ON THE INDEX AND DILATIONS OF COMPLETELY POSITIVE SEMIGROUPS

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Abstract. It is known that every semigroup of normal completely positive maps $P = \{P_t : t \geq 0\}$ of $\mathcal{B}(H)$, satisfying $P_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$, has a minimal dilation to an E_0-semigroup acting on $\mathcal{B}(K)$ for some Hilbert space $K \supseteq H$. The minimal dilation of P is unique up to conjugacy. In a previous paper a numerical index was introduced for semigroups of completely positive maps and it was shown that the index of P agrees with the index of its minimal dilation to an E_0-semigroup. However, no examples were discussed, and no computations were made.

In this paper we calculate the index of a unital completely positive semigroup whose generator is a *bounded* operator

$$L : \mathcal{B}(H) \to \mathcal{B}(H)$$

in terms of natrual structures associated with the generator. This includes all unital CP semigroups acting on matrix algebras. We also show that the minimal dilation of the semigroup $P = \{\exp tL : t \geq 0\}$ to an E_0-semigroup is is cocycle conjugate to a CAR/CCR flow.

Introduction.

In [4], a numerical index is introduced for semigroups $P = \{P_t : t \geq 0\}$ of normal completely positive maps of $\mathcal{B}(H)$. In the case where $P_t(\mathbf{1}) = \mathbf{1}$ for every t, a recent theorem of B. V. R. Bhat asserts that P can be "dilated" to an E_0-semigroup [5,6]; and it was shown in [4] that the index of P agrees with the index of its *minimal* E_0-semigroup dilation. However, no examples were discussed there. In particular, the results of [4] give no information about which E_0-semigroups can occur as the minimal dilations of unital completely positive semigroups acting on matrix algebras.

In this paper we consider the more general case of completely positive semigroups having bounded generator. We calculate the index of such semigroups in terms of basic structures associated with their generators (Theorem 2.3, Corollary 2.17) and

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in the case where the semigroup preserves the unit we show that their minimal dilations must be cocycle conjugate to a CAR/CCR flow (Corollary 4.21). The extent to which the index calculations of section 2 can be extended to semigroups with unbounded generators remains unclear at present; but certainly the description of their minimal dilations (e.g., Corollary 4.21) becomes false without strong hypotheses on the generator.

It is appropriate to point out that, using a completely different method, Powers [11] has independently shown that every unital completely positive semigroup acting on a matrix algebra dilates to a completely spatial E_0-semigroup and he calculates the index of the minimal dilation in that case.

1. Bounded generators, symbols, and metric operator spaces.

We are concerned with the structure of various linear mappings on the von Neumann algebra $M = \mathcal{B}(H)$, H being a separable Hilbert space. $\mathcal{L}(M)$ will denote the space of all bounded linear maps $L : M \to M$. The purpose of this section is to discuss the relationship of metric operator spaces [4] to the generators of completely positive semigroups and their symbols. Our methods in §§2–3, even the statement of the key Theorem 2.3, will involve metric operator spaces in an essential way.

We briefly recall the definition of the symbol of a linear map $L \in \mathcal{L}(M)$. Consider the bilinear mapping defined on $M \times M$ by

$$L(xy) - xL(y) - L(x)y + xL(\mathbf{1})y. \tag{1.1}$$

It is useful to regard this as a homomorphism of the bimodule Ω^2 of all noncommutative 2-forms into M, and that homomorphism of M-modules is the symbol of L. More explicitly, Ω^1 is defined as the submodule of the symmetric bimodule $M \otimes M$ (with operations $a(x \otimes y)b = ax \otimes yb$, $(x \otimes y)^* = y^* \otimes x^*$) generated by the range of the derivation $d : M \to M \otimes M$,

$$dx = x \otimes \mathbf{1} - \mathbf{1} \otimes x.$$

We have $(dx)^* = -d(x^*)$ and every element of Ω^1 is a finite sum of the form $adx_1 + \dots + adx_n$, $a_k, x_k \in M$. Ω^2 is defined by

$$\Omega^2 = \Omega^1 \otimes_M \Omega^1.$$

Every element of Ω^2 is a sum of the form $a_1 dx_1\, dy_1 + \dots + a_n dx_n\, dy_n$, and we have a natural multiplication

$$\omega_1, \omega_2 \in \Omega^1 \to \omega_1 \omega_2 \in \Omega^2$$

which satisfies the associative law $\omega_1(a\omega_2) = (\omega_1 a)\omega_2$ with respect to operators $a \in M$.

Given $L \in \mathcal{L}(M)$ there is a unique $\sigma_L \in \hom(\Omega^2, M)$ satisfying

$$\sigma_L(dx\, dy) = L(xy) - xL(y) - L(x)y + xL(\mathbf{1})y$$

(see [3]). σ_L is called the **symbol** of L, and it has the following basic properties:

(1.2) $\sigma_L = 0$ iff L has the form $L(x) = ax + xb$ for fixed elements $a, b \in M$.

(1.3) If $\sigma_L = 0$ and L satisfies $L(x^*) = L(x)^*$ for every x, then there is an element $a \in M$ such that $L(x) = ax + xa^*$.

(1.4) For every $x, y \in M$ we have

$$\|\sigma_L(dx\, dy)\| \le 4\|L\|\,\|x\|\,\|y\|.$$

Remark 1.5. Property (1.2) follows from the fact that if $\sigma_L = 0$ then the linear map $L_0(x) = L(x) - xL(\mathbf{1})$ is a derivation of M, and hence has the form $L_0(x) = ax - xa$. Property (1.3) follows from (1.2) after an elementary argument which uses the fact that if c is any operator satisfying $cx + xc^* = 0$ for every operator x then c must have the form $c = \sqrt{-1}\lambda\mathbf{1}$ where $\lambda \in \mathbb{R}$ (see Lemma 1.19 below).

The following proposition summarizes some known results that illustrate how properties of the symbol characterize the generators of semigroups of completely positive maps.

Proposition 1.6. *Suppose that A is a unital C^*-algebra and $L \in \mathcal{L}(A)$ is a bounded operator satisfying $L(x)^* = L(x^*)$, $x \in A$. Let $\{P_t : t \geq 0\}$ be the semigroup of linear operators on A obtained by exponentiation: $P_t = \exp(tL)$. Then the following are equivalent.*

1.6.1 *Each map P_t is completely positive.*

1.6.2 *If $a_1, x_1, \dots, a_n, x_n \in A$ satisfy $x_1a_1 + \dots x_na_n = 0$ then we have*

$$\sum_{k,j=1}^{n} a_j^* L(x_j^* x_k) a_k \geq 0.$$

1.6.3 $\sigma_L(\omega^*\omega) \leq 0$ *for all* $\omega_1, \omega_2 \in \Omega^1$.

proof. The equivalence of (1.6.1) and (1.6.2) is essentially a result of Evans and Lewis [8]. The equivalence of (1.6.2) and (1.6.3) is discussed in [3]. $\square$

It is a straightforward consequence of Stinespring's theorem that every normal completely positive linear map $P \in \mathcal{L}(M)$ can be expressed in the form

$$P(x) = \sum_k v_k x v_k^* \tag{1.7}$$

where $\{v_1, v_2, \dots\}$ is a (finite or infinite) sequence of operators in M. Since certain facts relating to this representation are fundamental to our approach, we offer the following comments. Let (π, V) be a pair consisting of a representation π of $M = \mathcal{B}(H)$ on some other Hilbert space space K and an operator $V \in \mathcal{B}(H, K)$ satisfying

$$P(x) = V^*\pi(x)V. \tag{1.8}$$

By cutting down to a subspace of K if necessary we can assume that K is spanned by $\{\pi(x)\xi : x \in M, \xi \in H\}$, and in this case the normality of P implies that π is a normal representation, necessarily nondegenerate. Thus by replacing (π, V) by an equivalent pair we may assume that $K = H^n$ is a countable direct sum of copies of H and π has the form $\pi(x) = x \oplus x \oplus \dots$. It follows that there is a sequence $v_1, v_2, \dots \in \mathcal{B}(H)$ such that

$$V\xi = (v_1\xi, v_2\xi, \dots)$$

and the representation (1.7) follows by taking $u_k = v_k^*$.

There is a natural operator space $\mathcal{E}_P$ associated with P, which can be defined in concrete terms as follows. Notice that if $\lambda = (\lambda_1, \lambda_2, \dots)$ is any sequence in ℓ^2 then the operator sum $\sum_k \lambda_k u_k$ is convergent in the strong operator topology (this sum represents the composition of V^* with the operator $\xi \in H \mapsto (\lambda_1\xi, \lambda_2\xi, \dots) \in H^n$), and because of the minimality of (π, V) we have

$$\lambda_1 u_1 + \lambda_2 u_2 \cdots = 0 \implies \lambda_1 = \lambda_2 = \cdots = 0, \tag{1.9}$$

for every $\lambda \in \ell^2$. We define

$$\mathcal{E}_P = \{\lambda_1 u_1 + \lambda_2 u_2 + \cdots : \lambda \in \ell^2\}. \tag{1.10}$$

$\mathcal{E}_P$ is not necessarily closed in the operator norm when it is infinite dimensional, but in all cases it is a Hilbert space with respect to the inner product defined on it by declaring $\{u_1, u_2, \dots\}$ to be an orthonormal basis. $\mathcal{E}_P$ has the following properties.

(1.11.1) An operator a belongs to $\mathcal{E}_P$ if and only if there is a constant $c \geq 0$ such that the mapping

$$x \in M \mapsto cP(x) - axa^*$$

is completely positive. In this case, $\langle a, a\rangle_{\mathcal{E}}$ is the smallest such constant c.

(1.11.2) If $w_1, w_2, \dots$ is any orthonormal basis for $\mathcal{E}_P$ then the sum $\sum_k w_k w_k^*$ converges strongly, and in fact

$$P(x) = \sum_k w_k x w_k^*, \qquad x \in M.$$

(1.11.3) If $z_1, z_2, \dots$ is any finite or infinite sequence of operators in M such that the series $\sum_k z_k z_k^*$ converges strongly, and which represents P in the sense that

$$P(x) = \sum_k z_k x z_k^*, \qquad x \in M,$$

then $\{z_1, z_2, \dots\}$ spans the Hilbert space $\mathcal{E}_P$. If, in addition, $z_1, z_2, \dots$ satisfies the linear independence condition (1.9), then it is an orthonormal basis for $\mathcal{E}_P$.

These properties are discussed more fully in [4].

Definition 1.12. *A* **metric operator space** *is a pair $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ consisting of a linear supspace $\mathcal{E} \subseteq M = \mathcal{B}(H)$ and an inner product $\langle\cdot,\cdot\rangle : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ with respect to which $\mathcal{E}$ is a separable Hilbert space with the following property: for any orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$ we have*

$$\|v_1^*\xi\|^2 + \|v_2^*\xi\|^2 + \cdots < \infty$$

for every $\xi \in H$.

It is apparent from the preceding remarks that the positive operator defined by the sum $\sum_k v_k v_k^*$ does not depend on the choice of orthonormal basis (v_k), and in fact we can associate with $\mathcal{E}$ a unique normal completely positive linear map $P_{\mathcal{E}}$ on M by

$$P_{\mathcal{E}}(x) = \sum_k v_k x v_k^*, \qquad x \in M.$$

The preceding remarks can now be summarized as follows: In the von Neumann algebra $M = \mathcal{B}(H)$, the association $\mathcal{E} \leftrightarrow P_{\mathcal{E}}$ defines a bijective correspondence between the set of metric operator spaces contained in M and the set of normal completely positive linear maps in $\mathcal{L}(M)$.

