ASYMPTOTIC LIFTS OF POSITIVE LINEAR MAPS

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Abstract. We show that the notion of asymptotic lift generalizes naturally to normal positive maps $\phi : M \to M$ acting on von Neumann algebras M. We focus on cases in which the domain of the asymptotic lift can be embedded as an operator subsystem $M_\infty \subseteq M$, and characterize when M_∞ is a Jordan subalgebra of M in terms of the asymptotic multiplicative properties of ϕ.

1. Introduction

Let $\phi : M \to M$ be a normal unit preserving positive linear map acting on a dual operator system M; we refer to such a pair (M, ϕ) as a *UP map*. While we are primarily interested in UP maps that act on von Neumann algebras M, it is useful to broaden the context as above. The powers of ϕ form an irreversible dynamical semigroup $\{\phi^n : n = 0, 1, 2, \dots\}$ acting on M. In this paper we generalize work begun in [Arv04] and [Arv06], together with complementary results in [Stø06], to further develop the asymptotic theory of such semigroups.

One may view UP maps as the objects of a category, in which a homomorphism from $\phi_1 : M_1 \to M_1$ to $\phi_2 : M_2 \to M_2$ is a UP map $E : M_1 \to M_2$ such that $E \circ \phi_1 = \phi_2 \circ E$. There is a natural notion of isomorphism in this category. Our first general result is that every normal unit-preserving positive linear map acting on a dual operator system has an asymptotic lift which is unique up to isomorphism. This generalizes one of the main results of [Arv06], which dealt with the subcategory in which the objects are unital normal *completely* positive maps (UCP maps) on dual operator systems, with UCP maps as morphisms.

We are primarily concerned with UP maps that act on von Neumann algebras M. In [Arv06] it was shown that the asymptotic lift (N, α, E) of a UCP map $\phi : M \to M$ acting on a von Neumann algebra M also acts on a von Neumann algebra N. Moreover, it was shown that the W^*-dynamical system (N, α) can be identified with the tail flow of the minimal dilation of ϕ to a $*$-endomorphism of a larger von Neumann algebra in most cases – namely those in which the dilation endomorphism has trivial kernel, which includes all UCP maps acting on factors M. Since the minimal dilation of a UCP map on a von Neumann algebra can be constructed explicitly in principle, that provided a concrete identification of the asymptotic lift.

It is significant that the asymptotic lift (N, α, E) of a UP map acting on a von Neumann algebra need not act on a von Neumann algebra N. In this paper we show that in general, N is order-isomorphic to a unique JW*-algebra – namely a dual operator system that is closed under the Jordan product $x \circ y = (xy + yx)/2$ – in such a way that α becomes a Jordan automorphism of N. Thus, *the asymptotic behavior of a UP map on a von Neumann algebra is always associated with a JW*-dynamical system* (N, α).

Naturally, one would like to identify (N, α) more concretely in terms of the asymptotic properties of the semigroup $\phi, \phi^2, \phi^3, \dots$. But since there is no dilation theory for semigroups of UP maps – and perhaps an effective dilation theory is impossible, there seems to be no candidate to replace the "tail flow" identification described above for UCP maps. Thus, this identification problem becomes a significant issue for UP maps.

In many cases, N can be embedded as an operator system $N \subseteq M$ in such a way that $\alpha = \phi \restriction_N$. While the operator system N is order-isomorphic to a JW*-algebra in general, it need not be closed under the ambient Jordan multiplication of M, and we address that issue in Section 4. We identify N concretely in terms of the action of ϕ on M, and when N is a Jordan subalgebra of M, we are able to go farther by identifying N with the *multiplicative core* of ϕ that was introduced in [Stø06].

The problem of finding a satisfactory concrete description of the JW*-dynamical system (N, α) when N does *not* embed in M remains untouched.

2. Asymptotic lifts of positive maps

In this section we describe how the notion of asymptotic lift (of a completely positive map) can be generalized to normal unit-preserving maps that are merely positive. We summarize the basic properties of asymptotic lifts, indicating briefly how proofs of [Arv06] should be modified.

