begin{aligned}\lim_{n\to\infty} e_n E(xae_n y)e_n &= \lim_{n\to\infty} E(e_n xae_n ye_n) = \lim_{n\to\infty} E(e_n xa)E(e_n ye_n)\\ &= \lim_{n\to\infty} e_n E(x)ae_n E(y)e_n = E(x)aE(y).\end{aligned}$$

Thus formula (5) follows from the uniqueness assertion of Proposition 2.1. The uniqueness of the α-expectation associated with φ is now apparent from formulas (5) and (1). $\square$

3. Existence of α-expectations

In this section we show that every completely positive map $\varphi : A \to A$, with $\|\varphi\| \leq 1$, gives rise to a σ-expectation $E : \mathcal{P}A \to A$ that is related to the moment polynomials of φ as in (5). This is established through a construction that exhibits $\mathcal{P}A$ as the enveloping C^*-algebra of a Banach $*$-algebra $\ell^1(\Sigma)$, in such a way that the desired conditional expectation appears as a completely positive map on $\ell^1(\Sigma)$. The details are as follows.

Let S be the set of finite sequences $\bar{n} = (n_1, n_2, \dots, n_k)$ of nonnegative integers, $k = 1, 2, \dots$ which have distinct neighbors,

$$n_1 \neq n_2, n_2 \neq n_3, \dots, n_{k-1} \neq n_k.$$

Multiplication and involution are defined in S as follows. The product of two elements $\bar{m} = (m_1, \dots, m_k), \bar{n} = (n_1, \dots, n_\ell) \in S$ is defined by conditional concatenation

$$\bar{m} \cdot \bar{n} = \begin{cases} (m_1, \dots, m_k, n_1, \dots, n_\ell), & \text{if } m_k \neq n_1,\\ (m_1, \dots, m_k, n_2, \dots, n_\ell), & \text{if } m_k = n_1,\end{cases}$$

where we make the natural conventions when $\bar{n} = (q)$ is of length 1, namely $\bar{m} \cdot (q) = (m_1, \dots, m_k, q)$ if $m_k \neq q$, and $\bar{m} \cdot (q) = \bar{m}$ if $m_k = q$. The involution in S is defined by reversing the order of components

$$(m_1, \dots, m_k)^* = (m_k, \dots, m_1).$$

One finds that S is an associative $*$-semigroup.

Fixing a C^*-algebra A, we attach a Banach space Σ_ν to every k-tuple $\nu = (n_1, \dots, n_k) \in S$ as follows

$$\Sigma_\nu = \underbrace{A \hat{\otimes} \cdots \hat{\otimes} A}_{k \text{ times}},$$

the k-fold projective tensor product of copies of the Banach space A. We assemble the Σ_ν into a family of Banach spaces over S, $p : \Sigma \to S$, by way of $\Sigma = \{(\nu, \xi) : \nu \in S, \xi \in E_\nu\}$, $p(\nu, \xi) = \nu$.

We introduce a multiplication in Σ as follows. Fix $\mu = (m_1, \dots, m_k)$ and $\nu = (n_1, \dots, n_\ell)$ in S and choose $\xi \in \Sigma_\mu$, $\eta \in \Sigma_\nu$. If $m_k \neq n_1$ then $\xi \cdot \eta$ is defined as the tensor product $\xi \otimes \eta \in \Sigma_{\mu \cdot \nu}$. If $m_k = n_1$ then we must tensor over A and make the obvious identifications. More explicitly, in this case there is a natural map of the tensor product $\Sigma_\mu \otimes_A \Sigma_\nu$ onto $\Sigma_{\mu \cdot \nu}$ by making identifications of elementary tensors as follows:

$$(a_1 \otimes \cdots \otimes a_k) \otimes_A (b_1 \otimes \cdots \otimes b_\ell) \sim a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k b_1 \otimes b_2 \otimes \cdots \otimes b_\ell.$$

With this convention $\xi \cdot \eta$ is defined by

$$\xi \cdot \eta = \xi \otimes_A \eta \in \Sigma_{\mu \cdot \nu}.$$

This defines an associative multiplication in the family of Banach spaces Σ. There is also a natural involution in Σ, defined on each Σ_μ, $\mu = (m_1, \dots, m_k)$ as the unique antilinear isometry to Σ_{μ^*} satisfying

$$((m_1, \dots, m_k), a_1 \otimes \cdots \otimes a_k)^* = ((m_k, \dots, m_1), a_k^* \otimes \cdots \otimes a_1^*).$$

This defines an isometric antilinear mapping of the Banach space Σ_μ onto Σ_{μ^*}, for each $\mu \in S$, and thus the structure Σ becomes an involutive $*$-semigroup in which each fiber Σ_μ is a Banach space.