Remark 1.13. Every normal completely positive linear map can be decomposed into a sum of the form $P_{\mathcal{E}_0} + c \cdot \iota_M$, where c is a nonnegative scalar, $\iota_M 4$ is the identity map of M, and $\mathcal{E}_0$ is a metric operator space satisfying the condition $\mathcal{E}_0 \cap \mathbb{C}\mathbf{1} = \{0\}$. To see that, suppose that $\mathcal{E}$ contains the identity operator and we set $\mathcal{E}_0 = \{v \in \mathcal{E} : \langle v, \mathbf{1}\rangle_{\mathcal{E}} = 0\}$. Then $\mathcal{E}_0$ is a metric operator space satisfying $\mathcal{E}_0 \cap \mathbb{C}\mathbf{1} = \{0\}$, and notice that there is a positive scalar c such that

$$P_{\mathcal{E}}(x) = P_{\mathcal{E}_0}(x) + cx.$$

Indeed, we can choose an orthonormal basis $v_0, v_1, v_2, \dots$ for $\mathcal{E}$ so that $v_0 = \lambda\mathbf{1}$ is a multiple of $\mathbf{1}$. Then $v_1, v_2, \dots$ is an orthonormal basis for $\mathcal{E}_0$ and we have

$$P_{\mathcal{E}}(x) = \sum_{k=0}^{\infty} v_k x v_k^* = \sum_{k=1}^{\infty} v_k x v_k^* + v_0 x v_0^* = P_{\mathcal{E}_0}(x) + |\lambda|^2 x.$$

Remark. While a normal completely positive map P on M determines its metric operator space $\mathcal{E}$ uniquely, that is not the case for the symbol of P. More precisely, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are two metric operator spaces with respective completely positive maps P_1 and P_2, then $\sigma_{P_1} = \sigma_{P_2}$ iff $\mathcal{E}_1 + \mathbb{C}\mathbf{1} = \mathcal{E}_2 + \mathbb{C}\mathbf{1}$. We will not make use of that fact, but we do require the following more explicit result from which it is easily deduced (see Theorem 3.3 and Remark 3.18 below for general results related to this issue).

Theorem 1.14. *Let $\mathcal{E}$ be a metric operator space satisfying $\mathcal{E} \cap \mathbb{C} \cdot \mathbf{1} = \{0\}$, and let $P = P_{\mathcal{E}}$ be its completely positive map. The most general normal completely positive linear map $Q \in \mathcal{L}(M)$ satisfying $\sigma_Q = \sigma_P$ has the form*

$$Q(x) = \sum_k (v_k + \lambda_k \mathbf{1}) x (v_k + \lambda_k \mathbf{1})^* + cx,$$

where $v_1, v_2, \dots$ is an orthonormal basis for $\mathcal{E}$, $(\lambda_1, \lambda_2, \dots)$ belongs to ℓ^2, and c is a nonnegative scalar.

proof. In view of (Remark 1.13), it suffices to prove the following assertion. Let $\mathcal{E}$, $\tilde{\mathcal{E}}$ be metric operator spaces such that $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \tilde{\mathcal{E}} \cap \mathbb{C}\mathbf{1} = \{0\}$, for which $P_{\mathcal{E}}$ and $P_{\tilde{\mathcal{E}}}$ have the same symbol. Then there is an orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$ and an ℓ^2 sequence $\lambda_1, \lambda_2, \dots$ such that $v_1 + \lambda_1\mathbf{1}, v_2 + \lambda_2\mathbf{1}, \dots$ is an orthonormal basis for $\tilde{\mathcal{E}}$.

To prove this, choose an orthonormal basis $u_1, u_2, \dots$ for $\mathcal{E}$ and let $n = \dim \mathcal{E}$. If we let π be the diagonal representation of $M = \mathcal{B}(H)$ on H^n

$$\pi(x) = x \oplus x \oplus \dots,$$

then we have

$$P_{\mathcal{E}}(x) = \sum_k u_k x u_k^* = V^* \pi(x) V,$$

where $V \in \mathcal{B}(H, H^n)$ is the operator

$$V\xi = (u_1^*\xi, u_2^*\xi, \dots).$$

Because of (1.9) we have $H^n = [\pi(x)\xi : x \in M, \xi \in H]$. However in this case, since $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$, we claim that in fact

$$H^n = [(Vx - \pi(x)V)\xi : x \in M, \xi \in H]. \tag{1.15}$$

To prove that let K be the subspace of H^n defined by the right side of (1.15). Since the map $x \in M \mapsto D(x) = Vx - \pi(x)V \in \mathcal{B}(H, K)$ satisfies

$$D(xy) = \pi(x)D(y) + D(x)y,$$

it follows that $K = [D(M)H]$ is invariant under operators in $\pi(M)$, and hence the projection p onto the orthocomplement of K belongs to the commutant of $\pi(M)$ and satisfies $pD(x) = 0$ for every $x \in M$. We have to show that $p = 0$. Considering the form of π we find that p is a matrix of scalar operators $p = (\lambda_{ij}\mathbf{1})$. Each row of the scalar matrix (λ_{ij}) belongs to ℓ^2, and the condition $pD(x) = 0$ implies that for every $i = 1, 2, \dots$ and every $x \in M$ we have

$$\sum_j \lambda_{ij}(u_j^*x - xu_j^*) = 0.$$

Thus for every i the operator $w_i = \sum_j \bar{\lambda}_{ij}u_j$ is an element of $\mathcal{E}$ which commutes with every operator in M, and is therefore a scalar multiple of the identity. Since $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$ we conclude that $w_1 = w_2 = \dots = 0$. Hence by (1.9) the matrix $p = (\lambda_{ij}\mathbf{1})$ is zero, proving (1.15).

Now let $m = \dim \tilde{\mathcal{E}}$ and let

$$\tilde{\pi}(x) = x \oplus x \oplus \dots$$

be the corresponding representation of M on H^m. Let $\tilde{u}_1, \tilde{u}_2, \dots$ be an orthonormal basis for $\tilde{\mathcal{E}}$ and let $\tilde{V} : H \to H^m$ be the associated operator

$$\tilde{V}\xi = (\tilde{u}_1^*\xi, \tilde{u}_2^*\xi, \dots).$$

Then there is a corresponding derivation $\tilde{D} : M \to \mathcal{B}(H, K)$ which is defined by $\tilde{D}(x) = \tilde{V}x - \tilde{\pi}(x)\tilde{V}$. We claim next that there is a (necessarily unique) unitary operator $W : H^n \to H^m$ satisfying

$$WD(x) = \tilde{D}(x), \qquad x \in M. \tag{1.16}$$

Noting that $H^n = [D(M)H]$ and $H^m = [\tilde{D}(M)H]$, it is clearly enough to show that for all $\xi, \eta \in H$ we have

$$\langle D(x)\xi, D(y)\eta\rangle = \left\langle \tilde{D}(x)\xi, \tilde{D}(y)\eta \right\rangle. \tag{1.17}$$

For that, we write

$$\begin{aligned} D(y)^*D(x) &= (Vy - \pi(y)V)^*(Vx - \pi(x)V) \\ &= V^*\pi(xy)V - y^*V^*\pi(x)V - V^*\pi(y^*)Vx + y^*V^*Vx \\ &= \sigma_{P_{\mathcal{E}}}(dy^*\, dx). \end{aligned}$$

By hypothesis the symbols of $P_{\mathcal{E}}$ and $P_{\tilde{\mathcal{E}}}$ agree, hence the right side is

$$\sigma_{\tilde{\mathcal{E}}}(dy^*\, dx) = \tilde{D}(y)^*\tilde{D}(x),$$

and formula (1.17) follows.

Note that $W\pi(x) = \tilde{\pi}(x)W$ for every x. Indeed, fixing x and choosing a vector in H^n of the form $\eta = D(y)\xi$ for $y \in M$, $\xi \in H$ we have

$$\begin{aligned} W\pi(x)\eta &= W\pi(x)D(y)\xi = WD(xy)\xi - WD(x)y\xi \\ &= \tilde{D}(xy)\xi - \tilde{D}(x)y\xi = \tilde{\pi}(x)\tilde{D}(y)\xi = \tilde{\pi}(x)WD(y)\xi. \end{aligned}$$

The assertion follows because H^n is spanned by such vectors η.

In particular, π and $\tilde{\pi}$ are equivalent representations of M. Hence $m = n$ and therefore $H^m = H^n$. Moreover, W belongs to the commutant of π and hence there is a unitary matrix (λ_{ij}) of complex scalars such that $W = (\lambda_{ij}\mathbf{1})$. If we now look at the components of the operator equation $WD(x) = \tilde{D}(x)$ we find that for every $i = 1, 2, \dots$

$$\sum_j \lambda_{ij}(u_j^*x - xu_j^*) = \tilde{u}_i^*x - x\tilde{u}_i^*.$$

Thus we can define a new orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$ by

$$v_i = \sum_j \bar{\lambda}_{ij}u_j.$$

The preceding equation relating the u_j to the $\tilde{u}_j$ becomes

$$v_i^*x - xv_i^* = \tilde{u}_i^*x - x\tilde{u}_i^*$$

for every operator $x \in M = \mathcal{B}(H)$. It follows that the operators $\tilde{u}_i - v_i$ commute with all bounded operators and therefore must be scalar multiples of the identity operator, say

$$\tilde{u}_i = v_i + \lambda_i\mathbf{1}. \tag{1.18}$$

The fact that both series $\sum_j v_jv_j^*$ and $\sum \tilde{u}_j\tilde{u}_j^*$ converge strongly implies that $\sum_j |\lambda|^2 < \infty$, and thus (1.18) provides an orthonormal basis $\tilde{u}_1, \tilde{u}_2, \dots$ for $\tilde{\mathcal{E}}$ of the required form. $\square$

Corollary. *Let $\mathcal{E}$ and $P = P_{\mathcal{E}}$ satisfy the hypotheses of Theorem 1.14, and let k be an operator in M. The following are equivalent:*

(1) *$Q(x) = P(x) + kx + xk^*$ is completely positive.*
(2) *k has a (necessarily unique) decompostion of the form $k = v + c\mathbf{1}$, where $v \in \mathcal{E}$ and c is a complex number satisfying $c + \bar{c} \geq \langle v, v\rangle_{\mathcal{E}}$.*

proof of (2) $\Longrightarrow$ (1). Let $k = v + 1/2\,\langle v, v\rangle_{\mathcal{E}} + d\mathbf{1}$ where $v \in \mathcal{E}$ and d is a complex number with nonnegative real part. Choose an orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$ and set $\lambda_k = \langle v, v_k\rangle$. Then for $\alpha = d + \bar{d} - \langle v, v\rangle_{\mathcal{E}} \geq 0$ we have

$$kx + xk^* = vx + xv^* + (\langle v, v\rangle_{\mathcal{E}} + \alpha)x = \sum_k (\lambda_k v_k x + \bar{\lambda}_k x v_k^* + |\lambda_k|^2 x) + \alpha x,$$

hence the map Q of (1) can be written

$$Q(x) = \sum_k (v_k + \bar{\lambda}_k \mathbf{1}) x (v_k + \bar{\lambda}_k \mathbf{1})^* + \alpha x$$

which is obviously completely positive.

Before proving the opposite implication we collect two elementary observations.

Lemma 1.19.

(1) *The only completely positive linear map of $M = \mathcal{B}(H)$ having symbol 0 is of the form $L(x) = cx$ where c is a nonnegative scalar.*
(2) *The only operator $k \in M$ for which $P(x) = kx + xk^*$ is a completely positive map is of the form $k = z\mathbf{1}$, where z is a complex number having nonnegative real part.*

proof. Suppose that P is a completely positive linear map on M for which $\sigma_P = 0$. Let $P(x) = V^*\pi(x)V$ be a Stinespring representation for P. From the definition of σ_P one finds that

$$\sigma_P(dx\, dy) = (V^*\pi(x) - xV^*)(\pi(y)V - Vy).$$

Therefore $(\pi(x)V - Vx)^*(\pi(x)V - Vx) = \sigma_P(d(x^*)\, dx) = 0$, hence $Vx = \pi(x)V$ for every x. It follows that $V^*Vx = xV^*V$ for all $x \in M = \mathcal{B}(H)$ so that there is a nonnegative scalar x such that $V^*V = c\mathbf{1}$. Thus $P(x) = V^*\pi(x)V = V^*Vx = cx$ as asserted.

