

THE DOMAIN ALGEBRA OF A CP -SEMIGROUP

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ABSTRACT. A CP -semigroup (or *quantum dynamical semigroup*) is a semigroup $\phi = \{\phi_t : t \geq 0\}$ of normal completely positive linear maps on $\mathcal{B}(H)$, H being a separable Hilbert space, which satisfies $\phi_t(\mathbf{1}) = \mathbf{1}$ for all t and is continuous in the natural sense.

Let \mathcal{D} be the natural domain of the generator L of ϕ , $\phi_t = \exp tL$. Since the maps ϕ_t need not be multiplicative \mathcal{D} is typically an operator space, but not an algebra. However, we show that the set of operators

$$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}$$

is a $*$ -subalgebra of $\mathcal{B}(H)$, indeed \mathcal{A} is the largest self-adjoint algebra contained in \mathcal{D} . Because \mathcal{A} is a $*$ -algebra one may consider its $*$ -bimodule of noncommutative 2-forms $\Omega^2(\mathcal{A}) = \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$, and any linear mapping $L : \mathcal{A} \rightarrow \mathcal{B}(H)$ has a symbol $\sigma_L : \Omega^2(\mathcal{A}) \rightarrow \mathcal{B}(H)$, defined as a linear map by

$$\sigma_L(a \, dx \, dy) = aL(xy) - axL(y) - aL(x)y + axL(\mathbf{1})y, \quad a, x, y \in \mathcal{A}.$$

The symbol is a homomorphism of \mathcal{A} -bimodules for any $*$ -algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ and any linear map $L : \mathcal{A} \rightarrow \mathcal{B}(H)$. When L is the generator of a CP -semigroup with domain algebra \mathcal{A} above, we show that the symbol is negative in that $\sigma_L(\omega^*\omega) \leq 0$ for every $\omega \in \Omega^1(\mathcal{A})$ ($-\sigma_L$ is in fact completely positive).

Examples are given for which the domain algebra \mathcal{A} is, and is not, strongly dense in $\mathcal{B}(H)$. We also relate the generator of a CP -semigroup to its commutative paradigm, the Laplacian of a Riemannian manifold.

1. Basic properties of \mathcal{A} . Let $\phi = \{\phi_t : t \geq 0\}$ be a CP -semigroup as defined in the abstract. We first recall four characterizations of the domain of the generator of ϕ .

Lemma 1. *Let $A \in \mathcal{B}(H)$. The following are equivalent.*

(i) *The limit*

$$L(A) = \lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(A) - A)$$

exists relative to the strong- $$ topology of $\mathcal{B}(H)$.*

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(ii) *the limit*

$$L(A) = \lim_{t \rightarrow 0+} \frac{1}{t} (\phi_t(A) - A)$$

exists relative to the weak operator topology of $\mathcal{B}(H)$.

(iii)

$$\sup_{t > 0} \frac{1}{t} \|\phi_t(A) - A\| \leq M < \infty.$$

(iv) *There is a sequence $t_n \rightarrow 0+$ for which*

$$\sup_n \frac{1}{t_n} \|\phi_{t_n}(A) - A\| \leq M < \infty.$$

proof. The implications (i) \implies (ii) and (iii) \implies (iv) are trivial, and (ii) \implies (iii) is a straightforward consequence of the Banach-Steinhaus theorem.

proof of (iv) \implies (i). Since the unit ball of $\mathcal{B}(H)$ is weakly sequentially compact, the hypothesis (iv) implies that there is a sequence $t_n \rightarrow 0+$ such that

$$\frac{1}{t_n} (\phi_{t_n}(A) - A) \rightarrow T \in \mathcal{B}(H)$$

in the weak operator topology. We claim: for every $s > 0$,

$$(1.1) \quad \int_0^s \phi_\lambda(T) d\lambda = \phi_s(A) - A.$$

The integral on the left is interpreted as a weak integral; that is, for $\xi, \eta \in H$,

$$\int_0^s \langle \phi_\lambda(T)\xi, \eta \rangle d\lambda = \langle \phi_s(A)\xi, \eta \rangle - \langle A\xi, \eta \rangle.$$

To see that, fix $\lambda > 0$. Since ϕ_λ is weakly continuous on bounded sets in $\mathcal{B}(H)$ we have

$$\frac{1}{t_n} (\phi_{\lambda+t_n}(A) - \phi_\lambda(A)) = \phi_\lambda \left(\frac{1}{t_n} (\phi_{t_n}(A) - A) \right) \rightarrow \phi_\lambda(T)$$

in the weak operator topology, as $n \rightarrow \infty$. By the bounded convergence theorem, we find that for fixed $\xi, \eta \in H$,

$$\lim_{n \rightarrow \infty} \frac{1}{t_n} \left(\int_0^s \langle \phi_{\lambda+t_n}(A)\xi, \eta \rangle d\lambda - \int_0^s \langle \phi_\lambda(A)\xi, \eta \rangle d\lambda \right) = \int_0^s \langle \phi_\lambda(T)\xi, \eta \rangle d\lambda.$$

Writing

$$\int_0^s f(\lambda + t_n) d\lambda - \int_0^s f(\lambda) d\lambda = \int_s^{s+t_n} f(\lambda) d\lambda - \int_0^{t_n} f(\lambda) d\lambda,$$

the left side of the preceding formula becomes

$$\lim_{n \rightarrow \infty} \left(\frac{1}{t_n} \int_s^{s+t_n} \langle \phi_\lambda(A)\xi, \eta \rangle d\lambda - \frac{1}{t_n} \int_0^{t_n} \langle \phi_\lambda(A)\xi, \eta \rangle d\lambda \right)$$

which, because of continuity of ϕ in the time parameter, is $\langle \phi_s(A)\xi, \eta \rangle - \langle A\xi, \eta \rangle$, as asserted in (1.1).

To prove the strong- $*$ convergence asserted in (i), fix $\xi \in H$ and use (1.1) to write

$$\begin{aligned} \left\| \frac{1}{s}(\phi_s(A)\xi - A\xi) - T\xi \right\| &= \frac{1}{s} \left\| \int_0^s \phi_\lambda(T)\xi \, d\lambda - \int_0^s T\xi \, d\lambda \right\| \\ &\leq \frac{1}{s} \int_0^s \|\phi_\lambda(T)\xi - T\xi\| \, d\lambda \leq \left(\frac{1}{s} \int_0^s \|\phi_\lambda(T)\xi - T\xi\|^2 \, d\lambda \right)^{1/2}. \end{aligned}$$

The integrand of the last term expands as follows

$$\begin{aligned} \|\phi_\lambda(T)\xi - T\xi\|^2 &= \langle \phi_\lambda(T)^* \phi_\lambda(T)\xi, \xi \rangle - 2\Re \langle \phi_\lambda(T)\xi, T\xi \rangle + \|T\xi\|^2 \\ &\leq \langle \phi_\lambda(T^*T)\xi, \xi \rangle - 2\Re \langle \phi_\lambda(T)\xi, T\xi \rangle + \|T\xi\|^2, \end{aligned}$$

the last inequality by the Schwarz inequality for unital CP maps. Since $\phi_\lambda(T^*T)$ (resp. $\phi_\lambda(T)$) tends weakly to T^*T (resp. T) as $\lambda \rightarrow 0+$, it follows that

$$\limsup_{s \rightarrow 0+} \frac{1}{s} \int_0^s \|\phi_\lambda(T)\xi - T\xi\|^2 \, d\lambda \leq \langle T^*T\xi, \xi \rangle - 2\langle T\xi, T\xi \rangle + \|T\xi\|^2 = 0,$$

and we conclude that $\frac{1}{s}(\phi_s(A) - A)$ tends strongly to T as $s \rightarrow 0+$.

