THE CANONICAL ANTICOMMUTATION RELATIONS

Lecture notes for Mathematics 208
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In these notes we discuss the canonical anticommutation relations, the C^*-algebra associated with them (the CAR algebra), second quantization, and the construction of KMS states for so-called free Fermi gasses. We only scratch the surface. For more, I refer you to Gert Pedersen's book *C^*-algebras and their automorphism groups* [3] and volume 2 of *Operator algebras and quantum statistical mechanics*, by Ola Bratteli and Derek Robinson [2].

Two operators X, Y are said to *anticommute* if $XY + YX = 0$. Suppose we are given two sets of self-adjoint operators $p_1, \dots, p_n$, $q_1, \dots, q_n$ acting on some Hilbert space (or more generally, belonging to some C^*-algebra) which satisfy the following

$$\begin{aligned} &p_k p_j + p_j p_k = q_k q_j + q_j q_k = 2\delta_{jk}\mathbf{1} \\ &p_k q_j + q_j p_k = 0 \end{aligned}$$

for all k, j. These are the canonical anticommutation relations in their self-adjoint form for a Fermionic quantum system having n degrees of freedom. Taking $j = k$ we find that $p_k^2 = q_k^2 = \mathbf{1}$ (a self-adjoint unitary operator is called a *reflection*). Thus, we simply have an even number of reflections which mutually anticommute with each other. The above equations make sense for infinite sequences $p_1, p_2, \dots,$ $q_1, q_2, \dots$ and we allow that possibility as well (indeed, most of the discussion to follow will be directed primarily to infinite systems).

These relations are best reformulated in their "complex" form, by introducing the sequence of n operators

$$a_k = \frac{1}{2}(q_k + ip_k).$$

After a straightforward calculation one finds that

$$a_k a_j + a_j a_k = 0, \tag{1}$$

$$a_k^* a_j + a_j a_k^* = \delta_{jk}\mathbf{1}, \tag{2}$$

for all $1 \le j, k \le n$ if n is finite, and for all $j, k \ge 1$ otherwise. The relations (1) and (2) are called the *canonical anticommutation relations* (abbreviated CARs) for n degrees of freedom.

It is also a good idea to carry this a step further, and reformulate the canonical anticommutation relations in a coordinate-free way. Assuming for the moment that n is finite and that $a_1, \dots, a_n \in \mathcal{B}(H)$ satisfy (1) and (2) we can define a linear map $a : \mathbb{C}^n \to \mathcal{B}(H)$ in the obvious way

$$a(z) = z_1 a_1 + \cdots + z_n a_n,$$

z denoting $(z_1, \dots, z_n) \in \mathbb{C}^n$, and we find that

$$a(z)a(w) + a(w)a(z) = 0, \tag{3}$$
$$a(w)^*a(z) + a(z)a(w)^* = \langle z, w\rangle \mathbf{1}, \tag{4}$$

where $\langle z, w\rangle$ denotes the usual inner product in $\mathbb{C}^n$, $z_1\bar{w}_1 + \cdots + z_n\bar{w}_n$. If n is infinite then we replace $\mathbb{C}^n$ with $\ell^2(\mathbb{N})$, define $a(z) = z_1a_1 + z_2a_2 + \dots$ for the dense subspace of all $z \in \ell^2(\mathbb{N})$ satisfying $z_k = 0$ for all but finitely many $k \in \mathbb{N}$. Equation (4) implies that the linear map $a(\cdot)$ is bounded and therefore extends uniquely to a linear map of $\ell^2(\mathbb{N})$ into $\mathcal{B}(H)$, and the relations (3) and (4) persist for all $z, w \in \ell^2(\mathbb{N})$.

We can now free ourselves of coordinates entirely by starting with a separable complex Hilbert space Z and a linear map $a : Z \to \mathcal{B}(H)$ which satisfies (3) and (4). Such a linear map is also called a representation of the CARs, and the number of degrees of freedom is the dimension of Z. If one chooses an orthonormal basis $\{e_1, e_2, \dots\}$ for Z and sets $a_k = a(e_k)$ then we recover equations (1) and (2), but of course there are many ways of choosing an orthonormal basis for Z.

There are a number of interesting things that one can deduce from (3) and (4) almost immediately. For example, the following result can be proved easily and I recommend that you supply that proof.