For the second assertion, suppose that $x \mapsto kx + xk^*$ is completely positive. The symbol of this map vanishes, so by what was just proved there is a nonnegative scalar c such that $kx + xk^* = cx$ for every x. Taking x to be an arbitrary projection p we find that $(\mathbf{1} - p)kp = 0$, hence k must be a scalar $k = \lambda\mathbf{1}$. The formula $kx + xk^* = cx$ implies that $\lambda + \bar{\lambda} = c \geq 0$, and the Lemma is proved. $\Box$

proof of (1) $\Longrightarrow$ (2). Suppose that k is an operator in M such that $Q(x) = P(x) + kx + xk^*$ is completely positive. Then Q and P have the same symbol, hence by Theorem 1.14 there is an ℓ^2 sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ and an orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$ such that

$$Q(x) = \sum_k (v_k + \lambda_k \mathbf{1}) x (v_k + \lambda_k \mathbf{1})^*.$$

Define an element $v \in \mathcal{E}$ by $v = \sum_k \bar{\lambda}_k v_k$, and write $|v|^2 = \langle v, v\rangle_{\mathcal{E}} = \sum_k |\lambda_k|^2$. We find that

$$\begin{aligned} Q(x) &= \sum_k v_k x v_k^* + vx + xv^* + |v|^2 x \\ &= P(x) + (v + 1/2|v|^2\mathbf{1})x + x(v + 1/2|v|^2\mathbf{1})^* \\ &= P(x) + kx + xk^*. \end{aligned}$$

It follows that the operator $\ell = k - v - 1/2|v|^2\mathbf{1}$ has the property that $x \mapsto \ell x + x\ell^*$ is a completely positive linear map on $M = \mathcal{B}(H)$. From part (2) of the preceding Lemma we find that there is a complex scalar z having nonnegative real part such that $\ell = z\mathbf{1}$, and the required representation

$$k = v + (1/2|v|^2 + z)\mathbf{1}$$

follows $\square$

We conclude this section by reformulating a known description of the bounded generators of semigroups of completely positive maps in terms of metric operator spaces.

Proposition 1.20. *Let $L \in \mathcal{L}(M)$ be an operator which generates a semigroup $P_t = \exp(tL)$, $t \geq 0$ of normal completely positive maps on M. Then there is a metric operator space $\mathcal{E}$ satisfying $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$, and an operator $z \in M$ such that*

$$L(x) = P_{\mathcal{E}}(x) + zx + xz^* \qquad x \in M.$$

proof. Using a general result of Christensen and Evans [7], one can find a completely positive linear map $Q : M \to M$ and an element $z \in M$ such that

$$L(x) = Q(x) + zx + xz^*, \qquad x \in M. \tag{1.21}$$

Since L is a bounded operator that generates a semigroup of normal completely positive maps, it must itself be a normal linear map on M; and we may conclude from (1.21) that Q is normal. Let $\mathcal{E}$ be a metric operator space such that $Q = P_{\mathcal{E}}$. The remark following Definition 1.12 shows that we can arrange $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$ by adjusting z if necessary $\square$

Remarks. Unlike the case of completely positive maps, the correspondence between metric operator spaces and generators of CP semigroups is not quite one-to-one. However, if $(\mathcal{E}_1, z_1)$ and $(\mathcal{E}_2, z_2)$ are two pairs which serve to represent a given generator L as in Proposition 1.20 and which satisfy

$$\mathcal{E}_1 \cap \mathbb{C}\mathbf{1} = \mathcal{E}_2 \cap \mathbb{C}\mathbf{1} = \{0\}, \tag{1.22}$$

then it follows from Theorem 1.14 that $\mathcal{E}_1 + \mathbb{C}\mathbf{1} = \mathcal{E}_2 + \mathbb{C}\mathbf{1}$, and hence $\dim \mathcal{E}_1 = \dim \mathcal{E}_2$. Thus we can make the following

Definition 1.23. *Let L be a bounded operator on $\mathcal{B}(H)$ which generates a semigroup of normal completely positive maps on $\mathcal{B}(H)$. The* **rank** *of L is defined as the dimension of any metric operator space $\mathcal{E}$ for which $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$ and which gives rise to a representation of L in the form*

$$L(x) = P_{\mathcal{E}}(x) + kx + xk^*, \qquad x \in \mathcal{B}(H)$$

where k is some operator in $\mathcal{B}(H)$.

We also point out that, with a little more care, one can recover the inner product on $\mathcal{E}$ from the properties of L (see Remark 3.18 below).

2. Units and covariance function.

In [4], a notion of index for CP semigroups P was introduced which directly generalizes the definition of numerical index of E_0-semigroups. Briefly, a **unit** of P is a strongly continuous semigroup $T = \{T(t) : t \geq 0\}$ of bounded operators in M for which there is a real constant k with the property that for every $t \geq 0$, the operator mapping

$$x \in M \mapsto e^{kt} P_t(x) - T(t)xT(t)^*$$

is completely positive. Let $\mathcal{U}_P$ be the set of all units of P. It is possible for $\mathcal{U}_P$ to be the empty set; indeed there are E_0-semigroups with this property [9,10]. But if $\mathcal{U}_P \neq \emptyset$ then one can define a function

$$c_P : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C},$$

called the **covariance function** of the semigroup P, as follows. Note first that for every positive t, there is a unique metric operator space associated with the completely positive map P_t; we will write $\mathcal{E}_P(t)$ for this metric operator space. The most elementary properties of the family

$$\mathcal{E}_P = \{\mathcal{E}_P(t) : t > 0\}$$

are as follows:

(2.1.1) Each $\mathcal{E}_P(t)$ is a separable Hilbert space.
(2.1.2) $\mathcal{E}_P(s+t)$ is spanned as a Hilbert space by the set of products

$$\{xy : x \in \mathcal{E}_P(s), y \in \mathcal{E}_P(t)\}.$$

Remarks. Regarding (2.1.2), it is shown in [4, Theorem 1.12] that operator multiplication

$$u \otimes v \in \mathcal{E}_P(s) \otimes \mathcal{E}_P(t) \mapsto uv \in \mathcal{E}_P(s+t)$$

extends uniquely to a bounded linear operator from the Hilbert space $\mathcal{E}_P(s)\otimes\mathcal{E}_P(t)$ to $\mathcal{E}_P(s+t)$ whose adjoint is an isometry from $\mathcal{E}_P(s+t)$ onto a closed subspace of $\mathcal{E}_P(s) \otimes \mathcal{E}_P(t)$.

In order to define the covariance function, choose $T_1, T_2 \in \mathcal{U}_P$, and fix $t > 0$. The condition (2.1) implies that both operators $T_1(t)$ and $T_2(t)$ belong to $\mathcal{E}_P(t)$ and thus we can form their inner product $\langle T_1(t), T_2(t)\rangle_{\mathcal{E}_P(t)}$ as elements of this Hilbert space. More generally, if we are given an arbitrary finite partition

$$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

of the interval $[0,t]$ then we define a function $f_{\mathcal{P},t} : \mathcal{U}_p \times \mathcal{U}_P \to \mathbb{C}$ as follows

$$f_{\mathcal{P},t}(T_1, T_2) = \prod_{k=1}^{n} \langle T_1((t_k - t_{k-1}), T_2(t_k - t_{k-1}))\rangle_{\mathcal{E}_P(t_k - t_{k-1})}.$$

It was shown in [4] that there is a unique complex number $c_P(T_1, T_2)$ such that for every $t > 0$

$$\lim_{\mathcal{P}} f_{\mathcal{P},t}(T_1, T_2) = e^{tc_P(T_1,T_2)}, \tag{2.2}$$

the limit being taken over the increasing directed set of all finite partitions $\mathcal{P}$ of $[0,t]$. That defines the covariance function $c_P : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ of any CP semigroup P for which $\mathcal{U}_P \neq \emptyset$.

The covariance function is conditionally positive definite in the sense that if $T_1, T_2, \dots, T_n \in \mathcal{U}_P$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ satisfy $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$, then

$$\sum_{i,j=1}^{n} \lambda_i \bar{\lambda}_j c_P(T_i, T_j) \geq 0$$

(see [cpindex, Proposition 2.7]). More generally, if we are given any nonempty set X and a conditionally positive definite function $c : X \times X \to \mathbb{C}$ then there is a natural way to construct a Hilbert space $H(X,c)$. Briefly, c defines a positive semidefinite sesquilinear form $\langle \cdot, \cdot \rangle$ on the vector space V of all finitely nonzero functions $f : X \to \mathbb{C}$ for which

$$\sum_{x \in X} f(x) = 0,$$

by way of

$$\langle f, g \rangle = \sum_{x,y \in X} f(x) \bar{g}(y) c(x,y),$$

and $H(X,c)$ is obtained by completing the inner product space obtained by promoting $\langle \cdot, \cdot \rangle$ to the quotient of V by the subspace

$$N = \{ f \in V : \langle f, f \rangle = 0 \}.$$

The index $d_*(P)$ of a CP semigroup P is defined by

$$d_*(P) = \dim H(\mathcal{U}_P, c_P)$$

in the case where $\mathcal{U}_P \neq \emptyset$, and is defined by $d_*(P) = 2^{\aleph_0}$ if $\mathcal{U}_P = \emptyset$. Notice that in order to calculate $d_*(P)$ one must calculate a) the set $\mathcal{U}_P$ of all units of P, and b) the covariance function $c_P : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$; moreover, this must be done explicitly enough so that the dimension of $H(\mathcal{U}_P, c_P)$ is apparent. The purpose of this section is to carry out these calculations for the case of CP semigroups with *bounded* generator in terms of the structures associated with the generator by Proposition 1.20. The principal result is Theorem 2.3 below.

Remarks. The covariance function $c_P : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ is a conditionally positive definite function having the property that for every $t > 0$,

$$T_1, T_2 \in \mathcal{U}_P \mapsto e^{t c_P(T_1, T_2)} - \langle T_1(t), T_2(t) \rangle_{\mathcal{E}_P(t)}$$

is a positive definite function. It is a simple exercise to show that if $d : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ is any other function for which

$$T_1, T_2 \in \mathcal{U}_P \mapsto e^{t d(T_1, T_2)} - \langle T_1(t), T_2(t) \rangle_{\mathcal{E}_P(t)}$$

is positive definite for every $t > 0$, then the difference $d - c_P$ is a positive definite function on $\mathcal{U}_P \times \mathcal{U}_P$. Thus the covariance function c_P is characterized in this sense as the "smallest" function $c : \mathcal{U}_P \times \mathcal{U}_P \to \mathbb{C}$ with the property that $e^{t c(T_1, T_2)}$ dominates the inner products $\langle T_1(t), T_2(t) \rangle_{\mathcal{E}_P(t)}$ for every $t > 0$.

Theorem 2.3. *Let $L \in \mathcal{L}(M)$ be a bounded operator which generates a CP semigroup on $M = \mathcal{B}(H)$. Let $\mathcal{E}$ be a metric operator space satisfying the conditions of Proposition 1.20, so that L has the form*

$$L(x) = P_{\mathcal{E}}(x) + kx + xk^* \tag{2.3.1}$$

for some $k \in M$. The units of the CP semigroup $P = \{\exp(tL) : t \geq 0\}$ are described in terms of $\mathcal{E}$ and k as follows. For every $(c, v) \in \mathbb{C} \times \mathcal{E}$, let $T_{(c,v)}$ be the operator semigroup

$$T_{(c,v)}(t) = e^{ct} \exp t(v + k), \qquad t \geq 0. \tag{2.3.2}$$

Then $T_{(c,v)}$ is a unit of P and the map $(c, v) \in \mathbb{C} \times \mathcal{E} \mapsto T_{(c,v)}$ is a bijection of $\mathbb{C} \times \mathcal{E}$ onto the set $\mathcal{U}_P$ of units of P.

The covariance function $c_P : \mathcal{U}_P \times \mathcal{U}_P \mapsto \mathbb{C}$ of P is given by

$$c_P(T_{(c_1,v_1)}, T_{(c_2,v_2)}) = c_1 + \bar{c}_2 + \langle v_1, v_2 \rangle_{\mathcal{E}}\,, \tag{2.3.2}$$

and the index of P is $d_*(P) = \dim \mathcal{E}$.

proof. Notice first that the map $(c, v) \to T_{(c,v)}$ from $\mathbb{C} \times \mathcal{E}$ to operator semigroups is one-to-one. Indeed, choosing complex numbers c_1, c_2 and elements $v_1, v_2 \in \mathcal{E}$ such that $T_{(c_1,v_1)}(t) = T_{(c_2,v_2)}(t)$ for every t, it follows that the generators of these two semigroups are equal. Using (2.3.2) we find that

$$c_1\mathbf{1} + v_1 + k = c_2\mathbf{1} + v_2 + k.$$

Cancelling k and using $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$, we obtain $v_1 = v_2$ and $c_1 = c_2$ as asserted.