Similarly, $\frac{1}{s}(\phi_s(A) - A)^* = \frac{1}{s}(\phi_s(A^*) - A^*)$ tends strongly to T^* . ■

Definition. Let \mathcal{D} be the set of all operators $A \in \mathcal{B}(H)$ for which the four conditions of Lemma 1 are satisfied. $L : \mathcal{D} \rightarrow \mathcal{B}(H)$ denotes the generator of ϕ ,

$$L(A) = \lim_{t \rightarrow 0+} \frac{1}{t}(\phi_t(A) - A), \quad A \in \mathcal{D}.$$

It is obvious that \mathcal{D} is a self-adjoint linear subspace of $\mathcal{B}(H)$, that $L(A^*) = L(A)^*$ for $A \in \mathcal{D}$, and a standard argument shows that \mathcal{D} is dense in $\mathcal{B}(H)$ in the σ -strong operator topology.

Lemma 2. For every operator $A \in \mathcal{D}$ we have

$$\|L(A)\| = \sup_{t > 0} \frac{1}{t} \|\phi_t(A) - A\|.$$

proof. The inequality \leq is clear from the fact that $L(A)$ is the weak limit of operators $\frac{1}{t}(\phi_t(A) - A)$ near $t = 0+$, i.e.,

$$\|L(A)\| \leq \limsup_{t \rightarrow 0+} \frac{1}{t} \|\phi_t(A) - A\| \leq \sup_{t > 0} \frac{1}{t} \|\phi_t(A) - A\|.$$

For \geq , set $T = L(A)$. Using (1.1), we can write for every $t > 0$

$$\frac{1}{t} \|\phi_t(A) - A\| = \frac{1}{t} \left\| \int_0^t \phi_\lambda(T) \, d\lambda \right\| \leq \frac{1}{t} \int_0^t \|\phi_\lambda(T)\| \, d\lambda \leq \|T\|,$$

since $\|\phi_\lambda\| \leq 1$ for every $\lambda \geq 0$. ■

Theorem A. $\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}$ is a $*$ -subalgebra of $\mathcal{B}(H)$.

proof. \mathcal{A} is obviously a self-adjoint set of operators. We have to show that \mathcal{A} is a vector space satisfying $\mathcal{A} \cdot \mathcal{A} \subseteq \mathcal{A}$.

Fix $t > 0$. By Stinespring's theorem we can write

$$(1.2) \quad \phi_t(X) = V_t^* \pi_t(X) V_t, \quad X \in \mathcal{B}(H)$$

where V_t is an isometry from H into some other Hilbert space H_t and where $\pi_t : \mathcal{B}(H) \rightarrow \mathcal{B}(H_t)$ is a *normal* $*$ -homomorphism of von Neumann algebras. $P_t = V_t V_t^*$ is a self-adjoint projection in $\mathcal{B}(H_t)$.

For $t > 0$ we will consider the seminorms p_t, q_t defined on $\mathcal{B}(H)$ as follows

$$\begin{aligned} p_t(X) &= t^{-1} \|\phi_t(X) - X\|, \\ q_t(X) &= t^{-1/2} \|P_t \pi_t(X) - \pi_t(X) P_t\|, \quad X \in \mathcal{B}(H). \end{aligned}$$

Lemma 3. For every operator $X \in \mathcal{B}(H)$ we have the following characterizations.

(i) $X \in \mathcal{D}$ iff

$$\sup_{t>0} p_t(X) < \infty,$$

and in that case $\|L(X)\| = \sup_{t>0} p_t(X)$.

(ii) $X \in \mathcal{A}$ iff both $\sup_{t>0} p_t(X)$ and $\sup_{t>0} q_t(X)$ are finite, and in that case

$$\max(\|\sigma_L(dX^* dX)\|^{1/2}, \|\sigma_L(dX dX^*)\|^{1/2}) \leq \limsup_{t \rightarrow 0^+} q_t(X),$$

where $\sigma_L(dX^* dX)$ and $\sigma_L(dX dX^*)$ are the operators in $\mathcal{B}(H)$ defined by

$$\begin{aligned} \sigma_L(dX^* dX) &= L(X^* X) - X^* L(X) - L(X^*) X, \\ \sigma_L(dX dX^*) &= L(X X^*) - X L(X^*) - L(X) X^* \end{aligned}$$

Remark. The second assertion of Lemma 3 requires clarification. By definition, an operator X belongs to \mathcal{A} iff all four operators $X, X^*, X^* X, X X^*$ belong to the domain of the generator L of $\phi = \{\phi_t : t \geq 0\}$. In that case both operators $\sigma_L(dX^* dX)$ and $\sigma_L(dX dX^*)$ are well defined by the above formulas. The ‘‘symbol’’ map σ_L will be discussed more fully in section 3.

proof of Lemma 3. The assertion (i) follows from Lemmas 1 and 2 above. In order to prove (ii) we require the following more concrete expression for the seminorm q_t ,

$$(1.3) \quad \begin{aligned} q_t(X) &= \\ &= \max\left(\left\|\frac{1}{t}(\phi_t(X^* X) - \phi_t(X)^* \phi_t(X))\right\|^{1/2}, \left\|\frac{1}{t}(\phi_t(X X^*) - \phi_t(X)^* \phi_t(X^*))\right\|^{1/2}\right). \end{aligned}$$

To prove (1.3) we decompose the commutator $\pi_t(X) P_t - P_t \pi_t(X)$ into a sum

$$\pi_t(X) P_t - P_t \pi_t(X) = (\mathbf{1} - P_t) \pi_t(X) P_t - P_t \pi_t(X) (\mathbf{1} - P_t).$$

Since the first term $(\mathbf{1} - P_t)\pi_t(X)P_t$ has initial space in P_tH_t and final space in $(\mathbf{1} - P_t)$, and the second term has the opposite property, it follows that

$$\|\pi_t(X)P_t - P_t\pi_t(X)\| = \max(\|(\mathbf{1} - P_t)\pi_t(X)P_t\|, \|P_t\pi_t(X)(\mathbf{1} - P_t)\|).$$

We have

$$\begin{aligned} \|(\mathbf{1} - P_t)\pi_t(X)P_t\|^2 &= \|V_t^*\pi_t(X^*)(\mathbf{1} - P_t)\pi_t(X)V_t\|^2 \\ &= \|V_t^*\pi_t(X^*X)V_t - V_t^*\pi_t(X^*)V_tV_t^*\pi_t(X)V_t\|^2 \\ &= \|\phi_t(X^*X) - \phi_t(X)^*\phi_t(X)\|. \end{aligned}$$

Similarly,

$$\|P_t\pi_t(X)(\mathbf{1} - P_t)\|^2 = \|V_t^*\pi_t(X)(\mathbf{1} - P_t)\pi_t(X^*)V_t\|^2 = \|\phi_t(XX^*) - \phi_t(X)^*\phi_t(X^*)\|,$$

and formula (1.3) follows from these two expressions.