Let $\ell^1(\Sigma)$ be the Banach $*$-algebra of summable sections. The norm and involution are the natural ones $\|f\| = \sum_{\mu \in \Sigma} \|f(\mu)\|$, $f^*(\mu) = f(\mu^*)^*$. Noting that $\Sigma_\lambda \cdot \Sigma_\mu \subseteq \Sigma_{\lambda \cdot \mu}$, the multiplication in $\ell^1(\Sigma)$ is defined by convolution

$$f * g(\nu) = \sum_{\lambda \cdot \mu = \nu} f(\lambda) \cdot g(\mu),$$

and one easily verifies that $\ell^1(\Sigma)$ is a Banach $*$-algebra.

For $\mu = (m_1, \dots, m_k) \in S$ and $a_1, \dots, a_k \in A$ we define the function

$$\delta_\mu \cdot a_1 \otimes \cdots \otimes a_k \in \ell^1(\Sigma)$$

to be zero except at μ, and at μ it has the value $a_1 \otimes \cdots \otimes a_k \in \Sigma_\mu$. These elementary functions have $\ell^1(\Sigma)$ as their closed linear span. Finally, there is a natural sequence of $*$-homomorphisms $\theta_0, \theta_1, \dots : A \to \ell^1(\Sigma)$ defined by

$$\theta_k(a) = \delta_{(k)} \cdot a, \qquad a \in A, \quad k = 0, 1, \dots,$$

and these maps are related to the generating sections by

$$\delta_{(n_1, \dots, n_k)} \cdot a_1 \otimes \cdots \otimes a_k = \theta_{n_1}(a_1) \theta_{n_2}(a_2) \cdots \theta_{n_k}(a_k).$$

The algebra $\ell^1(\Sigma)$ fails to have a unit, but it has the same representation theory as $\mathcal{P}A$ in the following sense. Given a sequence of representations $\pi_k : A \to \mathcal{B}(H)$, $k = 0, 1, \dots$, fix $\nu = (n_1, \dots, n_k) \in S$. There is a unique bounded linear operator $L_\nu : \Sigma_\nu \to \mathcal{B}(H)$ of norm 1 that is defined by its action on elementary tensors as follows

$$L_\nu(a_1 \otimes \cdots \otimes a_k) = \pi_{n_1}(a_1) \cdots \pi_{n_k}(a_k).$$

Thus there is a bounded linear map $\tilde{\pi} : \ell^1(\Sigma) \to \mathcal{B}(H)$ defined by

$$\tilde{\pi}(f) = \sum_{\mu \in S} L_\mu(f(\mu)), \qquad f \in \ell^1(\Sigma).$$

One finds that $\tilde{\pi}$ is a $*$-representation of $\ell^1(\Sigma)$ with $\|\tilde{\pi}\| = 1$. This representation satisfies $\tilde{\pi} \circ \theta_k = \pi_k$, $k = 0, 1, 2, , \dots$. Conversely, every bounded $*$-representation $\tilde{\pi}$ of $\ell^1(\Sigma)$ on a Hilbert space H is associated with a sequence of representations $\pi_0, \pi_1, \dots$ of A on H by way of $\pi_k = \tilde{\pi} \circ \theta_k$.

The results of the preceding discussion are summarized as follows:

Proposition 3.1. *The enveloping C^*-algebra $C^*(\ell^1(\Sigma))$, together with the sequence of homomorphisms $\tilde{\theta}_0, \tilde{\theta}_1, \dots : A \to C^*(\ell^1(\Sigma))$ defined by the maps $\theta_0, \theta_1, \dots : A \to \ell^1(\Sigma)$, has the same universal property as the infinite free product $\mathcal{P}A = A * A * \cdots$, and is therefore isomorphic to $\mathcal{P}A$.*

Notice that the natural shift endomorphism of $\ell^1(\Sigma)$ is defined by

$$\sigma : \delta_{(n_1,\dots,n_k)} \cdot \xi \mapsto \delta_{(n_1+1,\dots,n_k+1)} \cdot \xi, \qquad \nu = (n_1, \dots, n_k) \in \Sigma, \quad \xi \in \Sigma_\nu$$

and it promotes to the natural shift endomorphism of $\mathcal{P}A = C^*(\ell^1(\Sigma))$. The inclusion of A in $\ell^1(\Sigma)$ is given by the map $\theta_0(a) = \delta_{(0)} a \in \ell^1(\Sigma)$, and it too promotes to the natural inclusion of A in $\mathcal{P}A$.