Now fix $(c, v) \in \mathbb{C} \times \mathcal{E}$. We will show that $T_{(c,v)}$ is a unit of P. In order to do that, we must find a real constant α such that each mapping

$$x \in M \mapsto e^{t\alpha} P_t(x) - T_{(c,v)}(t) x T_{(c,v)}(t)^* \tag{2.4}$$

is completely positive, $t \geq 0$. Noting that $T_{(c,v)}(t) = e^{ct} T_{(0,v)}(t)$, it is clearly enough to prove (2.4) for $c = 0$; in that case we will show that (2.4) is true for the constant $\alpha = \langle v, v \rangle_{\mathcal{E}}$.

To that end, we set

$$L_0(x) = (v + k)x + x(v + k)^*,$$

and we claim first that for $\alpha = \langle v, v \rangle_{\mathcal{E}}$ we have

$$L_0 \leq L + \alpha \cdot \iota_M, \tag{2.5}$$

ι_M denoting the identity map of M. Equivalently, after choosing an orthonormal basis $v_1, v_2, \dots$ for $\mathcal{E}$, we want to show that the mapping

$$x \mapsto \sum_j v_j x v_j^* - vx - xv^* + \alpha x \tag{2.6}$$

is completely positive. Indeed, since $v \in \mathcal{E}$ and (v_j) is an orthonormal basis, we can find a sequence $\lambda \in \ell^2$ such that $v = \sum_j \bar{\lambda}_j v_j$ (note the complex conjugate). Then

$$\alpha = \langle v, v \rangle = \sum_j |\lambda_j|^2,$$

and hence the term on the right side of (2.6) can be collected as follows,

$$\sum_j (v_j - \lambda_j \mathbf{1}) x (v_j - \lambda_j)^*.$$

The latter formula obviously defines a completely positive map.

In order to pass from (2.5) to its exponentiated version (2.4), we require

Lemma 2.7. *Let L_1, L_2 belong to $\mathcal{L}(M)$. Suppose that both generate CP semigroups and that $L_2 - L_1$ is completely positive. Then for every $t \geq 0$ the map*

$$\exp(tL_2) - \exp(tL_1)$$

is completely positive.

proof. Since the hypotheses on L_1 and L_2 are invariant under scaling by positive constants, it is enough to prove the assertion for $t = 1$. We can write $L_2 = L_1 + R$ where R is a completely positive map. By the Lie product formula [12, page 245], we have

$$\exp(L_2) = \exp(L_1 + R) = \lim_{n\to\infty} (\exp(1/nL_1)\exp(1/nR))^n,$$

the limit on n existing relative to the operator norm on $\mathcal{L}(M)$. Thus it suffices to show that for every n, we have

$$(\exp(1/nL_1)\exp(1/nR))^n \geq \exp L_1. \tag{2.8}$$

To see the latter, note that for *completely positive* maps A_k, B_k, $k = 1, 2$ we have

$$B_1 \geq A_1 \text{ and } B_2 \geq A_2 \implies B_1B_2 \geq A_1A_2.$$

Indeed, this follows from the fact that a composition of completely positive maps is completely positive, so that $A_1(B_2 - A_2) \geq 0$ and $(B_1 - A_1)B_2 \geq 0$ together imply that $B_1B_2 \geq A_1B_2 \geq A_1A_2$, and the assertion follows.

We apply this to (2.8) as follows. Letting ι_M denote the identity map of M we have

$$\exp(1/nR) = \iota_M + 1/nR + 1/2(1/nR)^2 + \cdots \geq \iota_M$$

because R is completely positive. Hence

$$\exp(1/nL_1)\exp(1/nR) \geq \exp(1/nL_1).$$

For the same reason,

$$(\exp(1/nL_1)\exp(1/nR))^2 \geq (\exp(1/nL_1))^2,$$

and so on until we obtain

$$(\exp(1/nL_1)\exp(1/nR))^n \geq (\exp(1/nL_1))^n = \exp(L_1).$$

This establishes (2.8) and completes the proof of Lemma 2.7 $\square$

Applying Lemma 2.7 to (2.5) we obtain

$$\exp(tL_0) \leq \exp(t(L + \alpha\iota_M)) = e^{t\alpha}P_t.$$

Noting that $L_0(x) = (v + k)x + x(v + k)^*$ is the generator of the semigroup

$$x \mapsto T_{(0,v)}(t)xT_{(0,v)}(t)^*,$$

we obtain formula (2.4) for the case $c = 0$. This completes the proof that each operator semigroup of the form $T_{(c,v)} \in \mathcal{U}_P$ is a unit of P.

We show next that the map $(c, v) \in \mathbb{R} \times \mathcal{E} \mapsto T_{(c,v)} \in \mathcal{U}_P$. is surjective. For that, the following result is essential.

Lemma 2.9. *Let L be a bounded operator in $\mathcal{L}(M)$ which generates a unital CP semigroup. Then every semigroup $T \in \mathcal{U}_P$ has a bounded generator.*

proof. Let T be an operator semigroup with the property that for every $t \geq 0$ the mapping

$$x \mapsto e^{\alpha t} P_t(x) - T(t)xT(t)^* \tag{2.10}$$

is completely positive, P_t denotine $\exp(tL)$ and α being some real constant. To show that the generator of T is a bounded operator, it is enough to show that T is continuous relative to the operator norm on M

$$\lim_{t \to 0+} \|T(t) - \mathbf{1}\| = 0. \tag{2.11}$$

To prove (2.11) we make use of the symbol as follows. Fix $t > 0$ and consider the operator mapping

$$L_t(x) = T(t)xT(t)^*, \qquad x \in M.$$

The symbol of L_t is found to be

$$\sigma_{L_t}(dx\, dy) = (T(t)x - xT(t))(T(t)^*y - yT(t)^*).$$

Now the symbol of a completely positive map R satisfies $\sigma_R(dx\, dx^*) \geq 0$. Hence if R_1 and R_2 are completely positive maps satisfying $0 \leq R_1 \leq R_2$ then we have $0 \leq \sigma_{L_1}(dx\, dx^*) \leq \sigma_{L_2}(dx\, dx^*)$. Thus, (2,10) implies that for all $x \in M$,

$$\begin{aligned}(T(t)x - xT(t))(T(t)x - xT(t))^* = \sigma_{L_t}(dx\, dx^*) &\leq e^{t\alpha}\sigma_{P_t}(dx\, dx^*) \\ &= e^{t\alpha}\sigma_{P_t - \iota_M}(dx\, dx^*),\end{aligned}$$

the last equality resulting from the fact that the identity map ι_M of M has symbol zero. From (1.4) and the previous formula we conclude that for all $x \in M$ satisfying $\|x\| \leq 1$, we have

$$\|T(t)x - xT(t)\|^2 = e^{t\alpha}\|\sigma_{P_t - \iota_M}(dx\, dx^*)\| \leq 4e^{t\alpha}\|P_t - \iota_M\|,$$

$\|P_t - \iota_M\|$ denoting the norm of $P_t - \iota_M$ as an element of $\mathcal{L}(M)$. Now

$$\|P_t - \iota_M\| = \|\exp(tL) - \iota_M\| \to 0$$

as $t \to 0$ because L is bounded. Since the norm of a derivation of $M = \mathcal{B}(H)$ of the form $D(x) = Tx - xT$ satisfies inequalities of the form

$$\inf_{\lambda \in \mathbb{C}} \|T - \lambda\mathbf{1}\| \leq \|D\| \leq 2 \inf_{\lambda \in \mathbb{C}} \|T - \lambda\mathbf{1}\|,$$

it follows that

$$\inf_{\lambda \in \mathbb{C}} \|T(t) - \lambda\mathbf{1}\| \leq \sup_{\|x\| \leq 1} \|T(t)x - xT(t)\| \to 0,$$

as $t \to 0$. Thus there exist complex scalars λ_t such that $\|T(t) - \lambda_t\mathbf{1}\| \to 0$ as $t \to 0$. Since the semigroup $\{T(t) : t \geq 0\}$ is strongly continuous, $T(t)$ must tend to $\mathbf{1}$ in the strong operator topology as $t \to 0$; hence $\lambda_t \to 1$ as $t \to 0$ and (2.11) follows $\square$

Now choose any unit $T \in \mathcal{U}_P$. There is a real constant α such that for every t the mapping

$$x \mapsto e^{t\alpha} P_t(x) - T(t)xT(t)^*$$

is completely positive. We will show that there is an element $(c, v) \in \mathbb{C} \times \mathcal{E}$ such that $T = T_{(c,v)}$. By replacing T with the semigroup $\{e^{-\alpha t/2}T(t) : t \geq 0\}$ (and adjusting c accordingly), we may clearly assume that $\alpha = 0$. By Lemma 2.9, there is a *bounded* operator $a \in M$ such that

$$T(t) = \exp(ta), \qquad t \geq 0$$

and we have to show that a has the form

$$a = c\mathbf{1} + v + k \tag{2.12}$$

for some complex scalar c and some $v \in \mathcal{E}$. For that, we claim first that the operator mapping

$$R(x) = L(x) - ax - xa^* \tag{2.13}$$

is completely positive. Indeed, since for every $t > 0$ the map

$$x \mapsto P_t(x) - e^{ta}xe^{ta^*} = (P_t(x) - x) - (e^{ta}xe^{ta^*} - x)$$

is completely positive, we may divide the latter by t and take the limit as $t \to 0+$ to obtain (2.13), after noting that

$$\lim_{t\to 0+} t^{-1}(P_t(x) - x) = L(x), \qquad \text{and}$$
$$\lim_{t\to 0+} t^{-1}(e^{ta}xe^{ta^*} - x) = ax + xa^*.$$

Using (2.3.1) we can write R in the form

$$R(x) = P_{\mathcal{E}}(x) + (k - a)x + x(k - a)^*.$$

The Corollary of Theorem 1.14 implies that there is a complex number d and an element $u \in \mathcal{E}$ such that

$$k - a = d\mathbf{1} + u,$$

and the required representation (2.12) follows after taking $c = -d$ and $v = -u$.

It remains to compute the covariance function c_P of formula (2.2) in these coordinates.

Lemma 2.14. *Let $v_1, \dots, v_n \in \mathcal{E}$ and let T_j be the unit of P defined by*

$$T_j(t) = \exp t(v_j + k), \qquad t \geq 0, 1 \leq j \leq n.$$

Then for every $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{i,j} \lambda_i \bar{\lambda}_j \langle v_i, v_j \rangle_{\mathcal{E}} \leq \sum_{i,j} \lambda_i \bar{\lambda}_j c_P(T_i, T_j).$$

proof. Notice that if an n-tuple $\lambda_1, \dots, \lambda_n$ satisfies the required inequality and c is an arbitrary complex number, then so does $c\lambda_1, \dots, c\lambda_n$. Thus it is enough to prove Lemma 2.14 for n-tuples λ_k satisfying $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$.