Now if $X \in \mathcal{A}$ then all four operators X, X^*, X^*X, XX^* belong to \mathcal{D} , hence all four limits

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(X^*X) - X^*X) &= L(X^*X), \\ \lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(XX^*) - XX^*) &= L(XX^*), \\ \lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(X) - X) &= L(X), \\ \lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(X^*) - X^*) &= L(X^*) \end{aligned}$$

exist relative to the strong operator topology. Writing

$$(1.4) \quad \begin{aligned} &\phi_t(X^*X) - \phi_t(X)^*\phi_t(X) = \\ &(\phi_t(X^*X) - X^*X) - X^*(\phi_t(X) - X) - (\phi_t(X^*) - X^*)\phi_t(X) \end{aligned}$$

and using strong continuity of multiplication on bounded sets, we find that the limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(X^*X) - \phi_t(X^*)\phi_t(X)) = L(X^*X) - X^*L(X) - L(X^*)X = \sigma_L(dX^* dX)$$

exists relative to the strong operator topology.

In the same way we deduce the existence of the strong limit

$$\lim_{t \rightarrow 0^+} \frac{1}{t}(\phi_t(XX^*) - \phi_t(X)\phi_t(X^*)) = L(XX^*) - XL(X^*) - L(X)X^* = \sigma_L(dX dX^*).$$

It follows that for every $X \in \mathcal{A}$ the seminorms $q_t(X)$ are bounded for $t > 0$, and for such X we have

$$\max(\|\sigma_L(dX^* dX)\|^{1/2}, \|\sigma_L(dX dX^*)\|^{1/2}) \leq \limsup_{t \rightarrow 0^+} q_t(X).$$

Conversely, suppose we are given an operator $X \in \mathcal{D}$ for which the seminorms $q_t(X)$ are bounded for $t > 0$. We have to show that X^*X and XX^* belong to \mathcal{D} ; since \mathcal{D} is self-adjoint and the seminorms q_t are symmetric in that $q_t(X^*) = q_t(X)$, it is enough to show that X^*X belong to \mathcal{D} . (1.4) implies that for fixed $t > 0$,

$$(1.5) \quad \begin{aligned} & \phi_t(X^*X) - X^*X = \\ & (\phi_t(X^*X) - \phi_t(X^*)\phi_t(X)) + X^*(\phi_t(X) - X) + (\phi_t(X^*) - X^*)\phi_t(X) \end{aligned}$$

Because of (1.3), the first term on the right of (1.5) is bounded in norm by $M_1 \cdot t$ where M_1 is a positive constant. Similarly, since X and X^* belong to \mathcal{D} the second and third terms are bounded in norm by terms of the form $M_2 \cdot t$ and $M_3 \cdot t$ respectively, hence

$$\|\phi_t(X^*X) - X^*X\| \leq (M_1 + M_2 + M_3) \cdot t.$$

By Lemma 1, X^*X must belong to \mathcal{D} . ■

Turning now to the proof of Theorem A, (or more properly, to the proof that \mathcal{A} is an algebra), Lemma 3 tells us that \mathcal{A} consists of all operators $X \in \mathcal{B}(H)$ for which

$$\sup_{t>0} p_t(X) < \infty, \quad \text{and} \quad \sup_{t>0} q_t(X) < \infty.$$

Since p_t and q_t are both seminorms, it follows that \mathcal{A} is a complex vector space which is obviously closed under the $*$ -operation.

To see that \mathcal{A} is closed under multiplication, pick $X, Y \in \mathcal{A}$. According to Lemma 3, it is enough to show

$$(1.6) \quad \sup_{t>0} q_t(XY) < \infty$$

and

$$(1.7) \quad \sup_{t>0} p_t(XY) < \infty$$

To prove (1.6) we claim that

$$(1.8) \quad q_t(XY) \leq q_t(X)\|Y\| + \|X\|q_t(Y).$$

Indeed, writing $[A, B]$ for the commutator $AB - BA$ we have

$$[P_t, \pi_t(XY)] = [P_t, \pi_t(X)]\pi_t(Y) + \pi_t(X)[P_t, \pi_t(Y)],$$

and hence

$$\begin{aligned} q_t(XY) &= t^{-1/2} \|[P_t, \pi_t(XY)]\| \\ &\leq t^{-1/2} \|[P_t, \pi_t(X)]\| \cdot \|\pi_t(Y)\| + \|\pi_t(X)\| \cdot t^{-1/2} \|[P_t, \pi_t(Y)]\|, \end{aligned}$$

from which (1.8) is evident.

Finally, consider the condition (1.7). By definition of \mathcal{A} , $A \in \mathcal{A}$ implies $A^*A \in \mathcal{D}$. Since \mathcal{A} is now known to be a linear space we can assert that if $X, Y \in \mathcal{A}$ then for every $k = 0, 1, 2, 3$ we have $Y + i^k X \in \mathcal{A}$, hence $(Y + i^k X)^*(Y + i^k X) \in \mathcal{D}$ and by the polarization formula

$$X^*Y = \frac{1}{4} \sum_{k=0}^3 i^k (Y + i^k X)^*(Y + i^k X),$$

X^*Y must also belong to \mathcal{D} . Since $A^* = \mathcal{A}$, we can replace X^* with X to conclude that $XY \in \mathcal{D}$. (1.7) now follows from Lemma 3 (i). \blacksquare

Corollary. *Let \mathcal{D} be the domain of the generator of a CP -semigroup acting on $\mathcal{B}(H)$ and let A be a self-adjoint operator such that $A \in \mathcal{D}$ and $A^2 \in \mathcal{D}$. Then $p(A) \in \mathcal{D}$ for every polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$.*

2. Examples. In this section we describe two classes of examples which are in a sense at opposite extremes. In the first class of examples of CP -semigroups $\phi = \{\phi_t : t \geq 0\}$, each ϕ_t leaves the C^* -algebra \mathcal{K} of all compact operators invariant, $\phi_t(\mathcal{K}) \subseteq \mathcal{K}$, its domain algebra \mathcal{A} is strongly dense in $\mathcal{B}(H)$, and its generator restricts to a “second order” differential operator on \mathcal{A} (see formula (1.1) of [1]). In the second class of examples, the individual maps satisfy $\phi_t(\mathcal{K}) \cap \mathcal{K} = \{0\}$ for $t > 0$, \mathcal{A} is not strongly dense in $\mathcal{B}(H)$, and its generator is degenerate in the sense that it restricts to a *derivation* on \mathcal{A} .

We first recall the class of examples of CP -semigroups of [1], including the heat flow of the CCR algebra. While for simplicity we confine the discussion to the case of one degree of freedom, the reader will note that everything carries over verbatim to the case of n degrees of freedom, $n = 1, 2, \dots$.