Remark 2.1 (Dual operator systems, norm, and order). Every von Neumann algebra is the dual of a unique Banach space, and hence it carries a natural weak* topology (aka the ultraweak or σ-weak topology). A *dual operator system* is a linear space of operators $M \subseteq \mathcal{B}(H)$ that contains the identity $\mathbf{1}$, is self-adjoint $M^* = M$, and is closed in the weak*-topology of $\mathcal{B}(H)$. Such an M also has a unique predual M_*, and its intrinsic M_*-topology coincides with the relative weak*-topology of $\mathcal{B}(H)$. A map of dual operator systems $\phi : M \to N$ is called *normal* when it is weak*-continuous. There is an intrinsic characterization of dual operator systems that we do not require.

Let us recall the basic properties of unit-preserving normal positive linear maps $\phi : M \to N$ (UP maps) between dual operator systems. A UP map $\phi : M \to N$ defined on a C^*-algebra M must satisfy $\|\phi\| = 1$. That is most easily seen by making use of the Russo-Dye theorem [RD66] which implies that $\|\phi\| = \sup \|\phi(u)\|$, u ranging over the unitary group of M, together with the fact that for every unitary u, the restriction of ϕ to the commutative C^*-algebra $C^*(u)$ is completely positive and therefore satisfies the strong

Schwarz inequality $\phi(u)^*\phi(u) \leq \phi(u^*u) = \phi(\mathbf{1}_M) = \mathbf{1}_N$. More generally, if M is merely an operator system then one has $1 \leq \|\phi\| \leq 2$ in general, but the C^*-algebraic upper bound $\|\phi\| = 1$ often fails.

On the other hand, since the norm of a self-adjoint operator x is the smallest $\alpha \geq 0$ such that $-\alpha\mathbf{1} \leq x \leq \alpha\mathbf{1}$, the restriction of ϕ to the real Banach space $M^{\rm sa}$ of self-adjoint elements of M has norm 1. Conversely, if a linear map $\phi : M \to N$ carries self-adjoint elements to self-adjoint elements, maps $\mathbf{1}_M$ to $\mathbf{1}_N$, and satisfies $\|\phi \restriction_{M^{\rm sa}} \| = 1$, then ϕ must also preserve positivity. In particular, *an order isomorphism in the category of UP maps is characterized as a $*$-preserving linear map $\phi : M \to N$ of operator systems that carries $\mathbf{1}_M$ to $\mathbf{1}_N$ and restricts to an isometry of $M^{\rm sa}$ onto $N^{\rm sa}$.*

Definition 2.2. A *reversible lift* of a UP map $\phi : M \to M$ is a triple (N, α, E) consisting of a UP automorphism $\alpha : N \to N$ of another dual operator system N and a UP map $E : N \to M$ satisfying $E \circ \alpha = \phi \circ E$.

A reversible lift (N, α, E) of ϕ is said to be *nondegenerate* if

$$E(\alpha^{-n}(y)) = 0,\ n = 0, 1, 2, \dots \implies y = 0. \tag{2.1}$$

Significantly, if (2.1) fails, one can replace (N, α, E) with another reversible lift $(\tilde{N}, \tilde{\alpha}, \tilde{E})$ that is *nondegenerate*, as in Remark 2.3 of [Arv06].

Definition 2.3. Let $\phi : M \to M$ be a UP map on a dual operator system. An *asymptotic lift* of ϕ is a reversible lift (N, α, E) of ϕ that satisfies nondegeneracy (2.1), together with

$$\|\rho \circ E \restriction_{N^{\rm sa}} \| = \lim_{k\to\infty} \|\rho \circ \phi^k \restriction_{M^{\rm sa}} \|, \qquad \rho \in M_*. \tag{2.2}$$

Remark 2.4. We shall make use of the dual formulation of (2.2), and we record that now for later reference: For every nondegenerate reversible lift (N, α, E) of ϕ, (2.2) is equivalent to the following assertion:

$$E(\mathrm{ball}_r\, N^{\rm sa}) = \bigcap_{n=1}^{\infty} \phi^n(\mathrm{ball}_r\, M^{\rm sa}), \qquad r > 0, \tag{2.3}$$

where $\mathrm{ball}_r\, X$ denotes the closed ball of radius r in a real or complex Banach space X. The proof of equivalence of (2.2) and (2.3) follows the lines of the proof of the corresponding result of [Arv06].