Finally, we fix a contractive completely positive map $\varphi : A \to A$, and consider the moment polynomials associated with it by Proposition 2.1. A straightforward argument shows that there is a unique bounded linear map $E_0 : \ell^1(\Sigma) \to A$ satisfying

$$E_0(\delta_{(n_1,\dots,n_k)} \cdot a_1 \otimes \cdots \otimes a_k) = [n_1, \dots, n_k; a_1, \dots, a_k],$$

for $(n_1, \dots, n_k) \in S$, $a_1, \dots, a_k \in A$, $k = 1, 2, \dots$, and $\|E_0\| = \|\varphi\| \le 1$. Using the axioms MP1 and MP2, one finds that the map E_0 preserves the adjoint (see Equation (4)), satisfies the conditional expectation property $E_0(af) = aE_0(f)$ for $a \in A$, $f \in \ell^1(\Sigma)$, that the restriction of E_0 to the "hereditary" $*$-subalgebra of $\ell^1(\Sigma)$ spanned by $\theta_0(A)\ell^1(\Sigma)\theta_0(A)$ is multiplicative, and that it is related to φ by $E_0 \circ \sigma = \varphi \circ E_0$ and $E_0(\sigma(a)) = \varphi(a)$, $a \in A$. Thus, E_0 satisfies the axioms of Definition 1.2, suitably interpreted for the Banach $*$-algebra $\ell^1(\Sigma)$.

In view of the basic fact that a bounded completely positive linear map of a Banach $*$-algebra to A promotes naturally to a completely positive map of its enveloping C^*-algebra to A, the critical property of E_0 reduces to:

Theorem 3.2. *For every $n \geq 1$, $a_1, \dots, a_n \in A$, and $f_1, \dots, f_n \in \ell^1(\Sigma)$, we have*

$$\sum_{i,j=1}^{n} a_j^* E_0(f_j^* f_i) a_i \geq 0.$$

Consequently, E_0 extends uniquely through the completion map $\ell^1(\Sigma) \to \mathcal{P}A$ to a completely positive map $E_\varphi : \mathcal{P}A \to A$ that becomes a σ-expectation satisfying Equation (3).

We sketch the proof of Theorem 3.2, detailing the critical steps. Using the fact that $\ell^1(\Sigma)$ is spanned by the generating family

$$G = \{\delta_{(n_1,\dots,n_k)} \cdot a_1 \otimes \cdots \otimes a_k : (n_1, \dots, n_k) \in S, \quad a_1, \dots, a_k \in A, \quad k \geq 1\}$$

one easily reduces the proof of Theorem 3.2 to the following more concrete assertion: for any finite set of elements $u_1, \dots, u_n$ in G, the $n \times n$ matrix $(a_{ij}) = (E_0(u_j^* u_i)) \in M_n(A)$ is positive.

The latter is established by an inductive argument on the "maximum height" $\max(h(u_1), \dots, h(u_n))$, where the height of an element $u = \delta_{(n_1,\dots,n_k)} \cdot a_1 \otimes \cdots \otimes a_k$ in G is defined as $h(u) = \max(n_1, \dots, n_k)$. The general case easily reduces to that in which A has a unit e, and in that setting the inductive step is implemented by the following.

Lemma 3.3. *Choose $u_1, \dots, u_n \in G$ such that the maximum height $N = \max(h(u_1), \dots, h(u_n))$ is positive. For $k = 1, \dots, n$ there are elements $b_k, c_k \in A$ and $v_k \in G$ such that $h(v_k) < N$ and*

$$E_0(u_j^* u_i) = b_j^* \varphi(E_0(v_j^* v_i)) b_i + c_j^*(e - \varphi(e)) c_i, \qquad 1 \leq i, j \leq n. \tag{7}$$

Remark 3.4. Note that if an inductive hypothesis provides positive $n \times n$ matrices of the form $(E_0(v_j^* v_i))$ whenever $v_1, \dots, v_n \in G$ have height $< N$, then the $n \times n$ matrix whose ijth term is the right side of (7) must also be positive, because φ is a completely positive map and $0 \leq \varphi(e) \leq e$. It follows from Lemma 3.3 that $(E_0(u_j^* u_i))$ must be a positive $n \times n$ matrix whenever $u_1, \dots, u_n \in G$ have height $\leq N$.

proof of Lemma 3.3. We identify the unit e of A with its image $\delta_{(0)} \cdot e \in G$. Fix i, $1 \leq i \leq n$, and write $u_i e = \delta_{(n_1,\dots,n_k)} \cdot a_1 \otimes \cdots \otimes a_k$. Note that n_k must be 0 because u_i has been multiplied on the right by e.