Let $\{W_z : z \in \mathbb{R}^2\}$ be an irreducible Weyl system acting on a Hilbert space H . Thus, $z \in \mathbb{R}^2 \mapsto W_z$ is a strongly continuous mapping from \mathbb{R}^2 into the unitary operators on H which satisfies the canonical commutation relations in Weyl’s form

$$W_{z_1}W_{z_2} = e^{i\omega(z_1, z_2)}W_{z_1+z_2}, \quad z_1, z_2 \in \mathbb{R}^2,$$

ω denoting the symplectic form on \mathbb{R}^2 given by

$$\omega((x, y), (x', y')) = \frac{1}{2}(x'y - xy').$$

Let $\{\mu_t : t \geq 0\}$ be a one-parameter family of probability measures on \mathbb{R}^2 which is a semigroup under the natural convolution of measures

$$\mu * \nu(S) = \int_{\mathbb{R}^2 \times \mathbb{R}^2} \chi_S(z + w) d\mu(z) d\nu(w),$$

which satisfies $\mu_0 = \delta_{(0,0)}$, and which is measurable in t in the natural sense. It is convenient to define the Fourier transform of a measure μ in terms of the symplectic form ω as follows,

$$\hat{\mu}(z) = \int_{\mathbb{R}^2} e^{i\omega(z, \zeta)} d\mu(\zeta), \quad z \in \mathbb{R}^2.$$

Given such a semigroup of probability measures $\{\mu_t : t \geq 0\}$ there is a unique CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ which satisfies

$$\phi_t(W_z) = \hat{\mu}_t(z)W_z, \quad z \in \mathbb{R}^2, \quad t \geq 0$$

see [1], Proposition 1.7. Two cases of particular interest are

$$\text{(CCR heat flow)} \quad \phi_t(W_z) = e^{-t|z|^2}W_z, \quad t \geq 0$$

where $|(x, y)|$ denotes the Euclidean norm $(x^2 + y^2)^{1/2}$, and

$$\text{(Cauchy flow)} \quad \phi_t(W_z) = e^{-t|z|}W_z, \quad t \geq 0.$$

For both of these examples a straightforward estimate shows that for fixed $z \in \mathbb{R}^2$ there is a constant $M > 0$ such that

$$\|\phi_t(W_z) - W_z\| = |\hat{\mu}_t(z) - 1| \leq M \cdot t, \quad t > 0$$

and hence $W_z \in \mathcal{D}$. Since W_z is unitary, $\mathbf{1} = W_z^*W_z = W_zW_z^*$ belongs to \mathcal{D} , and hence W_z belongs to the domain algebra \mathcal{A} of ϕ for every $z \in \mathbb{R}^2$. We conclude that for these examples, the domain algebra is strongly dense in $\mathcal{B}(H)$.

Indeed, it is not hard to show that \mathcal{A} contains a $*$ -algebra of compact operators that is norm-dense in the algebra \mathcal{K} of all compact operators. Unlike the examples to follow, these flows leave \mathcal{K} invariant in the sense that $\phi_t(\mathcal{K}) \subseteq \mathcal{K}$ for all $t \geq 0$, and can therefore be considered as CP -semigroups which act on the separable C^* -algebra \mathcal{K} , rather than than as CP -semigroups acting on $\mathcal{B}(H)$.

We now describe a class of examples of CP semigroups whose domain algebras are *not* strongly dense in $\mathcal{B}(H)$. These examples are inspired by a class of CP semigroups that have emerged in recent work of Robert Powers, to whom we are indebted for useful discussions.

Let $H = L^2(0, \infty)$ and let $U = \{U_t : t \geq 0\}$ be the semigroup of isometries $U_t\xi(x) = \xi(x - t)$ for $x \geq t$, $U_t\xi(x) = 0$ for $0 \leq x < t$. Fix a real number $\alpha > 0$, and let f be the unit vector in $L^2(0, \infty)$ obtained by normalizing the exponential function $u(x) = e^{-\alpha x}$, $x \geq 0$. One has $U_t^*f = e^{-\alpha t}f$ for every $t \geq 0$, hence the vector state $\omega(A) = \langle Af, f \rangle$ satisfies $\omega(U_tAU_t^*) = e^{-2\alpha t}\omega(A)$, $A \in \mathcal{B}(H)$.

We consider the family of unit-preserving normal completely positive maps $\phi = \{\phi_t : t \geq 0\}$ defined on $\mathcal{B}(H)$ by

$$\phi_t(A) = \omega(A)E_t + U_tAU_t^*, \quad t \geq 0.$$

where $E_t = \mathbf{1} - U_tU_t^*$ is the projection on the subspace $L^2(0, t) \subseteq L^2(0, \infty)$. Since

$$\omega(E_t) = \omega(\mathbf{1}) - \omega(U_tU_t^*) = 1 - e^{-2\alpha t},$$

it follows that $\omega(\phi_t(A)) = \omega(A)$ for every A . A routine computation now shows that ϕ satisfies the semigroup property $\phi_s \circ \phi_t = \phi_{s+t}$, hence ϕ is a CP semigroup.

Let \mathcal{D} be the domain of the generator of ϕ and let \mathcal{A} be the domain algebra

$$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\}.$$

Theorem A implies that \mathcal{A} is a unital $*$ -algebra and we calculate its strong closure.

Proposition. *The strong closure of \mathcal{A} consists of all operators $B \in \mathcal{B}(H)$ such that B commutes with the rank-one projection $f \otimes \bar{f}$.*

Thus the strong closure of \mathcal{A} consists of all operators B such that both B and B^* have f as an eigenvector.

proof. By Lemma 1, the domain \mathcal{D} of the generator of ϕ consists of all operators A with the property

$$(2.1) \quad \|\phi_t(A) - A\| \leq M \cdot t, \quad \text{for all } t \geq 0,$$

where M is a positive constant depending on A .

First, we show that $f \otimes \bar{f}$ commutes with \mathcal{A} . Choose $A \in \mathcal{A}$. In order to show that A commutes with $f \otimes \bar{f}$, it is enough to show that

$$(2.2) \quad \omega(A^*A) = \omega(AA^*) = |\omega(A)|^2,$$

since (2.2) implies

$$\|Af - \omega(A)f\|^2 = \omega(A^*A) - 2|\omega(A)|^2 + |\omega(A)|^2 = 0,$$

and similarly $\|A^*f - \omega(A^*)f\| = 0$. Multiplying $\phi_t(A) - A$ on the right by E_t and using the fact that $\phi_t(A)E_t = \omega(A)E_t$ we conclude from (2.1) that

$$\lim_{t \rightarrow 0} \|\omega(A)E_t - AE_t\| = 0.$$

Replacing A with A^*A and AA^* one also finds

$$\lim_{t \rightarrow 0} \|\omega(A^*A)E_t - A^*AE_t\| = \lim_{t \rightarrow 0} \|\omega(AA^*)E_t - AA^*E_t\| = 0.$$

Taken together, these three limits imply that $\omega(A^*A) = \omega(AA^*) = |\omega(A)|^2$, as required.