Proposition 1. *Let $a : Z \to \mathcal{B}(H)$ be a representation of the canonical anticommutation relations. Then for every unit vector $z \in Z$ the operator $U = a(z)$ is a partial isometry such that UH and U^*H are orthocomplements of each other. The four operators $e_{11} = U^*U$, $e_{12} = U^*$, $e_{21} = U$, $e_{22} = UU^*$ are a system of 2×2 matrix units, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto ae_{11} + be_{12} + ce_{21} + de_{22}$ defines a $*$-isomorphism of $M_2(\mathbb{C})$ onto $C^*(U)$.*

Proposition 1 implies that the linear map $a(\cdot)$ associates to every nonzero vector $z \in Z$ a 2×2 matrix subalgebra of $\mathcal{B}(H)$, namely $C^*(a(z))$. Notice that the relation between these C^*-algebras for different z is quite complicated (for example, $C^*(a(z_1))$ neither commutes nor anticommutes with $C^*(a(z_2))$, even when $z_1 \perp z_2$). We will sort out the nature of this relationship presently.

It is also remarkable that the linear space of operators $a(Z)$ is actually a Hilbert space relative to the *operator* norm. Indeed, since each operator $a(z)$ has the form $\|z\|U$ where U is a partial isometry, we have

$$\|a(z)\| = \|z\|,$$

which shows that the operator norm on $a(Z)$ is given by the inner product structure of Z. We have seen this phenomenon before when studying the Cuntz C^*-algebras. Some people like to call linear spaces of operators with this property *operator Hilbert spaces.*

We now show how to construct representations of the CARs out of more elementary data, in a way that brings out the relation between the CARs and inductive limits of matrix algebras $M_{2^n}(\mathbb{C})$, $n = 1, 2, \dots$. Suppose we are given a sequence $A_1, A_2, \dots, A_n$ of C^*-subalgebras of $\mathcal{B}(H)$, each containing the identity operator, such that the operators in A_k commute with the operators in A_j for $k \neq j$, and such that each A_k is isomorphic to the 2×2 matrix algebra $M_2(\mathbb{C})$.

For each k we choose a partial isometry $u_k \in A_k$ satisfying

$$u_k u_k^* + u_k^* u_k = \mathbf{1}.$$

We know that A_k must contain such a partial isometry because it is isomorphic to $M_2(\mathbb{C})$. Notice first that, regardless of how one chooses u_k, it must generate A_k as a C^*-algebra (for example, noting that that $u_k^2 = 0$ we may construct a set of 2×2 matrix units as in Proposition 1 to see that $C^*(u_k)$ is itself isomorphic to $M_2(\mathbb{C})$ and thus both A_k and its subalgebra $C^*(u_k)$ are four-dimensional).

Fix $k = 1, \dots, n$. Since $u_k u_k^*$ is a projection, $(\mathbf{1} - 2u_k u_k^*)$ is a reflection which satisfies $u_k(\mathbf{1} - 2u_k u_k^*) = u_k$ and $(\mathbf{1} - 2u_k u_k^*)u_k = -u_k$. It follows that $(\mathbf{1} - 2u_k u_k^*)$ anticommutes with u_k,

$$(\mathbf{1} - 2u_k u_k^*)u_k + u_k(\mathbf{1} - 2u_k u_k^*) = 0,$$

while of course $\mathbf{1} - 2u_k u_k^*$ commutes with u_j for $j \neq k$. Thus

$$v_k = (\mathbf{1} - 2u_1 u_1^*)(\mathbf{1} - 2u_2 u_2^*) \dots (\mathbf{1} - 2u_k u_k^*), \qquad k = 1, \dots, n,$$

anticommutes with $u_1, \dots, u_k$ and commutes with the remaining ones $u_{k+1}, \dots, u_n$. The v_k are mutually commuting reflections.

With these relations in hand, one easily verifies that the sequence

$$a_k = u_k v_k, \qquad k = 1, 2, \dots, n$$

satisfies the CARs (1) and (2) for n degrees of freedom. Finally, we claim that

$$C^*(a_1, \dots, a_n) = C^*(u_1, \dots, u_n) = C^*(A_1 \cup \dots \cup A_n) \cong M_{2^n}(\mathbb{C}).$$

Indeed, since $a_k a_k^* = u_k u_k^*$ it follows that both sets $a_1, \dots, a_n$ and $u_1, \dots, u_n$ generate the same C^*-algebra; and by the preceding remarks this is the C^*-algebra generated by $A_1 \cup \dots \cup A_n$. It only remains to show that the latter is isomorphic to $M_{2^n}(\mathbb{C})$. We have already seen that $A_k \cong M_2(\mathbb{C})$, and hence we can find a set $\{e_{ij}(k) : 1 \leq i, j \leq 2\}$ of 2×2 matrix units for A_k, $k = 1, \dots, n$. Since these 2×2 systems mutually commute with each other their n-fold products

$$e_{i_1 j_1}(1) e_{i_1 j_2}(2) \dots e_{i_n j_n}(n), \qquad 1 \leq i_k, j_k \leq 2, \quad k = 1, \dots, n$$

defines a system of $2^n \times 2^n$ matrix units which generates $C^*(A_1 \cup \dots \cup A_n)$, hence all three C^*-algebras are isomorphic to $M_{2^n}(\mathbb{C})$.