If $n_1 > 0$ we choose ℓ, $1 < \ell < k$ such that $n_1, n_2, \dots, n_\ell$ are positive and $n_{\ell+1} = 0$. Setting $v_i = \delta_{(n_1-1,\dots,n_\ell-1)} \cdot a_1 \otimes \cdots \otimes a_\ell$ and $w_i = \delta_{(n_{\ell+1},\dots,n_k)} \cdot a_{\ell+1} \otimes \cdots \otimes a_k$, we obtain a factorization $u_i = \sigma(v_i) w_i$, and we define b_i and c_i by $b_i = E_0(w_i)$, $c_i = 0$. If $n_1 = 0$ then $u_i e$ cannot be factored in this way; still, we set $v_i = e$, and $b_i = c_i = E_0(u_i)$. This defines b_i, c_i and v_i.

One now verifies (7) in cases: where both $u_i e$ and $u_j e$ factor into a product of the form $\sigma(v) w$, when one of them so factors and the other does not, and when neither does. For example, if $u_i e = \sigma(v_i) w_i$ and $u_j e = \sigma(v_j) w_j$ both factor, then we can make use of the formulas $E_0(f) = eE_0(f)e = E_0(efe)$ for $f \in \ell^1(\Sigma)$, $E_0(fg) = E_0(f)E_0(g)$ for $f, g \in [A\ell^1(\Sigma)A]$, and $E_0 \circ \sigma = \varphi \circ E_0$, to write

$$\begin{aligned}E_0(u_j^*u_i) &= E_0(eu_j^*u_ie) = E_0((u_je)^*u_ie) = E_0(w_j^*\sigma(v_j^*v_i)w_i)\\ &= b_j^*E_0(\sigma(v_j^*v_i))b_i = b_j^*\varphi(E_0(v_j^*v_i))b_i.\end{aligned}$$

If $u_ie = \sigma(v_i)w_i$ so factors and $u_je = eu_je$ does not, then we write

$$\begin{aligned}E_0(u_j^*u_i) &= E_0((u_je)^*u_ie) = E_0(u_je)^*E_0(\sigma(v_i)w_i) = b_j^*E_0(\sigma(v_i))b_i\\ &= b_j^*\varphi(E_0(v_i))b_i = b_j^*(\varphi(E_0(v_j^*v_i))b_i,\end{aligned}$$

noting that in this case $v_j^* = e$. A similar string of identities settles the case $u_ie = eu_ie$, $u_je = \sigma(v_j)w_j$.

Note that in each of the preceding three cases, the terms $c_j^*(e-\varphi(e))c_i$ were all 0.

In the remaining case where $u_ie = eu_ie$ and $u_je = eu_je$, we can write $E_0(u_j^*u_i) = E_0(eu_j^*u_ie) = E_0((eu_je)^*eu_ie) = E_0(eu_je)^*E_0(eu_ie) = b_j^*b_i$. Formula (7) persists for this case too, since $v_i = v_j = e$ and we can write

$$b_j^*b_i = b_j^*\varphi(e)b_i + b_j^*(e-\varphi(e))b_i = b_j^*\varphi(E_0(v_j^*v_i))b_i + c_j^*(e-\varphi(e))c_i. \quad \square$$

4. THE HIERARCHY OF DILATIONS

Let (A,φ) be a pair consisting of an arbitrary C^*-algebra A and a completely positive linear map $\varphi : A \to A$ satisfying $\|\varphi\| \leq 1$.

Definition 4.1. A dilation of (A,φ) is an A-dynamical system (i,B,α) with the property that there is an α-expectation $E : B \to A$ satisfying

$$E(\alpha(a)) = \varphi(a), \qquad a \in A.$$

Notice that the α-expectation $E : B \to A$ associated with a dilation of (A,φ) is uniquely determined, by Theorem 1.3. The class of all dilations of (A,φ) is contained in the class of all A-dynamical systems, and it is significant that it is also a subcategory. More explicitly, if (i_1,B_1,α_1) and (i_1,B_2,α_2) are two dilations of (A,φ), and if $\theta : B_1 \to B_2$ is a homomorphism of A-dynamical systems, then the respective α-expectations E_1, E_2 must also transform consistently

$$E_2 \circ \theta = E_1. \tag{8}$$

This follows from Theorem 2.3, since both E_1 and $E_2\circ\theta$ are α_1-expectations that project $\alpha_1(a)$ to $\varphi(a)$, $a \in A$.