There are two fundamental results on asymptotic lifts of UP maps. The first concerns existence and uniqueness:

Theorem 2.5. *Every UP map $\phi : M \to M$ of a dual operator system has an asymptotic lift. If (N_1, α_1, E_1) and (N_2, α_2, E_2) are two asymptotic lifts for ϕ, then there is a unique UP-isomorphism of dual operator systems $\theta : N_1 \to N_2$ such that $\theta \circ \alpha_1 = \alpha_2 \circ \theta$ and $E_2 \circ \theta = E_1$.*

As in the case of completely positive maps, the existence issue is settled by a direct construction based on *inverse sequences*, namely bounded bilateral sequences (x_n) of elements of M that satisfy $x_n = \phi(x_{n+1})$, $n \in \mathbb{Z}$. The space of all inverse sequences is a dual operator system N, the bilateral shift $\alpha : (x_n) \mapsto (x_{n-1})$ is an automorphism of N, and the connecting map E carries (x_n) to x_0. The proof that (N, α, E) is an asymptotic lift is a minor (and somewhat simpler) variation of the proof of the corresponding result of [Arv06]. Similarly, the proof of uniqueness is a straightforward variation of arguments in [Arv06]; we omit the details.

If a UP map $\phi : M \to M$ on a von Neumann algebra M is completely positive, then its asymptotic lift (N, α, E) gives rise to a W^*-dynamical system (N, α) [Arv06]. While this need not be true for asymptotic lifts of UP maps, one can make the following assertion in general:

Theorem 2.6. *Let $\phi : M \to M$ be a UP map on a von Neumann algebra M and let (N, α, E) be its asymptotic lift. Then N is order isomorphic to a unique JW*-algebra in such a way that α is a Jordan automorphism of N.*

Again, the proof follows along the lines of arguments in [Arv06], by introducing a Jordan multiplication on the range of a positive idempotent map as in Corollary 1.6 of [ES79]. The key step is to show that the for constructed asymptotic lift (N, α, E) in which N is the space of inverse sequences, there is a projection of norm one $Q : \ell^\infty(M) \to N$. The existence of such a projection Q follows from the argument of [Arv06]. Once one has a positive projection (which in the current setting is typically not *completely* positive) with these properties, one can introduce a Jordan project in N by way of

$$x \circ y = Q(\frac{1}{2}(xy + yx)), \qquad x, y \in N,$$

and at this point one can show that N is order isomorphic to a JW*-algebra by imitating arguments in [Arv06].

3. Embeddable asymptotic lifts

Let (N, α, E) be the asymptotic lift of a UP map $\phi : M \to M$ acting on a dual operator system. In this section we fix attention on those cases in which the asymptotic lift can be embedded as a subsystem of M in the following particular way. We say that (N, α, E) *embeds in* M if E is restricts to an isometry on the self-adjoint part of N,

$$\|E(y)\| = \|y\|, \qquad y = y^* \in N. \tag{3.1}$$

This is equivalent to the assertion that E implements an order isomorphism of N onto $E(N)$. Since all asymptotic lifts of $\phi : M \to M$ are isomorphic, this definition does not depend on the particular choice of (N, α, E). In such cases, the range $E(N)$ of E is an operator subsystem of M with the property that ϕ restricts to an automorphism of $E(N)$, α is identified with $\phi \restriction_{E(N)}$, and E is identified with the inclusion map $\iota : E(N) \subseteq M$.

The purpose of this section is to make a precise summary of those facts, and especially to give an explicit description of $E(N)$ in terms of the action of

ϕ on M. We begin by introducing the following operator system $M_\infty \subseteq M$:

$$M_\infty = \bigcup_{r>0} \bigcap_{n=1}^{\infty} \phi^n(\operatorname{ball}_r M).$$

M_∞ consists of all $y \in M$ for which there is a *bounded* sequence $x_n \in M$ with $y = \phi^n(x_n)$, $n = 1, 2, \dots$. In general, M_∞ is a self-adjoint linear subspace of M containing the identity operator, it is invariant under ϕ, and in fact $\phi(M_\infty) = M_\infty$. Moreover, from (2.3) we may infer that

$$E(N^{\rm sa}) = \bigcup_{r>0} E(\operatorname{ball}_r N^{\rm sa}) = \bigcup_{r>0} \bigcap_{n=1}^{\infty} \phi^n(\operatorname{ball}_r M^{\rm sa}) = M_\infty^{\rm sa}, \tag{3.2}$$

hence $M_\infty = E(N)$ is precisely the range of E in all cases. We sometimes refer to M_∞ as the *tail operator system* of M.