To prove the opposite inclusion it is enough to show that for every self-adjoint operator $A \in \mathcal{B}(H)$ satisfying $Af = 0$ there is a sequence A_n of self-adjoint operators in \mathcal{A} which converges weakly to A (recall that \mathcal{A} is a self-adjoint algebra containing the identity). Fix such an A and, for every $\epsilon > 0$, set

$$A_\epsilon = \phi_\epsilon(A) = \omega(A)E_\epsilon + U_\epsilon A U_\epsilon^* = U_\epsilon A U_\epsilon^*.$$

A_ϵ converges weakly to A as $\epsilon \rightarrow 0$. Moreover, A_ϵ is supported in the interval (ϵ, ∞) in the sense that $A_\epsilon E_\epsilon = E_\epsilon A_\epsilon = 0$, and in addition we have $A_\epsilon f = 0$ since

$$A_\epsilon f = U_\epsilon A U_\epsilon^* f = e^{-\alpha\epsilon} U_\epsilon A f = 0.$$

We show that each A_ϵ can be weakly approximated by self-adjoint elements of the domain algebra.

Lemma. *Suppose $\epsilon > 0$ and let A be a self-adjoint operator in $\mathcal{B}(H)$ such that (i) $Af = 0$ and (ii) A is supported in (ϵ, ∞) in the sense that $AE_\epsilon = E_\epsilon A = 0$. Let u be a C^∞ function having compact support in $[0, \epsilon]$ and consider*

$$B = \int_0^\infty u(s)U_sAU_s^* ds = \int_0^\infty u(s)\phi_s(A) ds.$$

Then $B^n \in \mathcal{D}$ for every $n = 1, 2, \dots$, and in particular $B \in \mathcal{A}$.

proof. Observe first that B has both properties (i) and (ii), hence so does B^n for every n . Thus for $t < \epsilon$ we have

$$\phi_t(B^n) - B^n = U_t B^n U_t^* - B^n = U_t B^n U_t^* - B^n U_t U_t^* = (U_t B^n - B^n U_t) U_t^*.$$

This implies that for sufficiently small t

$$\|\phi_t(B^n) - B^n\| = \|U_t B^n - B^n U_t\|.$$

We conclude that $B^n \in \mathcal{D}$ iff there is a constant $K > 0$ such that

$$(2.3) \quad \|U_t B^n - B^n U_t\| \leq K \cdot t, \quad \text{for all } t > 0.$$

To prove (2.3), one uses the Leibniz rule for the derivation $D(X) = U_t X - X U_t$ to estimate $\|U_t B^n - B^n U_t\|$ in terms of $\|U_t B - B U_t\|$,

$$\|D(B^n)\| \leq n \cdot \|B\|^{n-1} \|D(B)\| = n \cdot \|B\|^{n-1} \|U_t B - B U_t\|.$$

Since B has been smoothed it belongs to the domain \mathcal{D} , hence there is a constant M such that $\|U_t B - B U_t\| \leq M \cdot t$, hence $\|U_t B^n - B^n U_t\| \leq nM \|B\|^{n-1} \cdot t$. \blacksquare

The proof of the Proposition is completed by choosing $A = A_\epsilon$ in the hypothesis of the Lemma and by choosing a sequence u_k of nonnegative C^∞ functions, each of which has integral 1, such that $u_k(x) = 0$ outside the interval $0 \leq x \leq 1/k$. A standard argument shows that the sequence of self-adjoint operators

$$B_k = \int_0^\infty u_k(s)\phi_s(A_\epsilon) ds$$

converges weakly to A_ϵ , and the Lemma implies that $B_k \in \mathcal{A}$ for $k > 1/\epsilon$. \blacksquare

Thus the strong closure \mathcal{A}^- of \mathcal{A} has the form $\mathcal{B}(H_0) \oplus \mathbb{C}$ where $H_0 \subseteq H$ is a subspace of codimension one in H , and the following implies that these examples are ‘‘almost’’ E_0 -semigroups in the sense that there is an E_0 -semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(H_0)$ such that ϕ_t acts as follows on \mathcal{A}^- ,

$$\phi_t(B \oplus \lambda) = \alpha_t(B) \oplus \lambda, \quad B \in \mathcal{B}(H_0), \quad \lambda \in \mathbb{C}.$$

Corollary. *Let $\bar{\mathcal{A}}$ be the strong closure of \mathcal{A} . Then $\phi_t(\bar{\mathcal{A}}) \subseteq \bar{\mathcal{A}}$ for every $t \geq 0$ and $\{\phi_t \upharpoonright_{\bar{\mathcal{A}}} : t \geq 0\}$ is a semigroup of endomorphisms of this von Neumann algebra.*

proof. We show that $\phi_t(\mathcal{A}) \subseteq \bar{\mathcal{A}}$, and for $A, B \in \mathcal{A}$ one has $\phi_t(AB) = \phi_t(A)\phi_t(B)$.

Choose $A \in \mathcal{A}$, and let f and $\omega(A) = \langle Af, f \rangle$ be as in the definition of ϕ_t ,

$$\phi_t(A) = \omega(A)E_t + U_tAU_t^*, \quad A \in \mathcal{A}, \quad t \geq 0.$$

Since f is an eigenvector for both A and A^* and $U_t^*f = e^{-\alpha t}f$, one can verify directly that $\phi_t(A)f = \omega(A)f$ and $\phi_t(A)^*f = \omega(A^*)f$, and the Proposition implies that $\phi_t(\mathcal{A}) \subseteq \bar{\mathcal{A}}$. Finally, for $A, B \in \mathcal{A}$ one has $\omega(AB) = \omega(A)\omega(B)$, and $\phi_t(AB) = \omega(AB)E_t + U_tABU_t^* = \omega(A)\omega(B)E_t + U_tAU_t^*U_tBU_t^* = \phi_t(A)\phi_t(B)$. By normality of ϕ_t , the formula $\phi_t(AB) = \phi_t(A)\phi_t(B)$ persists for operators A, B in the strong closure of \mathcal{A} . \blacksquare

3. The symbol of the generator: properties and structure.

There are two useful characterizations of the generators of *uniformly continuous CP-semigroups*, i.e., those whose generators are everywhere defined bounded linear maps on $\mathcal{B}(H)$. The first is due to Lindblad [24] and independently to Gorini *et al* [20] (also see [13], Theorem 4.2). The second characterization is due to Evans and Lewis [19], based on work of Evans [16]. These two results can be paraphrased as follows.

Theorem. *Let $L : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be a bounded linear map and let $\phi = \{\phi_t : t \geq 0\}$ be the semigroup defined on $\mathcal{B}(H)$ by $\phi_t = \exp(tL)$. The following are equivalent.*

- (1) ϕ_t is a completely positive map for every $t \geq 0$.
- (2) (Lindblad, Gorini *et al*) L admits a decomposition

$$L(A) = P(A) + BA + AB^*, \quad A \in \mathcal{B}(H)$$

where P is a completely positive linear map and $B \in \mathcal{B}(H)$.