Now we will show that such a sequence $a_1, \dots, a_n \in \mathcal{B}(H)$ is equivalent to any other representation $b_1, \dots, b_n \in \mathcal{B}(K)$ of the CARs in the sense that there is a unique $*$-isomorphism $\pi : C^*(a_1, \dots, a_n) \to C^*(b_1, \dots, b_n)$ such that $\pi(a_k) = b_k$ for every k. To see that, notice that since the two sets $\{b_k, b_k^*\}$ and $\{b_j, b_j^*\}$ mutually anticommute with each other for $k \neq j$, $b_j b_j^*$ must commute with $b_k b_k^*$ for $k \neq j$. Thus we can define mutually commuting reflections $\tilde{v}_1, \dots \tilde{v}_n$ by

$$\tilde{v}_k = (\mathbf{1} - 2b_1 b_1^*)(\mathbf{1} - 2b_2 b_2^*) \dots (\mathbf{1} - 2b_n b_n^*)$$

and corresponding operators $\tilde{u}_1, \dots, \tilde{u}_n$ by $\tilde{u}_k = b_k \tilde{v}_k$. Noting that $b_k = \tilde{u}_k \tilde{v}_k$, a calculation (essentially reversing what was done before) shows that $\tilde{u}_k$ is a partial isometry satisfying $\tilde{u}_k\tilde{u}_k^* + \tilde{u}_k^*\tilde{u}_k = \mathbf{1}$, and $\tilde{u}_k$ commutes with both $\tilde{u}_j$ and $\tilde{u}_j^*$ for all $j \neq k$. Thus we can make a $2^n \times 2^n$ system of matrix units out of the $\tilde{u}_k$ exactly as we made one out of the u_k above, and since now we are talking about two systems of $2^n \times 2^n$ matrix units, there is a unique $*$-isomorphism $\pi : C^*(u_1, \dots, u_n) \to C^*(\tilde{u}_1, \dots, \tilde{u}_n)$ such that $\pi(u_k) = \tilde{u}_k$ for $k = 1, \dots, n$. This π must carry a_k to b_k in view of the relations we have seen between the a_k, u_k, v_k and their bedfellows b_k, $\tilde{u}_k$, $\tilde{v}_k$.

What we have just done implies the following assertion about uniqueness even in the case of infinitely many degrees of freedom.

Theorem A. *Let $a_1, a_2, \dots \in \mathcal{B}(H)$ and $b_1, b_2, \dots \in \mathcal{B}(K)$ be two sequences of operators satisfying (1) and (2). Then there is a unique $*$-isomorphism*

$$\pi : C^*(a_1, a_2, \dots) \to C^*(b_1, b_2, \dots)$$

such that $\pi(a_k) = b_k$ for every $k = 1, 2, \dots$.

proof. For each $n = 1, 2, \dots$ let $A_n = C^*(a_1, \dots, a_n)$, $B_n = C^*(b_1, \dots, b_n)$. The previous argument implies that for every n there is a unique $*$-isomorphism $\pi_n : A_n \to B_n$ such that $\pi_n(a_k) = b_k$, $k = 1, 2, \dots, n$.

We have already seen that A_n is isomorphic to a full matrix algebra, hence it is simple; and being an injective morphism of C^*-algebras, π must be isometric. Because of the coherence property $\pi_{n+1} \restriction_{A_n} = \pi_n$, there is a unique isometric $*$-homomorphism of $\cup A_n$ onto $\cup B_n$ which provides a common extension of the π_n. The closure of the latter map is the desired isomorphism π of $C^*(a_1, a_2, \dots)$ onto $C^*(b_1, b_2, \dots)$. The uniqueness of π is clear. ∎

Theorem A implies that the C^*-algebra $\mathcal{A}$ generated by any representation of the infinite CARs is well-defined up to $*$-isomorphism. It is a separable C^*-algebra with unit. It also simple, in that it is obtained as the norm closure of an increasing sequence $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ of (unital) C^*-subalgebras such that each $\mathcal{A}_n$ is isomorphic to the full matrix algebra $M_{2^n}(\mathbb{C})$. $\mathcal{A}$ is called the CAR algebra (the acronym stands for the mouth-filling canonical anticommutation relations, but it is pronounced as in "car").