Theorem 1.3 implies that every pair (A,φ) can be dilated to the universal A-dynamical system $(i,\mathcal{P}A,\sigma)$. Let $E_\varphi : \mathcal{P}A \to A$ be the σ-expectation satisfying $E_\varphi(\sigma(a)) = \varphi(a)$, $a \in A$. The preceding remarks imply that for every other dilation (i,B,α), there is a unique surjective $*$-homomorphism $\theta : \mathcal{P}A \to B$ such that $\theta\circ\sigma = \alpha\circ\theta$, $E\circ\theta = E_\varphi$, and which fixes A elementwise. Thus, $(i,\mathcal{P}A,\sigma)$ *is a universal dilation of* (A,φ). The universal dilation is obviously too large, since its structure bears no relation to φ. Thus it is significant that there is a smallest (A,φ) dilation, whose structure is more closely tied to φ. We now discuss the basic properties of this minimal dilation; we examine its structure in section 5.

In general, every completely positive map of C^*-algebras $E : B_1 \to B_2$ gives rise to a norm-closed two-sided ideal $\ker E$ in B_1 as follows

$$\ker E = \{x \in B_1 : E(bxc) = 0, \quad \text{for all } b, c \in B_1\}.$$

In more concrete terms, if $B_2 \subseteq \mathcal{B}(H)$ acts concretely on some Hilbert space and $E(x) = V^*\pi(x)V$ is a Stinespring decomposition of E, where π is a representation of B_1 on some other Hilbert space K and $V : H \to K$ is a bounded operator such that $\pi(B_1)VH$ has K as its closed linear span, then one can verify that

$$\ker E = \{x \in B_1 : \pi(x) = 0\}. \tag{9}$$

Notice too that $\ker E = \{0\}$ iff E is *faithful on ideals* in the sense that for every two-sided ideal $J \subseteq B$, one has $E(J) = \{0\} \implies J = \{0\}$.

Proposition 4.2. *Let (i, B, α) be an A-dynamical system and let $E : B \to A$ be an α-expectation. Then $\ker E$ is an α-invariant ideal with the property $A \cap \ker E = \{0\}$.*

If (i_1, B_1, α_1) and (i_2, B_2, α_2) are two dilations of (A, φ) and $\theta : B_1 \to B_2$ is a homomorphism of A-dynamical systems, then

$$\ker E_1 = \{x \in B_1 : \theta(x) \in \ker E_2\}. \tag{10}$$

Proof. That $A \cap \ker E = \{0\}$ is clear from the fact that if $a \in A \cap \ker E$ then $AaA = E(AaA) = \{0\}$, hence $a = 0$. Relation (10) is also straightforward, since $\theta(B_1) = B_2$ and $E_2 \circ \theta = E_1$. Indeed, for each $x \in B_1$, we have

$$E_2(B_2\theta(x)B_2) = E_2(\theta(B_1)\theta(x)\theta(B_1)) = E_2(\theta(B_1xB_1)) = E_1(B_1xB_1),$$

from which (10) follows.

To see that $\alpha(\ker E) \subseteq \ker E$, choose $k \in \ker E$. Since B is spanned by all finite products of elements $\alpha^n(a)$, $a \in A$, $n = 0, 1, \dots$, it suffices to show that $E(y\alpha(k)x) = 0$ for all $y \in B$ and all x of the form $x = \alpha^{m_1}(a_1) \cdots \alpha^{m_k}(a_k)$. Being a completely positive contraction, E satisfies the Schwarz inequality

$$E(y\alpha(k)x)^*E(y\alpha(k)x) \le E(x^*\alpha(k^*)y^*y\alpha(k)x) \le \|y\|^2 E(x^*\alpha(k^*k)x);$$

hence it suffices to show that $E(x^*\alpha(k^*k)x) = 0$. To prove the latter, one can argue cases as follows. Assuming that $m_i > 0$ for all i, then $x = \alpha(x_0)$ for some $x_0 \in B$, and using $E \circ \alpha = \varphi \circ E$ one has

$$E(x^*\alpha(k^*k)x) = E(\alpha(x_0k^*kx_0)) = \varphi(E(x_0^*k^*kx_0)) = 0.$$

For the remaining case where some $m_i = 0$, notice that x must have one of the forms $x = a \in A$ (when all m_i are 0), or $x = ax_0$ with $x_0 \in B$ (when $m_1 = 0$ and some other m_j is positive), or $\alpha(x_1)ax_2$ with $a \in A$, $x_1, x_2 \in B$ (when $m_1 > 0, \dots, m_{r-1} > 0$ and $m_r = 0$), and in each case $E(x^*\alpha(k^*k)x) = 0$. If $x = \alpha(x_1)ax_2$, for example, then we make use of Formula (6) to write

$$E(x^*\alpha(k^*k)x) = E(x_2^*a^*\alpha(x_1^*k^*kx_1)ax_2) = E(x_2)^*a^*E(\alpha(x_1^*k^*kx_1)aE(x_2).$$