- (3) (Evans and Lewis) For every finite set of operators $A_1, \dots, A_n, B_1, \dots, B_n \in \mathcal{B}(H)$ which satisfy $A_1B_1 + \dots + A_nB_n = 0$, we have

$$\sum_{i,j=1}^n A_j^*L(B_j^*B_i)A_i \geq 0.$$

A linear map $L : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ satisfying property (3) of Theorem 3.1 is called *conditionally completely positive* [17]. While the characterization (2) tells us exactly which bounded linear maps generate *CP semigroups*, the cited decomposition of L into a sum of more familiar mappings is unfortunately not unique.

The purpose of this section is to make two observations. First, we point out that the notion of a conditionally completely positive linear map defined on a $*$ -algebra is more properly formulated in terms of the bimodule of noncommutative 2-forms over that algebra; and once that is done the “symbol” of the map becomes analogous to a Riemannian metric. Second, we show that by making use of the domain algebra of section 1, this notion becomes appropriate for the generators of *arbitrary CP-semigroups*.

Let \mathcal{A} be the domain algebra of a CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$

$$\mathcal{A} = \{A \in \mathcal{D} : A^*A \in \mathcal{D}, AA^* \in \mathcal{D}\},$$

where \mathcal{D} is the natural domain of the generator L of ϕ . We first recall the definition of the module of noncommutative 1-forms $\Omega^1(\mathcal{A})$, and 2-forms $\Omega^2(\mathcal{A})$. The algebraic tensor product of vector spaces $\mathcal{A} \otimes \mathcal{A}$ can be considered an involutive bimodule over \mathcal{A} , with

$$\begin{aligned} a(x \otimes y)b &= ax \otimes yb, \\ (x \otimes y)^* &= y^* \otimes x^*. \end{aligned}$$

The map $d : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ defined by $dx = \mathbf{1} \otimes x - x \otimes \mathbf{1}$ is a derivation for which $(dx)^* = -d(x^*)$, and it is a universal derivation of \mathcal{A} in the sense that if E is any \mathcal{A} -bimodule and $D : \mathcal{A} \rightarrow E$ is a linear map satisfying $D(xy) = xD(y) + D(x)y$ for all $x, y \in \mathcal{A}$, then there is a unique homomorphism of \mathcal{A} -modules $\theta : \Omega^1(\mathcal{A}) \rightarrow E$ such that $\theta \circ d = D$. Every element of $\Omega^1(\mathcal{A})$ is a finite sum of the form

$$\omega = \sum_{k=1}^r a_k dx_k,$$

and the involution in $\Omega^1(\mathcal{A})$ satisfies

$$(a dx)^* = -d(x^*)a^* = -d(x^*a^*) + x^*d(a^*).$$

Finally, $\Omega^1(\mathcal{A})$ is the kernel of the multiplication map $\mu : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ defined by $\mu(x \otimes y) = xy$, and thus we have an exact sequence of \mathcal{A} -modules

$$(3.1) \quad 0 \longrightarrow \Omega^1(\mathcal{A}) \longrightarrow \mathcal{A} \otimes \mathcal{A} \xrightarrow{\mu} \mathcal{A} \longrightarrow 0.$$

$\Omega^2(\mathcal{A})$ is defined by

$$\Omega^2(\mathcal{A}) = \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}),$$

and any element of $\Omega^2(\mathcal{A})$ can be written as a sum

$$\omega = \sum_{k=1}^r a_k dx_k dy_k.$$

The involution in $\Omega^2(\mathcal{A})$ satisfies

$$(a dx dy)^* = d(y^*)d(x^*)a^* = d(y^*)d(x^*a^*) - d(y^*x^*)d(a^*) + y^*d(x^*)d(a^*).$$

Since \mathcal{A} is a $*$ -subalgebra of $\mathcal{B}(H)$, we may also think of $\mathcal{B}(H)$ as an \mathcal{A} -bimodule. Now a straightforward argument shows that for every linear mapping $L : \mathcal{A} \rightarrow \mathcal{B}(H)$ there is a unique homomorphism of bimodules $\sigma_L : \Omega^2(\mathcal{A}) \rightarrow \mathcal{B}(H)$ which satisfies

$$(3.2) \quad \sigma_L(dx dy) = L(xy) - xL(y) - L(x)y + xL(\mathbf{1})y, \quad x, y \in \mathcal{A}$$

(see section 2 of [4] for more detail). $\sigma_L \in \text{hom}(\Omega^2, \mathcal{B}(H))$ is called the **symbol** of the linear map L .

Consider now the special case in which $\mathcal{A} = \mathcal{B}(H)$, $L : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is a *bounded* linear mapping, and let $\phi = \{\phi_t = \exp tL : t \geq 0\}$ is the semigroup of bounded operators on $\mathcal{B}(H)$ generated by L . The preceding theorem gives two characterizations of the maps L for which each $\phi_t = \exp tL$ is completely positive; however, the following characterization is perhaps more in spirit with the theory of differential operators on manifolds.

Theorem. *Let $L : \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be a bounded linear map. To the two characterizations (2) (3) above, one can append the following equivalent condition*

(4) *The symbol $\sigma_L : \Omega^2(\mathcal{B}(H)) \rightarrow \mathcal{B}(H)$ satisfies*

$$\sigma_L(\omega^*\omega) \leq 0, \quad \text{for every } \omega \in \Omega^1(\mathcal{B}(H)).$$

This characterization is Proposition 1.6 of [5]; a fuller discussion of these issues can be found in [4]. Notice that the sense of the inequality \leq is determined by the fact that the involution in Ω^1 satisfies $(dx)^* = -d(x^*)$, and hence for $\omega = dx$ we have $\omega^*\omega = -d(x^*)dx$. In particular, for $\omega = dx$ where x is a self-adjoint element we have $\sigma_L(\omega^2) \geq 0$ while $\sigma_L(\omega^*\omega) \leq 0$.

Remarks. There is a rather compelling analogy between this characterization of the generators of CP semigroups and the generator of the heat flow of a Riemannian manifold, namely the Laplacian. More precisely, let M be a complete (but not necessarily compact) Riemannian manifold and consider its natural Hilbert space $L^2(M)$. The Laplacian Δ acts naturally as a densely defined operator on $L^2(M)$ and generates a semigroup of bounded operators $\exp t\Delta$, $t \geq 0$, acting on $L^2(M)$ (the book of Davies [14] is a good reference). This semigroup maps bounded functions in $L^2(M)$ to bounded functions in $L^2(M)$, and the latter determines a semigroup of normal linear maps on the abelian von Neumann algebra $L^\infty(M)$ which carries nonnegative functions to nonnegative functions and fixes the constant functions.

In order to discuss the symbol of Δ we introduce local coordinates in some open set $U \subseteq M$ to identify U with an open region in \mathbb{R}^n . For clarity, we will be explicit with notation. At each point $x \in U$ the tangent space $T_x(M)$ is identified with \mathbb{R}^n , and for a smooth function f on M the differential df takes the following form

$$df(x, v) = \frac{d}{dt} f(x + tv)|_{t=0} = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x) v_k.$$

The metric gives rise to an operator function $x \in U \mapsto G(x)$ on \mathbb{R}^n by way of

$$\langle v, w \rangle_{T_x(M)} = \langle G(x)v, w \rangle_{\mathbb{R}^n}, \quad v, w \in T_x(M), \quad x \in U,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the Euclidean inner product on \mathbb{R}^n . $G(x)$ is an invertible positive operator on \mathbb{R}^n for every $x \in U$. For two vector fields ξ, η on M we have

$$\langle \xi(x), \eta(x) \rangle_{T_x(M)} = \langle G(x)\xi(x), \eta(x) \rangle_{\mathbb{R}^n} = \sum_{i,j=1}^n g_{ij}(x) \xi_j(x) \eta_i(x),$$

for $x \in U$, $(g_{ij}(x))$ being the matrix of $G(x)$ relative to the usual orthonormal basis for \mathbb{R}^n .