It is important to reformulate Theorem A in the following coordinate-free form.

Corollary 1. *Let Z_1, Z_2 be two (separable, infinite dimensional) Hilbert spaces and let $a : Z_1 \to \mathcal{B}(H)$ and $b : Z_2 \to \mathcal{B}(K)$ be linear maps satisfying (3) and (4). Then for every unitary operator $U : Z_1 \to Z_2$ there is a unique $*$-isomorphism $\alpha_U : C^*(a(Z_1)) \to C^*(b(Z_2))$ such that $\alpha_U(a(z)) = b(Uz)$ for every $z \in Z_1$.*

sketch of proof. Choose any orthonormal basis $e_1, e_2, \dots$ for Z_1 and let $f_1, f_2, \dots$ be the orthonormal basis for Z_2 defined by $f_k = Ue_k$, $k = 1, 2, \dots$. Then $a_k = a(e_k) \in \mathcal{B}(H)$, $b_k = b(f_k) \in \mathcal{B}(K)$ define two representations of the infinite CARs. Theorem A provides a $*$-isomorphism $\alpha_U : C^*(a_1, a_2, \dots) \to C^*(b_1, b_2, \dots)$ which carries a_k to b_k for every k. One verifies easily that these C^*-algebra are respectively $C^*(a(Z_1))$ and $C^*(b(Z_2))$, and that the required formula for α_U follows after taking linear combinations and limits. ∎

Let $a : Z \to \mathcal{B}(H)$ satisfy (3) and (4) and let $\mathcal{A} = C^*(a(Z))$ be the corresponding realization of the CAR algebra. Corollary 1 implies that every unitary operator in

$\mathcal{B}(Z)$ gives rise to a natural $*$-automorphism of $\mathcal{A}$. But what is more important for quantum physics is the following observation.

Corollary 2. *Let $a : Z \to \mathcal{B}(H)$ and $\mathcal{A}$ be as above. Let $U = \{U_t : t \in \mathbb{R}\}$ be a strongly continuous one-parameter unitary group acting on Z. Then there is a unique one-parameter group $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of $*$-automorphisms of $\mathcal{A}$ satisfying*

$$\alpha_t(a(z)) = a(U_t z), \qquad z \in Z, t \in \mathbb{R}.$$

Moreover, $(\mathcal{A}, \alpha, \mathbb{R})$ is a C^*-dynamical system.

proof. The only thing that is not immediate from Corollary 1 is continuity,

$$\lim_{t\to 0} \|\alpha_t(X) - X\| = 0, \qquad X \in \mathcal{A}. \tag{5}$$

But for $X = a(z)$, $z \in Z$, we have

$$\|\alpha_t(a(z)) - a(z)\| = \|a(U_t z) - a(z)\| = \|a(U_t z - z)\| = \|U_t z - z\|,$$

which tends to zero as $t \to 0$ by strong continuity of U. Since the set of elements $a(Z)$ generate $\mathcal{A}$ as a C^*-algebra, (5) follows from a now-familiar argument. ∎

Gauge group. Taking the scalar one-parameter unitary group $U_t z = e^{it} z$, we obtain a C^*-dynamical system $(\mathcal{A}, \gamma, \mathbb{R})$ by defining

$$\gamma_t(a(z)) = a(e^{it} z) = e^{it} a(z), \qquad z \in Z, t \in \mathbb{R}.$$

γ is called the *gauge group* of $\mathcal{A}$. It is periodic with period 2π: $\gamma_{t+2\pi} = \gamma_t$, $t \in \mathbb{R}$.

Hamiltonians. The one-parameter unitary groups $U = \{U_t : t \in \mathbb{R}\} \subseteq \mathcal{B}(Z)$ associated with the flow of time in quantum theory usually have spectrum that is bounded below in the sense that the spectral measure P associated with U,

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda}\, dP(\lambda),$$

satisfies $P((-\infty, \lambda_0)) = 0$ for some $\lambda_0 \in \mathbb{R}$. In such cases we can replace U with the phase-shifted version of itself

$$V_t = e^{-it\lambda_0} U_t,$$

and this new group V has nonnegative spectrum. If we choose λ_0 carefully (by taking it to be as large as possible) then we can ensure that the closed support of the spectral measure of V a) is contained in the nonnegative reals $[0, \infty)$ and b) contains 0. Of course, such a λ_0 is uniquely determined.

If we write $V_t = e^{itH}$, then H is an unbounded positive operator whose spectrum contains 0. We will make further restrictions on H presently (i.e., that it has discrete spectrum and its eigenvalues grow moderately rapidly), and for such a unitary group V we will show how to write down a KMS state for the C^*-dynamical system $(\mathcal{A}, \alpha, \mathbb{R})$ associated with it.