Choose $T_1, \dots, T_n \in \mathcal{U}_P$. By the remarks following (2.2) we have

$$\sum_{i,j} \lambda_i \bar{\lambda}_j \left\langle T_i(t), T_j(t) \right\rangle_{\mathcal{E}_P(t)} \leq \sum_{i,j} \lambda_i \bar{\lambda}_j e^{t c_P(T_i, T_j)}.$$

Equivalently, if for every $t > 0$ we set

$$A(t) = \lambda_1 T_1(t) + \dots + \lambda_n T_n(t),$$

then $A(t) \in \mathcal{E}_P(t)$ and we have

$$\left\langle A(t), A(t) \right\rangle_{\mathcal{E}_P(t)} \leq \sum_{i,j} \lambda_i \bar{\lambda}_j e^{t c_P(T_i, T_j)}.$$

It follows that the mapping

$$x \mapsto (\sum_{i,j} \lambda_i \bar{\lambda}_j e^{t c_P(T_i, T_j)}) P_t(x) - A(t) x A(t)^*$$

is completely positive. Since $\sum_j \lambda_j = 1$, this implies that

$$x \mapsto \sum_{i,j} \lambda_i \bar{\lambda}_j (e^{t c_P(T_i, T_j)} P_t(x) - x) - (A(t) x A(t)^* - x)$$

is completely positive. Notice that $A(t)$ is differentiable at $t = 0+$ and that $A'(0+) = \sum_j \lambda_j v_j + k$. Thus if we divide by t and take $\lim_{t \to 0+}$ we find that

$$x \mapsto (L(x) + \sum_{i,j} \lambda_i \bar{\lambda}_j c_P(T_i, T_j)) - ((\sum_j \lambda_j v_j) x + x (\sum_j \lambda_j v_j)^* + kx + xk^*)$$

is a completely positive map. Noting that $L(x) = P_{\mathcal{E}}(x) + kx + xk^*$, the terms involving k and k^* cancel and we are left with a completely positive map of the form

$$x \mapsto P_{\mathcal{E}}(x) - v(\lambda) x - x v(\lambda)^* + (\sum_{i,j} \lambda_i \bar{\lambda}_j c_P(T_i, T_j)) x.$$

where $v(\lambda) = \sum_j \lambda_j v_j \in \mathcal{E}$. From the corollary of Theorem 1.14 we deduce the required inequality

$$\sum_{i,j} \lambda_i \bar{\lambda}_j c_P(T_i, T_j) \geq \left\langle v(\lambda), v(\lambda) \right\rangle_{\mathcal{E}} = \sum_{i,j} \lambda_i \bar{\lambda}_j \left\langle v_i, v_j \right\rangle_{\mathcal{E}}.$$

□

Lemma 2.15. *Let $v \in \mathcal{E}$ and let $T \in \mathcal{U}_P$ be the semigroup $T(t) = \exp t(v + k)$, $t \geq 0$. Then*
$$c_P(T,T) \leq \langle v, v\rangle_{\mathcal{E}} \,.$$

proof. It has already been shown (see (2.4)) that the mapping

$$x \mapsto e^{t\langle v,v\rangle_{\mathcal{E}}} P_t(x) - T(t)xT(t)^*$$

is completely positive. It follows from the definition of $\mathcal{E}_P(t)$ that

$$\langle T(t), T(t)\rangle_{\mathcal{E}_P(t)} \leq e^{t\langle v,v\rangle_{\mathcal{E}}}$$

for every $t > 0$. Thus for every finite partition

$$\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$$

of the interval $[0,t]$ we have

$$\prod_{k=1}^{n} \langle T(t_k - t_{k-1}), T(t_k - t_{k-1})\rangle_{\mathcal{E}_P(t_k - t_{k-1})} \leq e^{t\langle v,v\rangle_{\mathcal{E}}}.$$

Noting the definition (2.2) of c_P we conclude that

$$e^{tc_P(T,T)} \leq e^{t\langle v,v\rangle_{\mathcal{E}}}$$

for every $t > 0$, and the asserted inequality follows □

To complete the proof of Theorem 2.3, choose complex numbers c_1, c_2, choose $v_1, v_2 \in \mathcal{E}$, and let T_1, T_2 be the units of P defined by

$$T_k(t) = T_{(0,v_k)} e^{ct} \exp t(v_k + k), \qquad t \geq 0.$$

Consider the self-adjoint 2×2 matrices $A = (a_{ij})$ and $B = (b_{ij})$ defined by

$$\begin{aligned} a_{ij} &= \langle v_i, v_j\rangle_{\mathcal{E}} \,, \\ b_{ij} &= c_P(T_i, T_j). \end{aligned}$$

Lemma 2.14 implies that $B - A \geq 0$, while Lemma 2.15 implies that the diagonal terms of $B - A$ are nonpositive. Hence the trace of $B - A$ is nonpositive and it follows that $A = B$. Comparing the off-diagonal terms we obtain

$$c_P(T_1, T_2) = \langle v_1, v_2\rangle_{\mathcal{E}} \,. \tag{2.16}$$

Now $T_k = T_{(0,v_k)}$. We bring in the c_k as follows. From the definition (2.2) of c_P and the fact that $T_{(c,v)}(t) = e^{ct} T_{(0,v)}(t)$, it follows that

$$c_P(T_{(c_1,v_1)}, T_{(c_2,v_2)}) = c_1 + \bar{c}_2 + c_P(T_{(0,v_1)}, T_{(0,v_2)}).$$

Together with (2.16), this implies

$$c_P(T_{(c_1,v_1)}, T_{(c_2,v_2)}) = c_1 + \bar{c}_2 + \langle v_1, v_2\rangle_{\mathcal{E}} \,,$$

as required.

Once one has this formula we obtain $d_*(P) = \dim \mathcal{E}$ by a straightforward calculation. Indeed, the preceding arguments show that we may identify $\mathcal{U}_P$ with $\mathbb{C} \times \mathcal{E}$ in such a way that the covariance function becomes

$$c_P((a, v), (b, w)) = a + \bar{b} + \langle v, w \rangle .$$

Now as in [1, Proposition 5.3] one computes directly that

$$d_*(P) = \dim H(\mathbb{C} \times \mathcal{E}, c_P) = \dim \mathcal{E}.$$

□

Together with the results of [4], Theorem 2.3 shows how to calculate the index of the minimal E_0-semigroup dilation of any unital CP semigroup having bounded generator:

Corollary 2.17. *Let L be a bounded linear map on $\mathcal{B}(H)$ which generates a semigroup $P = \{P_t = \exp(tL) : t \geq 0\}$ of normal completely positive maps satisfying $P_t(\mathbf{1}) = \mathbf{1}$ for every $t \geq 0$. Let $\alpha = \{\alpha_t : t \geq 0\}$ be the minimal E_0-semigroup-dilation of P. Then $\mathcal{U}_\alpha \neq \emptyset$ and the numerical index $d_*(\alpha)$ is the rank of L.*

proof. By Proposition 1.20, we find an operator k and a metric operator space $\mathcal{E}$ satisfying $\mathcal{E} \cap \mathbb{C}\mathbf{1} = \{0\}$, and

$$L(x) = P_{\mathcal{E}}(x) + kx + xk^*, \qquad x \in \mathcal{B}(H).$$

By [cpindex, Theorem 4.9] we have $d_*(\alpha) = d_*(P)$, while by Theorem 2.3 above we have $d_*(p) = \dim \mathcal{E} = \operatorname{rank}(L)$. □

Remarks. It is possible, of course, for the rank of L to be 0; equivalently, $\mathcal{E} = \{0\}$ and hence $\mathbb{C} \times \mathcal{E} \cong \mathbb{C}$. In this event P is a semigroup of $*$-automorphisms of $\mathcal{B}(H)$ and $\alpha = P$. This degenerate case is discussed more fully in the proof of Corollary 4.25 below.

3. Completeness of the covariance function. It is possible for different CP semigroups P, Q to have the same covariance function in the sense that P and Q have the same set of units and

$$c_P(S, T) = c_Q(S, T), \qquad S, T \in \mathcal{U}_P = \mathcal{U}_Q.$$

For example, let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $M = \mathcal{B}(H)$. For every $t > 0$ let $\mathcal{E}_\alpha(t)$ be the metric operator space associated with α_t. Since α_t is an endomorphism we have in this case

$$\mathcal{E}_\alpha(t) = \{T \in \mathcal{B}(H) : \alpha_t(x)T = Tx, \quad x \in \mathcal{B}(H)\},$$

and the inner product in $\mathcal{E}_\alpha(t)$ is defined by

$$\langle S, T \rangle_{\mathcal{E}_\alpha(t)} \mathbf{1} = T^* S.$$

Assuming that $\mathcal{U}_\alpha \neq \emptyset$, we can form a closed subspace $\mathcal{D}(t)$ of the Hilbert space $\mathcal{E}_\alpha(t)$ generated by all finite products obtained from units as follows

$$\mathcal{D}(t) = [u_1(t_1)u_2(t_2) \dots u_n(t_n) : u_k \in \mathcal{U}_\alpha, t_k > 0, t_1 + \dots + t_n = t, n \geq 1\}.$$

$\mathcal{D}(t)$ is itself a metric operator space, and it determines a $*$-endomorphism β_t of $\mathcal{B}(H)$ by way of

$$\beta_t(x) = \sum_k v_k x v_k^*, \qquad x \in \mathcal{B}(H)$$

$\{v_1, v_2 \dots\}$ denoting an orthonormal basis for $\mathcal{D}(t)$. Since $\mathcal{D}(s+t)$ is spanned by the set of products $\{ST : S \in \mathcal{D}(s), T \in \mathcal{D}(t)\}$ it follows that $\beta_{s+t} = \beta_s\beta_t$. If we set $\beta_0 = \iota_{\mathcal{B}(H)}$, then $\beta = \{\beta_t : t \geq 0\}$ is a semigroup of normal $*$-endomorphisms of $\mathcal{B}(H)$. The individual maps β_t are not necessarily unit preserving, but we do have

$$\beta_t \leq \alpha_t, \qquad t \geq 0, \tag{3.2}$$

and in fact the semigroup β is continuous. Moreover, the following conditions are equivalent for every $t > 0$,

CS1 H is spanned by $\{T\xi : T \in \mathcal{D}(t), \xi \in H\}$
CS2 $\beta_t(\mathbf{1}) = \mathbf{1}$
CS3 $\mathcal{E}_\alpha(t) = \mathcal{D}(t)$
CS4 $\beta_t = \alpha_t$

(see [1, §7]), and when these conditions are satisfied for some $t > 0$ then they are satisfied for every $t > 0$. When that is the case, α is called **completely spatial**. If α is any semigroup for which $\mathcal{U}_\alpha \neq \emptyset$ then α is called **spatial** and in this case we refer to its associated semigroup β as the **standard part** of α.

Now a straightforward computation shows that α and β have the same set of units and the same covariance function. So if α is any spatial E_0-semigroup which is *not* completely spatial, then α, β provide rather extreme examples of distinct CP semigroups which have the same covariance function. The existence of such E_0-semigroups is established in [10]. The following result asserts that this phenomenon cannot occur for CP semigroups which have bounded generators.

Theorem 3.3. *Let P_1, P_2 be CP semigroups with bounded generators L_1, L_2. Suppose that P_1 and P_2 have the same set of units and*

$$c_{P_1}(T, T') = c_{P_2}(T, T'), \qquad T, T' \in \mathcal{U}_{P_1} = \mathcal{U}_{P_2}.$$

Then $L_1 = L_2$, and hence $P_1 = P_2$.

poroof. By Proposition 1.20 we can find metric operator spaces $\mathcal{E}_1$, $\mathcal{E}_2$ and operators k_1, $k_2 \in \mathcal{B}(H)$ satisfying

$$\mathcal{E}_1 \cap \mathbb{C}\mathbf{1} = \mathcal{E}_2 \cap \mathbb{C}\mathbf{1} = \{0\} \tag{3.4}$$

and

$$L_j(x) = P_{\mathcal{E}_j}(x) + k_j x + x k_j^*, \qquad x \in \mathcal{B}(H)$$

for $j = 1, 2$. Theorem 2.3 asserts that the most general unit of P_j is a semigroup of the form $T(t) = \exp(ta)$, where a belongs to the set of operators $\mathcal{E}_j + \mathbb{C}\mathbf{1} + k_j$. Thus, the hypothesis $\mathcal{U}_{P_1} = \mathcal{U}_{P_2}$ is equivalent to the equality of the two sets

$$\mathcal{E}_1 + \mathbb{C}\mathbf{1} + k_1 = \mathcal{E}_2 + \mathbb{C}\mathbf{1} + k_2.$$

Now if E_1 and E_2 are linear subspaces of a vector space V and k_1, k_2 are elements of V such that $E_1 + k_1 = E_2 + k_2$ then we must have $E_1 = E_2$ and $k_2 - k_1 \in E_2$. Taking $E_j = \mathcal{E}_j + \mathbb{C}\mathbf{1}$ it follows that

$$\mathcal{E}_1 + \mathbb{C}\mathbf{1} = \mathcal{E}_2 + \mathbb{C}\mathbf{1}, \tag{3.6}$$

and

$$k_2 = k_1 + v_2 + \lambda\mathbf{1}, \tag{3.7}$$

where $v_2 \in \mathcal{E}_2$ and λ is a scalar.