The term on the right vanishes since $E(\alpha(x_1^*k^*kx_1) = \varphi(E(x_1^*k^*kx_1)) = 0$. The other cases are dealt with similarly, and $\alpha(\ker E) \subseteq \ker E$ follows. $\square$

We deduce the existence of minimal dilations and their basic characterization as follows. Fix a pair (A, φ), where $\varphi : A \to A$ is a completely positive contraction, let σ be the shift on $\mathcal{P}A$, and let $E_\varphi : \mathcal{P}A \to A$ be the unique σ-expectation satisfying $E_\varphi(\sigma(a)) = \varphi(a)$, $a \in A$. Proposition 4.2 implies that σ leaves $\ker E_\varphi$ invariant, thus it can be promoted to an endomorphism $\dot\sigma$ of the quotient C^*-algebra $\mathcal{P}A/\ker E_\varphi$. Moreover, since $A \cap \ker E_\varphi = \{0\}$, the inclusion of A in $\mathcal{P}A$ promotes to an inclusion of A in $\mathcal{P}A/\ker E_\varphi$. Thus we obtain an A-dynamical system $(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$ having a natural $\dot\sigma$-expectation $\dot E$ defined by $\dot E(x + \ker E_\varphi) = E_\varphi(x) + \ker E_\varphi$, which satisfies $\dot E(\dot\sigma(a)) = \varphi(a)$, $a \in A$. It is called the *minimal* dilation of (A, φ) in light of the following:

Corollary 4.3. *The dilation $(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$ of (A, φ) has the following properties.*

1. *$(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$ is subordinate to all other dilations of (A, φ).*
2. *The $\dot\sigma$-expectation $\dot E$ of $(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$ satisfies $\ker \dot E = \{0\}$.*
3. *Every dilation (i, B, α) of (A, φ) whose α-expectation E satisfies $\ker E = \{0\}$ is isomorphic to $(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$.*

Proof. (2) follows by construction of $(i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$, since the kernel ideal of its $\dot\sigma$-expectation has been reduced to $\{0\}$.

To prove (1), let (i, B, α) be an arbitrary dilation of (A, φ). By the universal property of $(i, \mathcal{P}A, \sigma)$ there is a surjective $*$-homomorphism $\theta : \mathcal{P}A \to B$ satisfying $\theta \circ \sigma = \alpha \circ \theta$; and by (8) one has $E \circ \theta = E_\varphi$. Formula (10) implies that $\ker E_\varphi$ contains $\ker \theta$, hence we can define a morphism of C^*-algebras $\omega : B \to \mathcal{P}A/\ker E_\varphi$ by way of $\omega(\theta(x)) = x + \ker E_\varphi$, for all $x \in \mathcal{P}A$. Obviously, ω is a homomorphism of A-dynamical systems, and we conclude that $(i, B, \alpha) \geq (i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$.

For (3), notice that if (i, B, α) is a dilation of (A, φ) whose α-expectation $E : B \to A$ satisfies $\ker E = \{0\}$ and $\theta : \mathcal{P}A \to B$ is the homomorphism of the previous paragraph, then Formula (10) implies that $\ker E_\varphi = \ker \theta$. Thus $\omega : B \to \mathcal{P}A/\ker E_\varphi$ has trivial kernel, hence it must implement an isomorphism of A-dynamical systems $(i, B, \alpha) \cong (i, \mathcal{P}A/\ker E_\varphi, \dot\sigma)$. $\square$

5. Structure of minimal dilations

Corollary 4.3 implies that minimal dilations of (A, φ) exist for every contractive completely positive map $\varphi : A \to A$, and that they are characterized by the fact that their α-expectations are faithful on ideals. The latter imposes strong requirements on the structure of minimal dilations, and we conclude by elaborating on these structural issues.

Definition 5.1. A *standard* dilation of (A, φ) is a dilation (i, B, α) such that $A = pBp$ is an essential corner of B whose projection $p \in M(B)$ satisfies $p\alpha(x)p = \varphi(pxp)$, $x \in B$.

In such cases, $E(x) = pxp$ is the α-expectation of B on A. Standard dilations are most transparent in the special case where A has a unit e and $\varphi(e) = e$. To illustrate that, let B be a C^*-algebra containing A and let α be an endomorphism of B with the following property:

$$\varphi(a) = e\alpha(a)e, \qquad a \in A, \tag{11}$$

e denoting the unit of A. We may also assume that B is generated by $A \cup \alpha(A) \cup \alpha^2(A) \cup \cdots$, so that (i, B, α) becomes an A-dynamical system.