Remark 3.1. It is clear that $M_\infty \subseteq \cap_n \phi^n(M)$, but the inclusion is typically proper. As a simple example, let $M = M_2(\mathbb{C})$ and consider the completely positive map $\phi : M \to M$ defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \lambda b \\ \lambda c & d \end{pmatrix}$$

where λ is a constant satisfying $0 < \lambda < 1$. One has $\cap_n \phi^n(M) = M$, but in this case M_∞ is the two-dimensional subalgebra of diagonal matrices.

Proposition 3.2. *Let (N, α, E) be an asymptotic lift of $\phi : M \to M$. Then $\ker E = \{0\}$ iff the restriction of ϕ to M_∞ is both injective and surjective.*

Proof. In general, one has $\phi(M_\infty) = M_\infty$ by definition of M_∞. Assuming that $\ker E = \{0\}$, choose $a \in M_\infty$ such that $\phi(a) = 0$. Since $E(N) = M_\infty$, there is a $y \in N$ such that $a = E(y)$, hence $0 = \phi(a) = \phi(E(y)) = E(\alpha(y))$, and therefore $\alpha(y) = 0$ because E is injective. $y = 0$ follows because α is an automorphism of N, hence $a = E(y) = 0$.

Conversely, if the restriction of ϕ to M_∞ is injective, choose a nonzero element $y \in N$. Since the norms $\|E(\alpha^{-n}(y))\|$ increase to $\|y\|$ as $n \uparrow \infty$ (see Lemma 3.7 of [Arv06]), we must have $E(\alpha^{-n}(y)) \neq 0$ for sufficiently large $n \geq 1$. Since each power ϕ^n restricts to an injective map on $M_\infty = E(N)$, it follows that

$$E(y) = \phi^n(E(\alpha^{-n}(y))) \neq 0$$

for large n, hence E is injective. □

We conclude that whenever $\ker E = \{0\}$, the restriction of ϕ to M_∞ defines an order-preserving linear bijection on M_∞. However, M_∞ itself need not be closed in any topology in general, and the restriction of ϕ to M_∞ need not be an order automorphism. The following result implies that when E restricts to an isometry on $N^{\rm sa}$, such anomalies cannot occur.

Theorem 3.3. *For every UP map $\phi : M \to M$ on a dual operator system, the following are equivalent.*

(i) *The asymptotic lift of* $\phi : M \to M$ *embeds in* M.
(ii) M_∞ *is weak***-closed and* ϕ *restricts to an order automorphism of it.*
(iii) *The asymptotic lift of* $\phi : M \to M$ *is isomorphic to the triple* $(M_\infty, \phi\restriction_{M_\infty}, \iota)$*, where* $\iota : M_\infty \subseteq M$ *is the inclusion map.*

Proof. Let (N, α, E) be an asymptotic lift of $\phi : M \to M$.

(i) $\implies$ (ii): By hypothesis, E restricts to an isometry of $N^{\rm sa}$ onto the self-adjoint part of $E(N) = M_\infty$. Since M_∞ is a self-adjoint linear space of operators and the adjoint operation is weak*-continuous, M_∞ will be weak*-closed provided we show that its self-adjoint part $M_\infty^{\rm sa}$ is weak*-closed. By a standard result on the weak*-closure of convex sets in dual Banach spaces, this will follow if we show that for every $r > 0$, the intersection $M_\infty^{\rm sa} \cap {\rm ball}_r\, M$ of $M_\infty^{\rm sa}$ with the r-ball of M is weak*-closed. Now the connecting map $E : N \to M$ restricts to a weak*-continuous isometry of $N^{\rm sa}$ onto $M_\infty^{\rm sa}$, and by (2.3), it carries ${\rm ball}_r\, N^{\rm sa}$ onto $M_\infty^{\rm sa} \cap {\rm ball}_r\, M$. Since the unit ball of N is weak*-compact, it follows that $M_\infty^{\rm sa} \cap {\rm ball}_r\, M = E({\rm ball}_r\, N^{\rm sa})$ is weak*-compact, hence weak*-closed. We conclude that $M_\infty^{\rm sa}$ is weak*-closed.

Making use of Proposition 3.2, let ψ be the linear automorphism of M_∞ inverse to $\phi\restriction_{M_\infty}$. Then since $E \circ \alpha = \phi \circ E$, we have $E \circ \alpha^{-1} = \psi \circ E$. Since both E and α^{-1} restrict to isometries on $N^{\rm sa}$, it follows that ψ restricts to an isometry on $M_\infty^{\rm sa} = E(N^{\rm sa})$. Since $\psi(\mathbf{1}) = \mathbf{1}$, ψ must also be order-preserving, and we conclude that $\phi\restriction_{M_\infty}$ is an order automorphism.