KMS states. Let $(A, \alpha, \mathbb{R})$ be a C^*-dynamical system. If A is unital then there are always states on A which are invariant under the action of the group of $*$-automorphisms α (see the current exercises). We want to single out a particularly important class of invariant states, called KMS states after the physicists Kubo, Martin, Schwinger. These are abstractions of the so-called Gibbs canonical ensembles to the setting of C^*-dynamical systems associated with flows. KMS states are very important not only for physical applications [2], but for the basic theory of C^*-dynamical systems as well (see [3]).

For $(A, \alpha, \mathbb{R})$ as above, one can always find a dense $*$-subalgebra A_0 of A whose elements are *entire* with respect to the action of α in the following sense: for every $a \in A_0$ and every bounded linear functional ρ on A, the function $t \in \mathbb{R} \mapsto \rho(\alpha_t(a))$ can be extended to an entire function of a complex variable t. One constructs elements of A_0 in the following way. Choose an arbitrary element $x \in A$ and let $f \in L^1(\mathbb{R})$ be an integrable function on $\mathbb{R}$ whose Fourier transform has compact support. It is not hard to show that the element

$$a = \int_{-\infty}^{\infty} f(t)\alpha_t(x)\, dt \in A$$

is entire with respect to α, and in fact one can take A_0 to be the subalgebra of A generated by such elements a (see [2]). For every element $a \in A_0$ the function $t \in \mathbb{R} \to \alpha_t(a)$ can be extended uniquely to an entire function from the complex plane $\mathbb{C}$ into A, and for $t \in \mathbb{C}$ we will write $\alpha_t(a)$ for the value of this entire function at t.

Fix a positive real number β. A state ω of A is called a KMS state (for the automorphism group α at inverse temperature β) if for every pair of elements $a, b \in A_0$ we have

$$\omega(a\alpha_{i\beta}(b)) = \omega(ba). \tag{6}$$

Notice that on the left side of (6) we have evaluated $\alpha_t(b)$ at a point $t = i\beta$ on the imaginary axis. We also point out that, when discussing KMS states, mathematicians frequently (but not always) take $\beta = 1$.

Here are two very instructive exercises about KMS states. Consider a matrix algebra $\mathcal{A} = M_n(\mathbb{C})$, $n = 2, 3, \dots$. Let H be a self-adjoint operator in $\mathcal{A}$ and consider the C^*-dynamical system $(\mathcal{A}, \alpha, \mathbb{R})$ defined by the automorphism group

$$\alpha_t(X) = e^{itH} X e^{-itH}, \qquad X \in \mathcal{A}, \quad t \in \mathbb{R}.$$

Exercise 1. Show that for every $\beta > 0$,

$$\omega_\beta(X) = \frac{\text{trace }(e^{-\beta H} X)}{\text{trace }(e^{-\beta H})}, \qquad X \in \mathcal{A}$$

is a KMS state for $(\mathcal{A}, \alpha, \mathbb{R})$ at inverse temperature β.

Notice that in this case standard power series methodology provides us with an entire function $z \in \mathbb{C} \mapsto e^{izH}$ without having to appeal to the subtler methods sketched in the previous paragraphs. Note too that the state ω_β is invariant under the action of the automorphism group $\alpha = \{\alpha_t : t \in \mathbb{R}\}$.

Exercise 2. Show that the state ω_β of exercise 1 is the only KMS state at inverse temperature β.

We now show how to construct KMS states associated with *some* of the the natural flows on the CAR algebra $\mathcal{A}$ that are defined by Corollary 2. In order to do that we must say something about the irreducible representations of $\mathcal{A}$. $\mathcal{A}$ is perhaps the most elementary example of a separable simple C^*-algebra which has uncountably many mutually inequivalent irreducible representations. Worse, we do not even know what its irreducible representations are (indeed, we can never know a complete list of mutually inequivalent irreducible representations which is "Borel measurable", since it is known that such a list does not exist for a C^*-algebra such as $\mathcal{A}$ [1]). This has serious consequences: if one needs to work with an irreducible representation of the infinite canonical anticommutation relations, which irreducible representation should one choose?

Fortunately, there is a "natural" irreducible representation, which we will now discuss. While it has many favorable properties and while it is possible to single out this particular representation in terms of certain natural requirements (axioms), it is still in some fundamental sense an arbitrary choice.