The inner product on the tangent space $T_x(M)$ promotes naturally to an inner product on the cotangent space $T_x^*(M)$. Indeed, the Riesz lemma implies that every linear functional ρ on $T_x(M)$ is associated with a unique vector $\rho_* \in T_x(M)$ via

$$\rho(v) = \langle v, \rho_* \rangle_{T_x(M)},$$

and the inner product in $T_x^*(M)$ is defined by

$$\langle \rho, \sigma \rangle_{T_x^*(M)} = \langle \rho_*, \sigma_* \rangle_{T_x(M)}.$$

With these conventions one finds that for a smooth function f and a point $x \in U$, $df(x, \cdot)_*$ becomes the vector in \mathbb{R}^n with components v_1, \dots, v_n ,

$$v_i = \sum_{j=1}^n g^{ij}(x) \frac{\partial f}{\partial x_j}(x),$$

$(g^{ij}(x)) = (g_{ij}(x))^{-1}$ being the matrix of the inverse operator $G(x)^{-1}$. For points $x \in U$ one has

$$(3.3) \quad \langle (df)_*, (dg)_* \rangle_{T_x(M)} = \sum_{i,j=1}^n g^{ij}(x) \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_i}.$$

We first recall that the dualized Riemannian metric (whose values are inner products on the cotangent spaces $T_x^*(M)$) can be linearized naturally so that it becomes a $C^\infty(M)$ -linear map of the the module $\Omega^{(2)}(M)$ of *symmetric* 2-forms. More explicitly, let $\Omega^1(M)$ be the usual module of 1-forms and let $\Omega^{(2)}(M)$ be the submodule of $\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M)$ consisting of all elements that are fixed under the action of the reflection R defined by $R : \omega_1 \otimes \omega_2 \mapsto \omega_2 \otimes \omega_1$. For $\omega_1, \omega_2 \in \Omega^1(M)$ we write $\omega_1 \omega_2$ for the symmetrized product

$$\omega_1 \omega_2 = \frac{1}{2}(\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) \in \Omega^{(2)}(M).$$

There is a unique homomorphism of $C^\infty(M)$ -modules $G^* : \Omega^{(2)}(M) \rightarrow C^\infty(M)$ satisfying $G^*(df dg)(x) = \langle df, dg \rangle_{T_x^*(M)}$ for all $x \in M$, and in local coordinates (3.3) implies that G^* has the form

$$(3.4) \quad G^*(df dg)(x) = \sum_{i,j=1}^n g^{ij}(x) \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i}, \quad x \in U.$$

If one knows G^* as a homomorphism of $C^\infty(M)$ -modules then one also knows the inner product in each cotangent space $T_x^*(M)$, and hence one can recover the original metric as an inner product on tangent spaces by duality and the Riesz lemma as above.

We now relate these remarks to the symbol of the Laplacian Δ of M . The symbol of any differential operator $L : C^\infty(M) \rightarrow C^\infty(M)$ of order at most 2 is associated with the bilinear form defined on $C^\infty(M)$ by

$$f, g \in C^\infty(M) \mapsto L(fg) - fL(g) - gL(f) + fgL(\mathbf{1}).$$

A straightforward argument shows that there is a (necessarily unique) homomorphism of $C^\infty(M)$ -modules $\sigma_L : \Omega^{(2)}(M) \rightarrow C^\infty(M)$ satisfying

$$\sigma_L(df dg) = L(fg) - fL(g) - gL(f) + fgL(\mathbf{1}).$$

In particular, this defines the symbol of any second order differential operator on $C^\infty(M)$, as an element of $\text{hom}(\Omega^2(M), C^\infty(M))$.

Restricting attention to the operator $L = \Delta$, one sees that for each $f \in C^\infty(M)$ the restriction of $\Delta(f)$ to U has the form

$$(3.5) \quad \Delta(f)(x) = \sum_{i,j=1}^n g^{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^n u_k(x) \frac{\partial f}{\partial x_k},$$

where u_1, \dots, u_n are appropriate smooth functions (see p. 147 of [14]). Using (3.5) one easily computes the symbol of Δ , and because of the local formula (3.4) for G^* one obtains $\sigma_\Delta = 2 \cdot G^*$. From these remarks we conclude that *the symbol of the Laplacian (considered as an element of $\text{hom}(\Omega^{(2)}(M), C^\infty(M))$) is precisely the Riemannian metric of M in its dualized form.*

Returning now to the case of a general CP semigroup $\phi = \{\phi_t : t \geq 0\}$ acting on $\mathcal{B}(H)$, let \mathcal{A} be the domain algebra of the generator of ϕ . Letting L be the restriction of the generator to \mathcal{A} , it is natural to ask the extent to which the generator can be identified with something analogous to a Riemannian metric (more precisely, to the homomorphism of $C^\infty(M)$ -modules $G^* : \Omega^{(2)}(M) \rightarrow C^\infty(M)$ that the dualized Riemannian metric determines). We have already defined the symbol $\sigma_L : \Omega^2(\mathcal{A}) \rightarrow \mathcal{B}(H)$ as a homomorphism of \mathcal{A} -modules, and the following asserts that σ_L does behave as if it were a (perhaps degenerate) Riemannian metric.

Proposition. *Let $\phi = \{\phi_t : t \geq 0\}$ be a CP semigroup acting on $\mathcal{B}(H)$ and consider the restriction L of the generator to the domain algebra $L : \mathcal{A} \rightarrow \mathcal{B}(H)$. Then the symbol of L satisfies*

$$\sigma_L(\omega^* \omega) \leq 0, \quad \omega \in \Omega^1(\mathcal{A});$$

and more generally for all $\xi_1, \dots, \xi_n \in H$ and $\omega_1, \dots, \omega_n \in \Omega^1(\mathcal{A})$ we have

$$\sum_{i,j=1}^n \langle \sigma_L(\omega_j^* \omega_i) \xi_i, \xi_j \rangle \leq 0.$$

The proof is a computation, facilitated by the following formula.

Lemma. *Let ω_1, ω_2 be elements of $\Omega^1(\mathcal{A})$ having the form*

$$\omega_k = \sum_{p=1}^s A_{kp} \otimes B_{kp}, \quad k = 1, 2,$$

where $A_{k1}B_{k1} + \dots + A_{ks}B_{ks} = 0$ for $k = 1, 2$. Then

$$\sigma_L(\omega_1 \omega_2) = - \sum_{p,q=1}^s A_{1p} L(B_{1p} A_{2q}) B_{2q}.$$

proof of Lemma. Since $\Omega^1(\mathcal{A})$ is spanned by elements of the form $A \cdot dX$, as well as by elements of the form $dY \cdot B$, $A, B, X, Y \in \mathcal{A}$, and since σ_L is a bimodule

homomorphism, it suffices to check the formula for ω_1, ω_2 of the particular form $\omega_1 = dX, \omega_2 = dY$.