(ii) $\implies$ (iii): Under the hypothesis (ii), $(M_\infty, \phi\restriction_{M_\infty}, \iota)$ becomes a reversible lift of $\phi : M \to M$ that is obviously nondegenerate. We show that it is the asymptotic lift of $\phi : M \to M$ by establishing (2.2). For that, consider the decreasing sequence of weak*-compact sets

$$\phi^n({\rm ball}\, M^{sa}), \qquad n = 1, 2, \dots$$

and choose a normal linear functional $\rho \in M_*$. By (2.3), we have

$$\bigcap_{n=1}^{\infty} \phi^n({\rm ball}\, M^{sa}) = E({\rm ball}\, N^{\rm sa}).$$

Since ρ is weak*-continuous, it follows from Lemma 3.5 of [Arv06]) that

$$\begin{aligned} \|\rho \circ \phi^n \restriction_{M_\infty^{\rm sa}}\| &= \sup\{\rho(x) : x \in \phi^n({\rm ball}\, M^{\rm sa})\} \downarrow \sup\{\rho(x) : x \in E({\rm ball}\, N^{\rm sa})\} \\ &= \|\rho \circ E \restriction_{N^{\rm sa}}\|, \end{aligned}$$

as $n \uparrow \infty$, and (iii) follows.

The implication (iii) $\implies$ (i) is obvious. □

Taken together, theorems 3.3 and 2.6 imply:

Corollary 3.4. *Let* $\phi : M \to M$ *be a UP map on a von Neumann algebra* M *whose asymptotic lift embeds in* M*. Then the tail operator system* M_∞ *is order-isomorphic to a JW***-algebra in such a way that the restriction of* ϕ *to* M_∞ *becomes a Jordan automorphism, and the asymptotic lift of* ϕ *is the triple* $(M_\infty, \phi \restriction M_\infty, \iota)$*,* ι *being the inclusion map* $\iota : M_\infty \subseteq M$.

The asymptotic lifts of many UP maps are embeddable. That is true for all examples covered by the hypotheses of [Arv04], and in Section 5 below we elaborate on the special case of maps on finite-dimensional algebras. Not all infinite-dimensional UP maps are embeddable, and the following example illustrates the fact.

Remark 3.5 (An Example). Let $L^2 = L^2(\mathbb{T}, \frac{d\theta}{2\pi})$, let $H^2 \subseteq L^2$ be the usual Hardy space, and let $s \in \mathcal{B}(H^2)$ be the unilateral shift. Consider the UCP map ϕ defined on $M = \mathcal{B}(H^2)$ by

$$\phi(x) = s^*xs, \qquad A \in \mathcal{B}(H^2).$$

Let $u \in \mathcal{B}(L^2)$ be the bilateral shift and let p be the projection of L^2 on H^2. Then the asymptotic lift of ϕ is the triple $(\mathcal{B}(L^2), \alpha, E)$, where $\alpha(y) = u^*bu$ and E is the compression map $E(y) = py\restriction_{H^2}$. Since $\ker E \neq \{0\}$, the asymptotic lift of ϕ is not embeddable. We omit the calculations.

4. Embedding and the multiplicative core

If the asymptotic lift of a UP map $\phi : M \to M$ acting on a von Neumann algebra embeds in M, then by Corollary 3.4, M_∞ is order isomorphic to a JW*-algebra and the restriction of ϕ to M_∞ becomes a Jordan automorphism of M_∞. Notice that this does not imply that M_∞ itself is closed under the Jordan product $x \circ y = \frac{1}{2}(xy + yx)$ inherited from M. Section 5 contains a simple example that illustrates the phenomenon.

In the present section we study the Jordan structure of M_∞ in more detail in the case when the asymptotic lift embeds in M. In particular, we show that if M_∞ is itself a JW*-algebra then it coincides with the *multiplicative core* C_ϕ of ϕ introduced in [Stø06]. We also give necessary and sufficient conditions for $(C_\phi, \phi\restriction_{C_\phi}, \iota)$ to be the asymptotic lift.