Antisymmetric Fock space $\mathcal{F}_-(Z)$. We start with a separable infinite dimensional Hilbert space Z. For every $n = 1, 2, \dots$ let $\bigwedge^n Z$ denote the antisymmetric subspace of the n-fold tensor product of Hilbert spaces $Z^{\otimes n}$. $\bigwedge^0 Z$ is defined as $\mathbb{C}$, and $\mathcal{F}_-(Z)$ is the direct sum

$$\mathcal{F}_-(Z) = \mathbb{C} \oplus Z \oplus (Z \wedge Z) \oplus \dots .$$

For each $n \geq 2$ and $z_1, \dots, z_n \in Z$, the wedge product $z_1 \wedge z_2 \wedge \cdots \wedge z_n$ is defined by projecting the elementary tensor $z_1 \otimes z_2 \otimes \cdots \otimes z_n \in Z^{\otimes n}$ onto the antisymmetric subspace $\bigwedge^n Z$. $\bigwedge^n Z$ is the closed linear span of all such expressions. If we pick an orthonormal basis $e_1, e_2, \dots$ for Z, then we obtain an orthonormal basis for $\bigwedge^n Z$ by taking all products of the form

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n}, \qquad 1 \leq i_1 < \cdots < i_n. \tag{7}$$

It is an instructive exercise to prove the following.

Proposition. *For every $z \in Z$ there is a unique bounded operator $a(z) \in \mathcal{B}(\mathcal{F}_-(Z))$ satisfying*

$$a(z) z_1 \wedge \cdots \wedge z_n = z \wedge z_1 \wedge z_2 \wedge \cdots \wedge z_n$$

for $n = 1, 2, \dots,$ and $a(z)1 = z$. This defines an irreducible representation of the canonical anticommutation relations on $\mathcal{F}_-(Z)$.

Remarks on other conventions and other mores. It is appropriate to comment here on the vagaries of notation. The operators $a(z)$ defined in the preceding proposition are called creation operators, since when z is a unit vector $a(z)$ creates a particle in the state z. Their adjoints $a(z)^*$ are called annihilation operators. Unfortunately, this terminology is not universal. Some people (physicists are the most likely perpetrators) like to think of the letter "a" appearing in $a(z)$ as representing "annihilation". Correspondingly, their $a(z)$ would be my $a(z)^*$. If one pursues that point of view, one is lead to define the anticommutation relations in terms of an antilinear map $a : z \in Z \to a(z) \in \mathcal{B}(H)$ rather than by a linear map as we have done. The problem arising from these other conventions are not serious, but they can be irritating for someone who is attached to linear maps.

Finally, none of this is significant for the mathematics of the situation since by the work we have already done, any representation $a_1, a_2, \dots$ of the canonical anticommutation relations is isomorphic to the sequence of adjoints $b_1 = a_1^*, b_2 = a_2^*, \dots$ since the b_k satisfy the same CARs as the a_k!

Second quantization. Let $A \in \mathcal{B}(Z)$ be a contraction, $\|A\| \leq 1$. The set of contractions is closed under operator multiplication, under the adjoint operation, and it has a natural strong and weak operator topology. Multiplication is strongly continuous (on the unit ball), but the adjoint operation is not.

The operator n-fold tensor product $A^{\otimes n}$ acts naturally on $Z^{\otimes n}$, and it leaves the antisymmetric subspace $\bigwedge^n Z$ invariant. Thus we can define a contraction operator $\Gamma_-(A)$ on $\mathcal{F}_-(Z)$ in a natural way

$$\Gamma_-(A) = \mathbf{1} \oplus A \oplus (A \otimes A \restriction_{\bigwedge^2 Z}) \oplus (A \otimes A \otimes A \restriction_{\bigwedge^3 Z}) \oplus \dots .$$

One has $\Gamma_-(A) z_1 \wedge \cdots \wedge z_n = Az_1 \wedge \cdots \wedge Az_n$ for every $n \geq 1$, and it is a simple matter to verify that $\Gamma_-(AB) = \Gamma_-(A)\Gamma_-(B)$, $\Gamma_-(A^*) = \Gamma_-(A)^*$, $\Gamma_-(\mathbf{1}) = \mathbf{1}$, and that $A \mapsto \Gamma_-(A)$ is strongly continuous. The mapping Γ_- is called *second quantization* (for Fermions).