Writing

$$\begin{aligned} dX dY &= (X \otimes \mathbf{1} - \mathbf{1} \otimes X)(Y \otimes \mathbf{1} - \mathbf{1} \otimes Y) \\ &= (X \otimes \mathbf{1})(Y \otimes \mathbf{1}) - (\mathbf{1} \otimes X)(Y \otimes \mathbf{1}) - (X \otimes \mathbf{1})(\mathbf{1} \otimes Y) + (\mathbf{1} \otimes X)(\mathbf{1} \otimes Y) \end{aligned}$$

the right side of the asserted formula for $\sigma_L(dX dY)$ has the form

$$\begin{aligned} -(XL(Y) - L(XY) - XL(\mathbf{1})Y + L(X)Y) = \\ L(XY) - XL(Y) - L(X)Y + XL(\mathbf{1})Y = \sigma_L(dX dY), \end{aligned}$$

as required. ■

proof of Proposition. Because of the exact sequence (3.1), every element $\omega \in \Omega^1(\mathcal{A})$ can be written

$$\omega = A_1 \otimes B_1 + \cdots + A_s \otimes B_s,$$

where A_k, B_k are elements of \mathcal{A} satisfying $A_1 B_1 + \cdots + A_s B_s = 0$. Choose elements $\omega_1, \dots, \omega_n \in \Omega^1(\mathcal{A})$ of the form

$$\omega_k = \sum_{p=1}^s A_{kp} \otimes B_{kp}, \quad k = 1, \dots, n$$

where $\sum_p A_{kp} B_{kp} = 0$ for $k = 1, \dots, n$. We have

$$\omega_k^* = \sum_{p=1}^s B_{kp}^* \otimes A_{kp}^*$$

so that the product $\omega_k^* \omega_j \in \Omega^2(\mathcal{A})$ is given by

$$\omega_k^* \omega_j = \sum_{p,q=1}^s (B_{kp}^* \otimes A_{kp}^*)(A_{jq} \otimes B_{jq}).$$

The Lemma implies that

$$(3.6) \quad \sigma_L(\omega_k^* \omega_j) = - \sum_{p,q=1}^s B_{kp}^* L(A_{kp}^* A_{jq}) B_{jq}.$$

If we now choose vectors $\xi_k \in H, k = 1, \dots, n$ then we find that

$$\begin{aligned} (3.7) \quad \sum_{k,j} \langle \sigma_L(\omega_k^* \omega_j) \xi_j, \xi_k \rangle &= - \sum_{i,j,p,q} \langle L(A_{kp}^* A_{jq}) B_{jq} \xi_j, B_{kp} \xi_k \rangle \\ &= - \sum_{\alpha,\beta} \langle L(A_\beta^* A_\alpha) \eta_\alpha, \eta_\beta \rangle, \end{aligned}$$

where in the third term, α and β run over all pairs $\{(k, p) : 1 \leq k \leq n, 1 \leq p \leq s\}$ and where the η_α are defined by $\eta_{(k,p)} = B_{kp}\xi_k$. Finally, the last term on the right of (3.7) can be rewritten in terms of the $n \cdot s \times n \cdot s$ operator matrix \tilde{A} having the entries A_α along a single row and zeros along all the other rows as follows

$$- \left\langle L^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle,$$

where $\tilde{\eta}$ is the column vector with components η_α , and where $L^{(n \cdot s)}$ is the natural map induced by L on matrices over \mathcal{A} by applying L to the elements of the matrix term-by-term.

Thus we have to show that $\left\langle L^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle \geq 0$. Recalling that the definition of L on elements of \mathcal{A} is

$$L(X) = \lim_{t \rightarrow 0^+} \frac{1}{t} (\phi_t(X) - X)$$

and the fact that \mathcal{A} is a $*$ -subalgebra of the domain of L , it follows that

$$\left\langle L^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle = \lim_{t \rightarrow 0^+} \frac{1}{t} (\left\langle \phi_t^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle - \left\langle \tilde{A}^* \tilde{A}\tilde{\eta}, \tilde{\eta} \right\rangle).$$

Notice that $\left\langle \tilde{A}^* \tilde{A}\tilde{\eta}, \tilde{\eta} \right\rangle = 0$. Indeed, by inspection of the components of the column vector $\tilde{A}\tilde{\eta}$ we find that it is the column vector having a single (possibly) nonzero component and that component is

$$\sum_{\alpha} A_{\alpha} \eta_{\alpha} = \sum_{k,p} A_{kp} B_{kp} \xi_k = \sum_{k=1}^n (\sum_{p=1}^s A_{kp} B_{kp}) \xi_k = 0,$$

since $\sum_p A_{kp} B_{kp} = 0$ for every k . Thus we have to show that

$$(3.8) \quad \lim_{t \rightarrow 0^+} \frac{1}{t} \left\langle \phi_t^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle \geq 0.$$

Now since for each $t > 0$ the map ϕ_t is unital and completely positive, the Schwarz inequality for completely positive maps implies

$$\phi_t^{(n \cdot s)}(\tilde{A}^* \tilde{A}) \geq \phi_t^{(n \cdot s)}(\tilde{A})^* \phi_t^{(n \cdot s)}(\tilde{A}),$$

and hence for positive t we have

$$\frac{1}{t} \left\langle \phi_t^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle \geq \frac{1}{t} \left\langle \phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta}, \phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta} \right\rangle = \frac{1}{t} \|\phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta}\|^2.$$

We claim that the term on the right tends to zero as $t \rightarrow 0^+$. Indeed, since the operator matrix \tilde{A} belongs to the domain of the generator of the CP semigroup $\phi^{(n \cdot s)} = \{\phi_t^{(n \cdot s)} : t \geq 0\}$, Lemma 1 implies that there is a constant $M > 0$ such that for every positive t , $\|\phi_t^{(n \cdot s)}(\tilde{A}) - \tilde{A}\| \leq M \cdot t$. It follows that

$$\|\phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta}\| = \|\phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta} - \tilde{A}\tilde{\eta}\| \leq M \cdot t \cdot \|\tilde{\eta}\|$$

and hence

$$\limsup_{t \rightarrow 0^+} \frac{1}{t} \|\phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta}\|^2 \leq \lim_{t \rightarrow 0^+} \frac{1}{t} (M^2 \cdot t^2 \cdot \|\tilde{\eta}\|^2) = 0.$$

It follows that

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \left\langle \phi_t^{(n \cdot s)}(\tilde{A}^* \tilde{A})\tilde{\eta}, \tilde{\eta} \right\rangle \geq \lim_{t \rightarrow 0^+} \frac{1}{t} \|\phi_t^{(n \cdot s)}(\tilde{A})\tilde{\eta}\|^2 = 0,$$

and the inequality (3.8) follows. \blacksquare

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