Associated with any pair of operator spaces $\mathcal{E}_1$, $\mathcal{E}_2$ satisfying (3.4) and (3.6) there is an isomorphism of vector spaces $\theta : \mathcal{E}_1 \to \mathcal{E}_2$. Indeed, since every element $v \in \mathcal{E}_1$ has a unique decomposition

$$v = v' + \lambda\mathbf{1}$$

where $v' \in \mathcal{E}_2$ and $\lambda \in \mathbb{C}$, we can define a linear functional $f : \mathcal{E}_1 \to \mathbb{C}$ and a linear isomorphism $\theta : \mathcal{E}_1 \to \mathcal{E}_2$ by

$$v = \theta(v) + f(v)\mathbf{1}. \tag{3.8}$$

We will show first that the linear isomorphism $\theta : \mathcal{E}_1 \to \mathcal{E}_2$ defined by (3.8) is a unitary operator in that for any pair of elements $v, v' \in \mathcal{E}_1$ we have $\langle \theta(v), \theta(v')\rangle_{\mathcal{E}_2} = \langle v, v'\rangle_{\mathcal{E}_1}$. To that end, fix $v, v' \in \mathcal{E}_1$ and consider the units T, T' of P_1 defined by

$$T(t) = \exp t(v + k_1), \quad T'(t) = \exp t(v' + k_1).$$

Combining formula (2.2) with Theorem 2.3, we find that for every $t > 0$,

$$c_{P_1}(T, T') = \langle v, v'\rangle_{\mathcal{E}_1}. \tag{3.9}$$

Now we must consider T and T' relative to the coordinates associated with P_2. In order to do that, we use (3.7) and (3.8) to write

$$v + k_1 = \theta(v) + f(v)\mathbf{1} + k_1 = (\theta(v) - v_2) + (f(v) - \lambda)\mathbf{1} + k_2,$$

and similarly

$$v' + k_1 = (\theta(v') - v_2) + (f(v') - \lambda)\mathbf{1} + k_2.$$

Considering T and T' as units of P_2, we have in the notation of formula (2.3.3),

$$T = T_{(f(v)-\lambda, \theta(v)-v_2)}, \quad T' = T_{(f(v')-\lambda, \theta(v')-v_2)},$$

and corresponding to (3.9) we have

$$c_{P_2}(T, T') = f(v) - \lambda + \bar{f}(v') - \bar{\lambda} + \langle \theta(v) - v_2, \theta(v') - v_2\rangle_{\mathcal{E}_2}. \tag{3.10}$$

Since $c_{P_1} = c_{P_2}$, we may equate the right sides of (2.23) and (2.24) to obtain

$$\langle v, v'\rangle_{\mathcal{E}_1} = f(v) - \lambda + \bar{f}(v') - \bar{\lambda} + \langle \theta(v) - v_2, \theta(v') - v_2\rangle_{\mathcal{E}_2}. \tag{3.11}$$

The identity (3.11) implies that θ is unitary. To see that, consider the sesquilinear form representing the difference

$$D(v,v') = \langle v, v'\rangle_{\mathcal{E}_1} - \langle \theta(v), \theta(v')\rangle_{\mathcal{E}_2}.$$

We can rewrite (2.25) in the form

$$D(v,v') = g(v) + \bar{g}(v'), \tag{3.12}$$

where $g : \mathcal{E}_1 \to \mathbb{C}$ is the function $g(v) = f(v) - \langle \theta(v), v_2\rangle_{\mathcal{E}_1} + 1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} - \lambda$. For every $t > 0$ we can write

$$D(v,v') = t^{-2}D(tv,tv') = t^{-2}(g(tv) + \bar{g}(tv')),$$

and clearly

$$t^{-2}g(tv) = t^{-1}(f(v) - \langle \theta(v), v_2\rangle) + t^{-2}(1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} - \lambda)$$

tends to zero as $t \to \infty$. Thus,

$$D(v,v') = 0 = g(v) + \bar{g}(v') \tag{3.13}$$

for every $v, v' \in \mathcal{E}_1$.

We claim next that (3.13) implies

$$\lambda = 1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} + ic, \qquad \text{and} \tag{3.14}$$

$$f(v) = \langle \theta(v), v_2\rangle_{\mathcal{E}_2}, \tag{3.15}$$

where $i = \sqrt{-1}$ and c is a real number. Indeed, setting $v = v' = 0$ in the equation $g(v) + \bar{g}(v') = 0$ (3.13) we obtain

$$1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} - \lambda + 1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} - \bar{\lambda} = 0,$$

hence (3.14). Thus the linear functional $\rho(v) = f(v) - \langle \theta(v), v_2\rangle_{\mathcal{E}_2}$ satisfies

$$\rho(v) + \bar{\rho}(v') = g(v) + \bar{g}(v') = 0$$

for all $v, v' \in \mathcal{E}_1$ and (3.15) follows after setting $v' = 0$.

From (3.7) and (3.14) we obtain

$$k_2 = k_1 + v_2 + 1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2}\mathbf{1} + ic\mathbf{1},$$

so for all $x \in \mathcal{B}(H)$ we have

$$k_2x + xk_2^* = k_1x + xk_1^* + v_2x + xv_2^* + \langle v_2, v_2\rangle_{\mathcal{E}_2} x. \tag{3.16}$$

Since $L_2(x) = P_{\mathcal{E}_2}(x) + k_2x + xk_2^*$ it follows that

$$L_2(x) = P_{\mathcal{E}_2}(x) + v_2x + xv_2^* + \langle v_2, v_2\rangle_{\mathcal{E}_2} x + k_1x + xk_1^*. \tag{3.17}$$

We will show now that the right side of (3.17) is $L_1(x)$; equivalently, we claim that

$$P_{\mathcal{E}_2}(x) + v_2x + xv_2^* + \langle v_2, v_2\rangle_{\mathcal{E}_2} x = P_{\mathcal{E}_1}(x). \tag{3.18}$$

To see that, let $u_1, u_2, \dots$ be an orthonormal basis for $\mathcal{E}_1$. Then $\theta(u_1), \theta(u_2)\dots$ is an orthonormal basis for $\mathcal{E}_2$ and if we set $\mu_k = \langle v_2, \theta(u_k)\rangle_{\mathcal{E}_2}$ then the sequence $(\mu_1, \mu_2, \dots)$ belongs to ℓ^2 and $v_2 = \sum_k \mu_k\theta(u_k)$. Thus we have

$$v_2x + xv_2^* + \langle v_2, v_2\rangle_{\mathcal{E}_2} x = \sum_k (\mu_k\theta(u_k)x + x\bar{\mu}_k\theta(u_k)^* + |\mu_k|^2 x),$$

while

$$P_{\mathcal{E}_2}(x) = \sum_k \theta(u_k)x\theta(u_k)^*,$$

so that the left side of (3.18) can be written

$$\sum_k (\theta(u_k) + \bar{\mu}_k\mathbf{1})x(\theta(u_k) + \bar{\mu}_k\mathbf{1})^*.$$

Noting that for each k,

$$\theta(u_k) + \bar{\mu}_k\mathbf{1} = \theta(u_k) + \langle\theta(u_k), v_2\rangle_{\mathcal{E}_2} \mathbf{1} = \theta(u_k) + f(u_k)\mathbf{1} = u_k$$

by the definition (3.8) of θ and f, we find that the last expression reduces to

$$\sum_k u_kxu_k^* = P_{\mathcal{E}_1}(x),$$

as asserted. That completes the proof of Theorem 3.3 $\square$

Remark 3.18. The proof of Theorem 3.3 gives somewhat more information than is contained in its statement. For example, suppose that $\mathcal{E}_1$, $\mathcal{E}_2$ are two metric operator spaces satisfying $\mathcal{E}_1 \cap \mathbb{C}\mathbf{1} = \mathcal{E}_2 \cap \mathbf{1} = \{0\}$, and let k_1, k_2 be two operators such that the corresponding generators are the same:

$$P_{\mathcal{E}_1}(x) + k_1x + xk_1^* = P_{\mathcal{E}_2} + k_2x + xk_2^*, \qquad x \in \mathcal{B}(H).$$

Then the proof of Theorem 3.3 implies that

$$\mathcal{E}_1 + \mathbb{C}\mathbf{1} = \mathcal{E}_2 + \mathbb{C}\mathbf{1}. \tag{3.19}$$

(3.19) allows one to define a linear isomorphism $\theta : \mathcal{E}_1 \to \mathcal{E}_2$ and a linear functional f on $\mathcal{E}_1$ by

$$v = \theta(v) + f(v)\mathbf{1}, \qquad v \in \mathcal{E}_1. \tag{3.20}$$

The same argument shows that θ is a unitary operator and that the unique element $v_2 \in \mathcal{E}_2$ defined by

$$f(v) = \langle\theta(v), v_2\rangle_{\mathcal{E}_2}, \qquad v \in \mathcal{E}_1 \tag{3.21}$$

satisfies

$$k_2 = k_1 + v_2 + (1/2\,\langle v_2, v_2\rangle_{\mathcal{E}_2} + ic)\mathbf{1} \tag{3.22}$$

where c is a real constant.

Conversely, if we start with two pairs $(\mathcal{E}_1, k_1)$, $(\mathcal{E}_2, k_2)$ satisfying (3.19)–(3.22) (so that the linear isomorphism θ defined by (3.20) is a unitary operator) along with $\mathcal{E}_1 \cap \mathbb{C}\mathbf{1} = \mathcal{E}_2 \cap \mathbb{C}\mathbf{1} = \{0\}$, then both $(\mathcal{E}_1, k_1)$ and $(\mathcal{E}_2, k_2)$ give rise to the same generator

$$L(x) = P_{\mathcal{E}_1}(x) + k_1x + xk_1^* = P_{\mathcal{E}_2}(x) + k_2x + xk_2^*.$$

4. Minimal dilations.

Every unital CP semigroup $P = \{P_t : t \geq 0\}$ acting on $\mathcal{B}(H)$ has a minimal dilation to an E_0-semigroup $\alpha = \{\alpha_t : t \geq 0\}$ acting on $\mathcal{B}(K)$, where K is a Hilbert space which contains H as a subspace [2,5,6]. Recall that an E_0-semigroup α is said to be completely spatial if there are "enough" units in the sense that the equivalent conditions CS1–CS4 of the preceding section are satisfied. The completely spatial E_0-semigroups constitute the best-understood class. All of the basic examples are of this type, and they are classified up to cocycle conjugacy by their numerical index $d_*(\alpha)$ [1, Corollary of Proposition 7.2]. Thus, if one knows that an E_0-semigroup α is completely spatial and has numerical index $n = 1, 2, \ldots, \infty$, then α must be conjugate to a cocycle perturbation of the CAR/CCR flow of index n.

The purpose of this section is to show that if a CP semigroup has a *bounded* generator then its minimal dilation is completely spatial (Theorem 4.8 below). The proof of Theorem 4.8 is based on Theorem 3.3 and the following result.

Theorem 4.1. *Let $P = \{P_t : t \geq 0\}$ be a CP semigroup acting on $\mathcal{B}(H)$ which has a bounded generator. Let $\{Q_t : t > 0\}$ be a family of normal completely positive maps on $\mathcal{B}(H)$ satisfying the two conditions*

$$Q_t \leq P_t, \tag{4.1.1}$$

$$Q_{s+t} = Q_s Q_t, \tag{4.1.2}$$

for all $s, t > 0$, and which is not the trivial family $Q_t = 0$, $t > 0$. Let Q_0 be the identity map of $\mathcal{B}(H)$. Then $\{Q_t : t \geq 0\}$ is also a CP semigroup having bounded generator.

Our proof of this result requires the following estimate.

Lemma 4.2. *Let P be a normal completely positive linear map on $M = \mathcal{B}(H)$ and let σ_P be its symbol. Then we have*

$$\inf_{\lambda > 0} \|P - \lambda \iota_M\| \leq 6\|P\|^{1/2} \sup_{\|x\| \leq 1} \|\sigma_P(dx^*\, dx)\|^{1/2},$$

where ι_M denotes the identity map of M.