In particular, if $U = \{U_t : t \in \mathbb{R}\}$ is a strongly continuous one-parameter unitary group acting on the one-particle space Z, then $V_t = \Gamma_-(U_t)$ defines a strongly continuous one-parameter unitary group acting on $\mathcal{F}_-(Z)$. More significantly, from the definition of the operator $a(z)$ and the formulas above, we see that $V_t a(z) V_t^* = a(U_t z) = \alpha_t(a(z))$ for every $t \in \mathbb{R}$, $z \in Z$. It follows that the "second quantized" unitary group V implements the action of the automorphism group α of Corollary 2 on the CAR algebra $\mathcal{A} = C^*(a(Z)) \subseteq \mathcal{B}(\mathcal{F}_-(Z))$ in the sense that

$$V_t X V_t^* = \alpha_t(X), \qquad X \in \mathcal{A}, t \in \mathbb{R}. \tag{8}$$

Let us look more carefully at (8) and the relation between U_t and $V_t = \Gamma_-(U_t)$. If we write H and K for the generators of U and V respectively,

$$U_t = e^{itH}, \quad V_t = e^{itK}, \qquad t \in \mathbb{R},$$

then by definition of V we have $e^{itK} = \Gamma_-(e^{itH})$. If one likes to think by analogy with the exponential map of Lie group theory, then one can interpret the preceding formula as $K = d\Gamma_-(H)$, meaning that the unbounded self-adjoint operator $d\Gamma_-(H)$ is *defined* as the generator of the one parameter unitary group $\Gamma_-(e^{itH})$ by way of

$$\Gamma_-(e^{itH}) = e^{itd\Gamma_-(H)}, \qquad t \in \mathbb{R}.$$

We will not have to make use of this formalism.

More significantly, suppose now that the generator H of $\{U_t : t \in \mathbb{R}\}$ is positive in the sense described in the preceding pages. Letting P be the spectral measure of H, we have

$$U_t = \int_0^\infty e^{it\lambda}\, dP(\lambda),$$

and thus for every $s \geq 0$ in $\mathbb{R}$ we can define a contraction $e^{-sH} \in \mathcal{B}(H)$ by

$$e^{-sH} = \int_0^\infty e^{-s\lambda}\, dP(\lambda).$$

The family of operators $\{e^{-sH} : s \geq 0\}$ is in fact a strongly continuous semigroup of positive contractions in $\mathcal{B}(Z)$, which is *formally* related to the original unitary group U by $e^{-sH} = U_{is}$. While the latter can be made precise, we will not have to do so.

If one now second quantizes this semigroup, one would expect that $\Gamma_-(e^{-sH})$ should agree with e^{-sK}, that is, that K should have nonnegative spectrum, and that its associated semigroup satisfies the expected relation with e^{-sH}. This is the case, it is not hard to prove, but we omit the argument, see [3].

We now make the key hypothesis on H, namely that for some positive $s_0 > 0$, $e^{-s_0 H}$ belongs to the trace class

$$\text{trace } e^{-s_0 H} < \infty. \tag{9}$$

(9) implies that the spectrum of H is discrete, and that if we enumerate its eigenvalues (including multiplicity) in increasing order $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ then we have

$$\sum_{n=1}^{\infty} e^{-s_0 \lambda_n} < \infty. \tag{10}$$

This means that the eigenvalues λ_n must grow with n, but slow growth is good enough. For example, the sequence $\lambda_n = \log(n)$ satisfies (10) for any $s_0 > 1$.

What we require is that the quantized semigroup e^{-sK} should also be trace-class for sufficiently large s, and the following result establishes this. This remarkable result is associated with the antisymmetry of $\mathcal{F}_-(Z)$, and in fact the corresponding property fails for symmetric "Bosonic" second quantization (the symmetric second quantization of a rank-one projection is a projection of infinite rank!).

Theorem B. *Let $A \in \mathcal{B}(H)$ be a positive contraction with finite trace, and let $\lambda_1, \lambda_2, \dots$ be the eigenvalues of A, repeated according to multiplicity. Then*

$$\textit{trace}\ \Gamma_-(A) = \prod_{k=1}^{\infty} (1 + \lambda_k), \tag{11}$$

and in particular, trace $\Gamma_-(A) \leq e^{\text{trace } A} < \infty$.

Remarks. The term on the right of (11) is the product of the eigenvalues of the operator $\mathbf{1} + A$, and this suggests that one can think of (11) as the formula trace $\Gamma_-(A) = \det(\mathbf{1} + A)$ (of course, we have not actually given a coherent definition of the determinant of an operator on an infinite dimensional Hilbert space).

Note too that since $\Gamma_-(A)$ is a positive operator which restricts to A on the one-particle subspace $Z \subseteq \mathcal{F}_-(Z)$, we also have the inequality trace $A \leq$ trace $\Gamma_-(A)$. Conclusion: for any positive contraction $A \in \mathcal{B}(H)$ one has

$$\text{trace } A < \infty \iff \text{trace } \Gamma_-(A) < \infty.$$

proof. We can find an orthonormal basis $e_1, e_2, \dots$ for Z such that $Ae_k = \lambda_k e_k$, $k = 1, 2, \dots$. Let P_n be the projection on the span of $e_1, \dots, e_n$, and let $A_n = AP_n$. Then A_n has finite rank and $A_n \uparrow A$ (strongly) as $n \to \infty$.