Proposition 5.2. *The projection $e \in B$ satisfies $\alpha(e) \geq e$, $A = eBe$ is a hereditary subalgebra of B, and the map $E(x) = exe$ defines an α-expectation from B to A. If, in addition, A is an essential subalgebra of B, then (i, B, α) is a standard dilation of (A, φ).*

Sketch of proof. Formula (11) implies that $e\alpha(e)e = \varphi(e) = e$, hence $\alpha(e) \geq e$. It follows immediately that $e\alpha(exe)e = e\alpha(x)e$ for $x \in B$.

At this point, a simple induction establishes $e\alpha^n(a)e = \varphi^n(a)$, $a \in A$, $n = 0, 1, 2, \dots$. An argument similar to the proof of Theorem 2.3 allows one to evaluate more general expectation values as in (5)

$$e\alpha^{n_1}(a_1)\alpha^{n_2}(a_2)\cdots\alpha^{n_k}(a_k)e = [n_1, \dots, n_k; a_1, \dots, a_k],$$

which implies $eBe \subseteq A$. Hence $A = eBe$ is a hereditary subalgebra of B.

With these formulas in hand one finds that the conditional expectation $E(x) = exe$ satisfies axioms E1 and E2 of Definition 4.1. Hence (i, B, α) is a standard dilation of (A, φ) whenever A is an essential subalgebra of B. $\square$

We remark that the converse is also true: given (A, φ) for which A has a unit e and $\varphi(e) = e$, then every standard dilation has the properties of Proposition (5.2). The description of standard dilations in general, where A is unital and $\|\varphi\| \leq 1$ or is perhaps nonunital, becomes more subtle.

The universal dilation $(i, \mathcal{P}A, \sigma)$ of (A, φ) is *not* a standard dilation. For example, when A has a unit e one can make use of the universal property of $\mathcal{P}A = A * A * \cdots$ to exhibit representations $\pi : \mathcal{P}A \to \mathcal{B}(H)$ such that $\pi(e)$ and $\pi(\sigma(e))$ are nontrivial orthogonal projections. Hence $\sigma(e) \not\geq e$. Moreover, A is not a hereditary subalgebra of $\mathcal{P}A$, and the conditional expectation of Theorem 1.3 is never of the form $x \mapsto exe$.

On the other hand, we now show that minimal dilations of (A, φ) must be standard. This is based on the following characterization of essential corners in terms of conditional expectations.

Proposition 5.3. *For every inclusion of C^*-algebras $A \subseteq B$, the following are equivalent.*

(i) *A is an essential corner pBp of B.*
(ii) *There is a conditional expectation $E : B \to A$ whose restriction to $[ABA]$ is multiplicative, and which satisfies* $\ker E = \{0\}$.

Moreover, the conditional expectation $E : B \to A$ of (ii) is the compression map $E(x) = pxp$, and it is unique. The projection $p \in M(B)$ satisfies

$$\lim_{n\to\infty} \|xe_n - xp\| = 0, \qquad x \in B,$$

where e_n is any approximate unit for A, and it defines the closed left ideal generated by A as follows: $[BA] = Bp$.

Proof. The implication (i) $\implies$ (ii) is straightforward, since the compression map $E(x) = pxp$ obviously defines a conditional expectation of B on $A = pBp$ that is multiplicative on $[ABA]$. If $x \in B$ satisfies $E(BxB) = \{0\}$ then $pBx^*xBp = \{0\}$, hence $xBA = xBpBp = \{0\}$, and therefore $x = 0$ because $[BAB]$ is assumed to be an essential ideal in (i).

(ii) $\implies$ (i). Given a conditional expectation $E : B \to A$ satisfying (ii), we may assume that $A \subseteq \mathcal{B}(H)$ acts nondegenerately on some Hilbert space (e.g., represent B faithfully on some Hilbert space and take H to be the closed linear span of the ranges of all operators in A). Thus $E : B \to \mathcal{B}(H)$ becomes an operator-valued completely positive map of norm 1, having a Stinespring decomposition $E(x) = V^*\pi(x)V$, with π a representation of B on a Hilbert space K, and $V : H \to K$ a contraction with $[\pi(B)VH] = K$.