Remark. In proving this estimate we will make use of the following bit of lore. Let $N \subseteq \mathcal{B}(K)$ be an amenable von Neumann algebra and let T be an operator on K. Then there is an operator T' in the commutant of N such that

$$\|T - T'\| = \sup\{\|Tx - xT\| : x \in N, \|x\| \leq 1\}. \tag{4.3}$$

Indeed, the operator T' may be obtained by a familiar averaging process, in which one uses an invariant mean on a suitable subgroup G of the unitary group in N to average quantities of the form uTu^*, $u \in G$, after noting that for every such unitary operator u, $\|uTu^* - T\|$ is dominated by the right side of (4.3).

proof of Lemma 4.2. Because of Stinespring's theorem there is a Hilbert space K, a normal representation $\pi : M \to \mathcal{B}(K)$ and an operator $V \in \mathcal{B}(H, K)$ such that $P(x) = V^*\pi(x)V$, $x \in M$. As in the proof of Lemma 1.19, the symbol of P is related to V and π by way of

$$(\pi(x)V - Vx)^*(\pi(x)V - Vx) = \sigma_P(dx^*\, dx),$$

and hence

$$\|Vx - \pi(x)V\|^2 = \|\sigma_P(dx^* \, dx)\|.$$

Setting $\|\sigma_P\| = \sup_{\|x\|\leq 1} \|\sigma_P(dx^* \, dx)\|$, we obtain

$$\sup_{\|x\|\leq 1} \|Vx - \pi(x)V\| \leq \|\sigma_P\|^{1/2}. \tag{4.4}$$

Now consider the von Neumann algebra $N \subseteq \mathcal{B}(H \oplus K)$ and the operator $T \in \mathcal{B}(H \oplus K)$ defined by

$$N = \{\begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix} : x \in M\}, \qquad T = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

One finds that

$$T\begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix} - \begin{pmatrix} x & 0 \\ 0 & \pi(x) \end{pmatrix}T = \begin{pmatrix} 0 & -(Vx - \pi(x)V)^* \\ Vx - \pi(x)V & 0 \end{pmatrix}.$$

The norm of the operator on the right is $\|Vx - \pi(x)V\|$, hence

$$\begin{aligned} &\sup\{\|Ty - yT\| : y \in N, \|y\| \leq 1\} = \\ &\sup\{\|Vx - \pi(x)V\| : x \in M, \|x\| \leq 1\} \leq \|\sigma_P\|^{1/2}. \end{aligned}$$

Using (4.3) we find an operator $T' \in N'$ satisfying $\|T - T'\| \leq \|\sigma_P\|^{1/2}$. A straightforward computation shows that operators in the commutant of N must have the form

$$T' = \begin{pmatrix} A & Y^* \\ X & B \end{pmatrix}$$

where A is a scalar multiple of the identity of $\mathcal{B}(H)$, B belongs to the commutant of $\pi(M)$, and X and Y are intertwining operators, $Xx = \pi(x)X$, $Yx = \pi(x)Y$, $x \in M$. It follows that there is such an X for which

$$\|V - X\| \leq \|T - T'\| \leq \|\sigma_P\|^{1/2}. \tag{4.5}$$

Since X^*X commutes with $M = \mathcal{B}(H)$ we must have $X^*X = \lambda\mathbf{1}_H$ for some scalar $\lambda \geq 0$, and hence

$$\|P(x) - \lambda \cdot x\| = \|V^*\pi(x)V - X^*\pi(x)X\| \leq 2\|V - X\| \cdot \|x\| \max(\|V\|, \|X\|).$$

Note that $\max(\|V\|, \|X\|) \leq 3\|P\|^{1/2}$. Indeed, since $V^*V = P(\mathbf{1})$ we have $\|V\| \leq \|P\|^{1/2}$, and by (1.4) we can estimate $\|X\|$ by way of

$$\|X\| \leq \|V\| + \|V - X\| \leq \|P\|^{1/2} + \|\sigma_P\|^{1/2} \leq 3\|P\|^{1/2}.$$

Using (4.5) we arrive at the desired inequality $\|P - \lambda \cdot \iota_M\| \leq 6\|P\|^{1/2}\|\sigma_P\|^{1/2}$ $\square$

proof of Theorem 4.1. Let $\{Q_t : t > 0\}$ satisfy (4.1.1) and (4.1.2). We will show that

$$\lim_{t\to 0+} \|Q_t - \iota\| = 0, \tag{4.6}$$

ι denoting the identity map $\iota(x) = x$, $x \in \mathcal{B}(H)$. Theorem 4.1 follows immediately since under these conditions the semigroup $\{Q_t : t \geq 0\}$ becomes a continuous semigroup of elements in the Banach algebra of all normal linear mappings L on $\mathcal{B}(H)$ with the uniform norm

$$\|L\| = \sup_{\|x\| \leq 1} \|L(x)\|.$$

In order to prove (4.6), we claim first that there is a family λ_t, $t > 0$ of non-negative numbers such that $\|Q_t - \lambda_t \iota\| \to 0$ as $t \to 0+$. Indeed, since $P_t - Q_t$ is completely positive for every $t > 0$ we have $0 \leq \sigma_{Q_t}(dx\, dx^*) \leq \sigma_{P_t}(dx\, dx^*)$ for every $x \in M$; hence

$$\|\sigma_{Q_t}(dx\, dx^*)\| \leq \|\sigma_{P_t}(dx\, dx^*)\| = \|\sigma_{P_t - \iota}(dx\, dx^*)\| \leq 4\|P_t - \iota\|\, \|x\|^2.$$

Using Lemma 4.2 together with the latter inequality we find that

$$\begin{aligned} \inf_{\lambda > 0} \|Q_t - \lambda\iota\| \leq 6\|Q_t\|^{1/2} \sup_{\|x\| \leq 1} \|\sigma_{Q_t}(dx\, dx^*)\|^{1/2} \leq \\ 12\|Q_t\|^{1/2}\|P_t - \iota\|^{1/2} \leq 12\|P_t\|^{1/2}\|P_t - \iota\|^{1/2} \end{aligned}$$

and the claim follows because $\|P_t - \iota\|$ tends to 0 as $t \to 0+$.

It remains to prove that $\lambda_t \to 1$ as $t \to 0+$. To that end, we claim

$$\lim_{t \to 0+} \|Q_t\| = 1. \tag{4.7}$$

Indeed, since $\|Q_t\| \leq \|P_t\|$ and $\|P_t - \iota\| \to 0$ as $t \to 0+$, we have

$$\limsup_{t \to 0+} \|Q_t\| \leq \limsup_{t \to 0+} \|P_t\| = \|\iota\| = 1.$$

So if (4.7) fails then we must have

$$\liminf_{t \to 0+} \|Q_t\| < 1,$$

and in that event we can pick $r < 1$ such that $\liminf_{s \to 0+} \|Q_s\| < r$. Let $R > 1$ be close enough to 1 so that $rR < 1$. Then for sufficiently small t we have both $\|Q_t\| \leq R$ and $\inf_{0<s<t} \|Q_s\| \leq r$. For such a t we can find $0 < s < t$ such that $\|Q_s\| \leq r$, and hence

$$\|Q_t\| \leq \|Q_s\| \cdot \|Q_{t-s}\| \leq rR.$$

Thus $\|Q_t\| \leq \|Q_{t/n}\|^n \leq (rR)^n$ for every $n = 1, 2, \dots$ and hence $\|Q_t\| = 0$ for all sufficiently small t. Because of the semigroup property it follows that $\|Q_t\|$ is identically zero, contradicting our hypothesis on Q and proving (4.7).

To see that $\lambda_t \to 1$ as $t \to 0+$ write

$$\lambda_t = \|(\lambda_t \iota - Q_t) + Q_t\|,$$

and use $\lim_{t \to 0+} \|Q_t - \lambda_t \iota\| = 0$ together with (4.7) $\square$

Following is the principal result of this section.

Theorem 4.8. *Let $P = \{P_t : t \geq 0\}$ be a unital CP semigroup having a bounded generator. Then the minimal dilation of P to an E_0-semigroup is completely spatial.*

proof. Let $\alpha = \{\alpha_t : t \geq 0\}$ be an E_0-semigroup acting on $\mathcal{B}(K)$, $K \supseteq H$ which is the minimal dilation of P. Let $\beta = \{\beta_t : t \geq 0\}$ be the standard part of α as defined in section 3. Letting $\{\mathcal{D}(t) : t > 0\}$ be the family of metric operator spaces of (3.1), we have to show that

$$K = [A\xi : A \in \mathcal{D}(t), \xi \in K],$$

or equivalently, that $\beta_t(\mathbf{1}_K) = \mathbf{1}_K$ for every t.

Now in general, since β is a semigroup of endomorphisms of $\mathcal{B}(K)$, the projections $\beta_t(\mathbf{1}_K)$ decrease as t increases, and the limit projection

$$e_\infty = \lim_{t \to \infty} \beta_t(\mathbf{1}_K) \tag{4.9}$$

is fixed under the action of β. The compression β^∞ of the semigroup β to the corner $e_\infty \mathcal{B}(K) e_\infty$ can be viewed as a semigroup of *unit-preserving* endomorphisms of $\mathcal{B}(e_\infty K)$. It is not quite obvious that β^∞ is an E_0-semigroup since we have not proved that β is continuous. While it is possible to establish that directly, we will not have to do so since the following result implies that β^∞ is actually a compression of α.

Lemma 4.10. *The projection e_∞ satisfies $\alpha_t(e_\infty) \geq e_\infty$ for every $t \geq 0$, and the compression of α_t to $e_\infty \mathcal{B}(K) e_\infty$ is β_t^∞ for every $t \geq 0$. In particular, β^∞ defines an E_0-semigroup acting on $\mathcal{B}(e_\infty K)$.*

proof. We claim first that for every $x \in \mathcal{B}(K)$ we have

$$\alpha_t(x)\beta_t(\mathbf{1}_K) = \beta_t(x) = \beta_t(\mathbf{1}_K)\alpha_t(x). \tag{4.11}$$

Indeed, if we let $u_1, u_2, \dots$ be an orthonormal basis for $\mathcal{D}(t)$ then we have

$$\beta_t(x) = \sum_j u_j x u_j^*, \qquad x \in \mathcal{B}(K).$$

Since $\mathcal{D}(t) \subseteq \mathcal{E}_\alpha(t)$ it follows that for every j,

$$\alpha_t(x) u_j u_j^* = u_j x u_j^* = u_j u_j^* \alpha_t(x),$$

and (4.11) follows by summing on j. Taking $x = e_\infty$ in (4.11) and using $e_\infty = \beta_t(e_\infty)e_\infty$, we obtain

$$\alpha_t(e_\infty)e_\infty = \alpha_t(e_\infty)\beta_t(e_\infty)e_\infty = \beta_t(e_\infty)e_\infty = e_\infty,$$

hence $\alpha_t(e_\infty) \geq e_\infty$.

Now choose an operator $x \in e_\infty \mathcal{B}(K) e_\infty$. We have

$$\begin{aligned} e_\infty \alpha_t(x) e_\infty &= e_\infty \beta_t(\mathbf{1}_K)\alpha_t(x) e_\infty = e_\infty \beta_t(x) e_\infty \\ &= \beta_t(e_\infty)\beta_t(x)\beta_t(e_\infty) = \beta_t(e_\infty x e_\infty) = \beta_t(x), \end{aligned}$$

as asserted □

Let $p_0 \in \mathcal{B}(K)$ be the projection onto H. Then of course $\alpha_t(p_0) \geq p_0$, and after identifying $\mathcal{B}(H)$ with $p_0\mathcal{B}(K)p_0$ we have

$$P_t(x) = p_0\alpha_t(x)p_0, \qquad t \geq 0, x \in \mathcal{B}(H). \tag{4.12}$$

We will prove that

$$p_0 \leq e_\infty = \lim_{t\to\infty} \beta_t(\mathbf{1}_K). \tag{4.13}$$

Granting (4.13) for a moment, then Lemma 4.10 asserts that the compression of α to $e_\infty\mathcal{B}(K)e_\infty$ is an E_0-semigroup which is obviously an intermediate dilation of P. By the *definition* of minimal dilation given in [2] we may conclude that $e_\infty = \mathbf{1}_K$. In particular, $\beta_t(\mathbf{1}_K) = \mathbf{1}_K$ for every t, proving that α is completely spatial.

Thus the proof of Theorem 4.8 is reduced to establishing (4.13). This will be done indirectly, by considering the family of maps $Q = \{Q_t : t \geq 0\}$ on $\mathcal{B}(H)$ obtained by compressing the semigroup β:

$$Q_t(x) = p_0\beta_t(x)p_0, \qquad x \in \mathcal{B}(H), t \geq 0. \tag{4.14}$$

We will show that Q is a CP semigroup with bounded generator, which has the same set of units and the same covariance function as P. Theorem 3.3 will then imply that $Q_t = P_t$ for all t, and in particular $Q_t(p_0) = P_t(P_0) = p_0$. From the definition (4.14) of Q we conclude that

$$p_0 \leq \beta_t(p_0) \leq \beta_t(\mathbf{1}_K)$$

for every $t \geq 0$, and (4.13) will follow.