Fix $m = 1, 2, \dots$ and consider the restriction of $\Gamma_-(A_n)$ to $\bigwedge^m Z$. Since

$$\{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_m} : 1 \leq i_1 < i_2 < \dots < i_m\}$$

is an orthonormal basis for $\bigwedge^m Z$ and $A_n e_i = 0$ for $i > n$, we see that $\Gamma_-(A_n)$ vanishes on $\bigwedge^m Z$ for $m > n$, and for $m \leq n$ we have

$$\begin{aligned}\operatorname{trace} \Gamma_-(A_n) \restriction_{\bigwedge^m Z} &= \sum_{1 \leq i_1 < \dots < i_m \leq n} \langle Ae_{i_1} \wedge \dots \wedge Ae_{i_m}, e_{i_1} \wedge \dots \wedge e_{i_m} \rangle \\ &= \sum_{1 \leq i_1 < \dots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{trace} \Gamma_-(A_n) &= 1 + \sum_{m=1}^{n} \operatorname{trace} \Gamma_-(A_n) \restriction_{\bigwedge^m Z} = 1 + \sum_{m=1}^{n} \sum_{1 \leq i_1 < \dots < i_m \leq n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \\ &= (1 + \lambda_1)(1 + \lambda_2) \dots (1 + \lambda_n).\end{aligned}$$

Since $A_n \uparrow A$ strongly as $n \to \infty$ we have $\Gamma_-(A_n) \uparrow \Gamma_-(A)$, and since the trace is lower semicontinuous we conclude that

$$\operatorname{trace} \Gamma_-(A) = \lim_{n \to \infty} (1 + \lambda_1)(1 + \lambda_2) \dots (1 + \lambda_n) = \prod_{k=1}^{\infty} (1 + \lambda_k).$$

Since $1 + \lambda \leq e^{\lambda}$ for every real number λ, we can estimate the right side of the preceding formula in the obvious way

$$\prod_{k=1}^{\infty} (1 + \lambda_k) \leq e^{\lambda_1 + \lambda_2 + \dots} \leq e^{\operatorname{trace} A}$$

to complete the proof. ∎

Returning now to the case $e^{-sK} = \Gamma_-(e^{-sH})$ under discussion, let us assume that there is a $\beta > 0$ such that trace $e^{-\beta H} < \infty$. Theorem B implies that $e^{-\beta K}$ is a positive trace class operator on $\mathcal{F}_-(Z)$, and hence we can define a normal state ω_β of $\mathcal{B}(\mathcal{F}_-(Z))$ by

$$\omega_\beta(X) = \frac{\operatorname{trace}\,(e^{-\beta K} X)}{\operatorname{trace}\,(e^{-\beta K})}, \qquad X \in \mathcal{B}(\mathcal{F}_-(Z)). \tag{12}$$

ω_β is called the Gibbs state (at inverse temperature β). It can be shown that it is the only normal state of $\mathcal{B}(\mathcal{F}_-(Z))$ whose restriction to the CAR algebra $\mathcal{A}$ satisfies the KMS condition (at inverse temperature β), relative to the automorphism group $\alpha = \{\alpha_t : t \in \mathbb{R}\}$ of $\mathcal{A}$ defined by

$$\alpha_t(a(z)) = a(e^{itH} z), \qquad z \in Z, t \in \mathbb{R}$$

(the proof can be found around page 40 of [2]).

Concluding Remarks. Let us look back over what has been done. We started with a one parameter unitary group U acting on Z and formed the associated C^*-dynamical system $(\mathcal{A}, \alpha, \mathbb{R})$ by way of Corollary 2. Our goal was to construct a KMS state for $(\mathcal{A}, \alpha, \mathbb{R})$. We got going on this by first picking a particular irreducible representation of $\mathcal{A}$ in which the action of α could be implemented spatially by a one-parameter unitary group (obtained through the second quantization procedure).

In order to proceed further, we had to make additional hypotheses on $U_t = e^{itH}$, namely that a) H is positive, and b) $e^{-\beta H}$ is trace-class for some $\beta > 0$. Under these conditions, Theorem B gave us the machinery we needed to write down the Gibbs state of (12) (we did not verify the KMS condition, but it is verified in [2]).

I do not know how to prove either existence or uniqueness of KMS states for the C^*-dynamical systems obtained in this way without the additional hypotheses of the preceding paragraph. More information can be found in [3].

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