Let P be the projection on $[\pi(A)K]$. We claim that V is an isometry with $VV^* = P$. To prove that, choose $a \in A$, $b \in B$, and let e_n be an approximate unit for A. Since E is multiplicative on $[ABA]$ we can write

$$\begin{aligned} V^*\pi(b^*a^*)(VV^* - \mathbf{1})\pi(ab)V^* &= E(b^*a)E(ab) - E(b^*a^*ab) = \\ &\lim_n e_n(E(b^*a^*)E(ab) - E(b^*a^*ab))e_n = \\ &\lim_n(E(e_nb^*a^*)E(abe_n) - E(e_nb^*a^*abe_n)) = 0. \end{aligned}$$

It follows that $VV^* - \mathbf{1}$ vanishes on the closed linear span of $\pi(A)\pi(B)VH$, namely $[\pi(A)K]$; hence $VV^* \geq P$. On the other hand, for $a \in A$ we have $Va = VE(a) = VV^*\pi(a)V = \pi(a)V$. Thus $VH \subseteq [VAH] = [\pi(A)VH] \subseteq PK$; hence $VV^* \leq P$. That V is an isometry follows from the fact that for $a \in A$, $V^*Va = V^*VE(a) = V^*VV^*\pi(a)V = V^*\pi(a)V = E(a) = a$, and by nondegeneracy H is the closed linear span of $\{a\xi : a \in A, \quad \xi \in H\}$.

We claim that $P = VV^*$ belongs to the multiplier algebra of $\pi(B)$. For that, choose an approximate unit e_n for A. Since both $\pi(e_n)$ and VV^* are self-adjoint, it suffices to show that for every $b \in B$, $\pi(b)\pi(e_n) \to \pi(b)VV^*$ in norm as $n \to \infty$. Using $VV^*\pi(e_n) = \pi(e_n)VV^* = \pi(e_n)$, we can write

$$\begin{aligned} &\|\pi(b)(\pi(e_n) - VV^*)\|^2 = \|(\pi(e_n) - VV^*)\pi(b^*b)(\pi(e_n) - VV^*)\| = \\ &\|VV^*(\pi(e_nb^*be_n) - \pi(e_nb^*b) - \pi(b^*be_n) + \pi(b^*b))VV^*\| = \\ &\|V(E(e_nb^*be_n) - E(e_nb^*b) - E(b^*be_n) + E(b^*b))V^*\| \leq \\ &\|e_nE(b^*b)e_n - e_nE(b^*b) - E(b^*b)e_n + E(b^*b)\|, \end{aligned}$$

and the last term tends to 0 as $n \to \infty$ because e_n is an approximate unit for A and $E(b^*b) \in A$. It follows that $\pi(B)VV^* = \pi(A)P$ is the closed left ideal in $\pi(B)$ generated by $\pi(A)$.

We claim next that $\pi(A) = P\pi(B)P$ is a corner of $\pi(B)$. Indeed,

$$VV^*\pi(B)VV^* = VE(B)V^* = \pi(E(B))VV^* = \pi(A)VV^* = \pi(A).$$

It is essential because for any operator $T \in \mathcal{B}(K)$ for which $T\pi(BA) = \{0\}$ we must have $T\pi(BA)VV^* = \{0\}$. But since K is spanned by vectors of the form $\pi(b)Va\xi = \pi(b)\pi(a)V\xi$ for $a \in A$, $b \in B$, the only possibility is $T = 0$.

Finally, π must be a faithful representation because $\pi(x) = 0$ implies

$$E(BxB) = V^*\pi(B)\pi(x)\pi(B)V = \{0\},$$

and the latter implies $x = 0$ by hypothesis (ii). The preceding assertions can now be pulled back through the isomorphism $\pi : B \to \pi(B)$ to give (i). □

Combining Corollary 4.3 with Proposition 5.3, we obtain:

Theorem 5.4. *For every (A, φ) as above, the minimal dilation of (A, φ) is a standard dilation satisfying the assertions of Proposition 5.3. All standard dilations of (A, φ) are equivalent to the minimal one.*

We remark that Theorem 5.4, together with a theorem of Larry Brown [**Bro77**], implies that the C^*-algebra B of the *minimal* dilation (i, B, α) of (A, φ) can be embedded in the multiplier algebra of $A \otimes \mathcal{K}$.

Concluding Remarks. It is appropriate to review some highlights of the literature on noncommutative dilation theory, since it bears some relationship to the contents of §§4–5. Several approaches to dilation theory for semigroups of completely positive maps have been proposed since the mid 1970s, including work of Evans and Lewis [**EL77**], Accardi et al [**AL82**], Kümmerer [**Küm85**], Sauvageot [**Sau86**], and many others. Our attention was drawn to these developments by work of Bhat and Parthasarathy [**BP94**], in which the first dilation theory for CP semigroups acting on $\mathcal{B}(H)$ emerged that was effective for our work on E_0-semigroups [**Arv97**], [**Arv00**]. SeLegue [**SeL97**] showed how to apply multi-operator dilation theory to obtain the Bhat–Parthasarathy results, and he calculated the expectation values of the n-point functions of such dilations. Recently, Bhat and Skeide [**BS00**] have initiated an approach to the subject that is based on Hilbert modules over C^*-algebras and von Neumann algebras.

We intend to take up applications to semigroups of completely positive maps elsewhere.

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