We show first that the family Q of (4.14) is a semigroup. To that end, we claim that for every $t \geq 0$,

$$p_0\beta_t(p_0) = p_0\beta_t(\mathbf{1}_K), \quad \text{and } \beta_t(p_0)p_0 = \beta_t(\mathbf{1}_K)p_0. \tag{4.15}$$

Indeed, since $\beta_t(x) = \alpha_t(x)\beta_t(\mathbf{1}_K)$ for every $x \in \mathcal{B}(K)$ we have

$$p_0\beta_t(\mathbf{1}_K - p_0) = p_0\alpha_t(\mathbf{1}_K - p_0)\beta_t(\mathbf{1}_K) = p_0(\mathbf{1}_K - \alpha_t(p_0))\beta_t(\mathbf{1}_K)$$

and the right side is zero because $\alpha_t(p_0) \geq p_0$. Similarly, $\beta_t(\mathbf{1}_K - p_0)p_0 = 0$. Thus for $s, t > 0$ and $x \in \mathcal{B}(H)$ we have

$$Q_sQ_t(x) = p_0\beta_s(p_0\beta_t(x)p_0)p_0 = p_0\beta_s(p_0)\beta_{s+t}(x)\beta_s(p_0)p_0.$$

Because of (4.15) we have $p_0\beta_s(p_0) = p_0\beta_s(\mathbf{1}_K)$ and $\beta_s(p_0)p_0 = \beta_s(\mathbf{1}_K)p_0$, hence

$$Q_sQ_t(x) = p_0\beta_s(\mathbf{1}_K)\beta_{s+t}(x)\beta_s(\mathbf{1}_K)p_0.$$

Finally, since $\beta_{s+t}(\mathbf{1}_K) \leq \beta_s(\mathbf{1}_K)$, we have

$$\beta_s(\mathbf{1}_K)\beta_{s+t}(x)\beta_s(\mathbf{1}_K) = \beta_{s+t}(x),$$

and the formula $Q_sQ_t(x) = Q_{s+t}(x)$ follows.

Now since $\beta_t \leq \alpha_t$ for every $t \geq 0$ we have $Q_t \leq P_t$. Thus the hypotheses (4.1.1) and (4.1.2) of Theorem 4.1 are satisfied. Notice that Q_t cannot vanish for every $t > 0$. Indeed, Corollary 2.17 implies that α must have units. Choose $u \in \mathcal{U}_\alpha$, and let T be the corresponding unit of P defined by

$$T(t)^* = u(t)^* \restriction_H, \qquad t \geq 0.$$

There is a real constant k such that $\langle u(t), u(t)\rangle_{\mathcal{E}_\alpha(t)} = e^{kt}$. Since $u(t) \in \mathcal{D}(t)$ we have

$$u(t)u(t)^* \leq e^{kt}\beta_t(\mathbf{1}_K),$$

and hence by (4.15)

$$T(t)T(t)^* = p_0u(t)u(t)^*p_0 \leq e^{kt}p_0\beta_t(\mathbf{1}_K)p_0 = e^{kt}p_0\beta_t(p_0)p_0 = Q_t(p_0).$$

Since $T(t)T(t)^*$ tends weakly to $p_0 = \mathbf{1}_H$ as $t \to 0+$, so does $Q_t(p_0)$.

From Theorem 4.1 we conclude that the semigroup Q has a bounded generator. Finally, we claim that $Q_t = P_t$ for all $t \geq 0$. According to Theorem 3.3, it is enough to show that $\mathcal{U}_P = \mathcal{U}_Q$ and that P and Q have the same covariance function. For that, we require a general observation:

Lemma 4.16. *Let P, Q be two CP semigroups acting on $\mathcal{B}(H)$, and suppose that $Q_t \leq P_t$ for every $t \geq 0$. Then $\mathcal{U}_Q \subseteq \mathcal{U}_P$, and for every finite set $T_1, T_2, \dots, T_n \in \mathcal{U}_Q$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ we have*

$$\sum_{i,j} \lambda_i\bar{\lambda}_j c_P(T_i, T_j) \leq \sum_{i,j} \lambda_i\bar{\lambda}_j c_Q(T_i, T_j),$$

where c_P and c_Q are the covariance functions of P and Q.

Remarks. Let X be a set and $f : X \times X \to \mathbb{C}$ a function. Recall that f is *positive definite* if for every $x_1, x_2, \dots, x_n \in X$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ we have

$$\sum_{i,j} \lambda_i\bar{\lambda}_j f(x_i, x_j) \geq 0.$$

Given two functions $f, g : X \times X \to \mathbb{C}$, we will write $f \lesssim g$ or $g \gtrsim f$ if $g - f$ is positive definite. We will make use of the following elementary facts about positive definite functions.

$$f \gtrsim 0, \quad g \gtrsim 0 \implies fg \gtrsim 0, \tag{4.17}$$

where fg denotes the pointwise product $fg(x, y) = f(x, y)g(x, y)$. (4.13), together with transitivity of the relation $\lesssim$, implies that for any four complex-valued functions f_1, f_2, g_2, g_2 on $X \times X$ we have

$$0 \lesssim f_i \lesssim g_i, i = 1, 2 \implies f_1f_2 \lesssim g_1g_2. \tag{4.18}$$

proof of Lemma 4.16. Choose $T \in \mathcal{U}_Q$. Then there is a real constant k such that the semigroup $R_t(x) = T(t)xT(t)^*$ satisfies

$$R_t \leq e^{kt}Q_t \leq e^{kt}P_t, \qquad t \geq 0,$$

hence $T \in \mathcal{U}_P$.

Fix $S, T \in \mathcal{U}_Q$, and for every $t > 0$ define functions f, g by

$$f(S,T;t) = \langle S(t), T(t) \rangle_{\mathcal{E}_Q(t)},$$
$$g(S,T;t) = \langle S(t), T(t) \rangle_{\mathcal{E}_P(t)}.$$

Notice first that

$$0 \lesssim f(\cdot,\cdot;t) \lesssim g(\cdot,\cdot;t) \tag{4.19}$$

as functions on $\mathcal{U}_Q \times \mathcal{U}_Q$. Indeed, if $T_1, T_2, \dots, T_n \in \mathcal{U}_Q$, $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ and we set $A(t) = \sum_k \lambda_k T_k(t)$, then $A(t)$ belongs to $\mathcal{E}_Q(t)$ and hence the mapping $x \mapsto A(t)xA(t)^*$ is dominated by

$$\langle A(t), A(t) \rangle_{\mathcal{E}_Q(t)} Q_t \le \langle A(t), A(t) \rangle_{\mathcal{E}_Q(t)} P_t.$$

It follows that

$$0 \le \langle A(t), A(t) \rangle_{\mathcal{E}_P(t)} \le \langle A(t), A(t) \rangle_{\mathcal{E}_Q(t)}$$

and (4.19) follows after expanding the inner products in the obvious way.

For every partition $\mathcal{P} = \{0 = t_0 < t_1, \dots < t_n = t\}$ of the interval $[0,t]$, set

$$f_{\mathcal{P}}(S,T;t) = \prod_{k-1}^{n} \langle S(t_k - t_{k-1}), T(t_k - t_{k-1}) \rangle_{\mathcal{E}_P(t)},$$
$$g_{\mathcal{P}}(S,T;t) = \prod_{k-1}^{n} \langle S(t_k - t_{k-1}), T(t_k - t_{k-1}) \rangle_{\mathcal{E}_Q(t)}.$$

By (4.17) and (4.18), we have

$$0 \lesssim f_{\mathcal{P}}(\cdot,\cdot;t) \lesssim g_{\mathcal{P}}(\cdot,\cdot;t)$$

for every partition $\mathcal{P}$ of $[0,t]$. After taking the limit on $\mathcal{P}$ we obtain

$$e^{tc_P} \lesssim e^{tc_Q},$$

for every $t > 0$. It follows that $e^{tc_P} - 1 \lesssim e^{tc_Q} - 1$ for every $t > 0$ and hence

$$c_P = \lim_{t \to 0+} \frac{e^{tc_P} - 1}{t} \lesssim \lim_{t \to 0+} \frac{e^{tc_Q} - 1}{t} = c_Q,$$

as required $\square$

We claim now that

$$\mathcal{U}_P \subseteq \mathcal{U}_Q, \qquad \text{and} \tag{4.19}$$
$$c_Q(T,T) \le c_P(T,T), \qquad \text{for every } T \in \mathcal{U}_P. \tag{4.20}$$

To see that, choose any unit $T \in \mathcal{U}_P$. By [4, Theorem 3.6] there is a unit $u \in \mathcal{U}_\alpha$ such that

$$T(t)^* = u(t)^* \restriction_H, \qquad t > 0;$$

moreover, $c_P(T,T) = c_\alpha(u,u)$ because of the minimality of α. Since $u(t) \in \mathcal{D}(t)$ and $\langle u(t), u(t)\rangle_{\mathcal{D}(t)} = e^{tc_\alpha(u,u)}$, the map

$$x \in \mathcal{B}(H) \mapsto e^{tc_\alpha(u,u)} Q_t(x) - T(t)xT(t)^*$$

is completely positive. (4.19) follows. We may also conclude from this argument that

$$\langle T(t), T(t)\rangle_{\mathcal{E}_Q(t)} \le e^{tc_P(T,T)}.$$

Hence for every partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ of the interval $[0,t]$,

$$\prod_{k=1}^{n} \langle T(t_k - t_{k-1}), T(t_k - t_{k-1})\rangle_{\mathcal{E}_Q(t)} \le e^{tc_P(T,T)},$$

and after passing to the limit on $\mathcal{P}$ we obtain

$$e^{tc_Q(T,T)} \le e^{tc_P(T,T)}$$

for every $t > 0$, from which (4.20) is immediate.

Together with Lemma 4.12, (4.19) implies that $\mathcal{U}_P = \mathcal{U}_Q$. We claim now that $c_P = c_Q$. To see that, fix T_1, $T_2 \in \mathcal{U}_P$, and consider the 2×2 matrix $A = (a_{ij})$ defined by

$$a_{ij} = c_Q(T_i, T_j) - c_P(T_i, T_j).$$

Lemma 4.16 implies that $A \ge 0$; while (4.20) implies that both diagonal terms of A are nonpositive, so that the trace of A is nonpositive. It follows that $A = 0$. In particular,

$$c_Q(T_1, T_2) - c_P(T_1, t_2) = a_{ij} = 0.$$

We may now apply Theorem 3.3 to obtain $P = Q$. As we have already pointed out, one may deduce from this the required inequality (4.13). That completes the proof of Theorem 4.8 $\square$

Corollary 4.21. *Let $P = \{P_t : t \ge 0\}$ be a unital CP semigroup acting on $\mathcal{B}(H)$ whose generator is bounded and which is not a semigroup of $*$-automorphisms. Then the minimal dilation of P is a cocycle perturbation of a CAR/CCR flow of positive index $r = 1, 2, \dots, \infty$. The index r is the rank of the generator of P.*

proof. By Theorem 4.8, the minimal dilation of P is a completely spatial E_0-semigroup. The classification results of [1, Corollary of Proposition 7.2] imply that this E_0-semigroup is cocycle conjugate to a CAR/CCR flow. Its index is the rank of the generator of P by Corollary 2.17.

Thus we only have to check that the generator cannot have rank zero. Let L be the generator of P and suppose that L has rank zero. Then the metric operator space associated with L is $\{0\}$, and L must have the form

$$L(x) = kx + xk^*$$

for some $k \in M_n(\mathbb{C})$ (see Definition 1.23). Since P is unital we have $L(\mathbf{1}) = 0$, hence $k + k^* = 0$. It follows that $k = ih$ for some self adjoint matrix h and $L(x) = (ih)x - x(ih)$. Thus

$$P_t(x) = e^{ith} x e^{-ith}$$

is a semigroup of $*$-automorphisms, contrary to hypothesis $\square$

Remarks. In particular, Corollary 4.21 leads to a description of the minimal dilations of all unital CP semigroups which act on a matrix algebra $M_n(\mathbb{C})$, $n = 2, 3, \dots$. If the semigroup is nontrivial then its minimal dilation α is a cocycle perturbation of a CAR/CCR flow of finite positive index $d_*(\alpha)$. Considering the relation between generators and metric operator spaces (Proposition 1.20), we find that for fixed n the possible values of $d_*(\alpha)$ are $1, 2, \dots, n^2 - 1$.

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