Following [Stø06], the *definite set* of a UP map $\phi : M \to M$ is defined by

$$M_\phi = \{x \in M : \phi(x^* \circ x) = \phi(x)^* \circ \phi(x)\}; \tag{4.1}$$

it is a JW*-algebra, and we have

$$\phi(x \circ y) = \phi(x) \circ \phi(y), \qquad x \in M_\phi, \quad y \in M.$$

Continuing as in [Stø06], one can show that

$$B_\phi = \{x \in M : \phi^n(x) \in M_\phi, \quad n = 0, 1, 2, \dots\}$$

is a JW*-algebra on which ϕ restricts to a Jordan endomorphism, and one forces surjectivity on ϕ by restricting it to the "tail" subalgebra

$$C_\phi = \bigcap_{n=1}^{\infty} \phi^n(B_\phi).$$

C_ϕ is called the *multiplicative core* of ϕ. We require the following variation of Lemma 6 of [Stø06].

Lemma 4.1. *The multiplicative core is characterized as the largest JW*-subalgebra $N \subseteq M$ with the following two properties:*

(i) $\phi(N) = N$.
(ii) $\phi(x \circ y) = \phi(x) \circ \phi(y)$ *for all* $x, y \in N$.

Proof. Let N be a JW*-subalgebra of M with properties (i) and (ii), and choose $x \in N$. Then we have

$$\phi(\phi^n(x)^* \circ \phi^n(x)) = \phi^{n+1}(x)^* \circ \phi^{n+1}(x), \qquad n \geq 0,$$

because ϕ is a Jordan endomorphism of N, hence $N \subseteq B_\phi$. It follows that $N = \phi^n(N) \subseteq \phi^n(B_\phi)$ for every $n \geq 0$, so that $N \subseteq \cap_n \phi^n(B_\phi) = C_\phi$. □

The following result associates the multiplicative core with asymptotic lifts that are embeddable as Jordan subalgebras:

Theorem 4.2. *For every UP map* $\phi : M \to M$ *acting on a von Neumann algebra* M*, the tail operator system* M_∞ *contains the multiplicative core* C_ϕ*.*

Assume that the asymptotic lift embeds in M*. Then* M_∞ *is a Jordan subalgebra of* M *precisely when* $M_\infty = C_\phi$*.*

Proof. By definition, $C_\phi = \cap_n \phi^n(B_\phi)$, and ϕ restricts to a Jordan endomorphism of B_ϕ. Since each power ϕ^n of ϕ restricts to a Jordan endomorphism on B_ϕ and a Jordan homomorphism maps the unit ball of the self-adjoint part of its domain onto the unit ball of the self-adjoint part of its range, we have $\phi^n(\mathrm{ball}_r\, B_\phi^{\mathrm{sa}}) = \mathrm{ball}_r(\phi^n(B_\phi)^{\mathrm{sa}})$ for every $r > 0$ and every $n = 1, 2, \ldots$, hence

$$\mathrm{ball}_r\, C_\phi^{\mathrm{sa}} = \bigcap_{n=1}^{\infty} \mathrm{ball}_r\, \phi^n(B_\phi^{\mathrm{sa}}) = \bigcap_{n=1}^{\infty} \phi^n(\mathrm{ball}_r\, B_\phi^{\mathrm{sa}}) \subseteq M_\infty.$$

The asserted inclusion $C_\phi = C_\phi^{\mathrm{sa}} + iC_\phi^{\mathrm{sa}} \subseteq M_\infty$ follows after taking the union over all positive r.

Assume that the asymptotic lift of $\phi : M \to M$ embeds in M. If M_∞ is closed under the Jordan multiplication of M, then by Theorem 3.3 (iii), $\phi \restriction_{M_\infty}$ is an order automorphism of M_∞, and an application of Kadison's Schwarz inequality for positive linear maps implies that ϕ is a Jordan automorphism of M_∞. By Lemma 4.1, $M_\infty \subseteq C_\phi$, and we conclude that $M_\infty = C_\phi$. The converse is trivial. □

It is significant that when M_∞ is not itself closed under the Jordan product, it contains a largest JW*-algebra that is characterized as follows.

Proposition 4.3. *Let* $\phi : M \to M$ *be a UP map whose asymptotic lift embeds in* M*. Then the weak**-*closed linear span of all projections in* M_∞ *is a JW**-*algebra.*

Proof. By Corollary 3.4, there is a JW*-algebra B and an order isomorphism α of B onto M_∞. Viewing α as a UP map of B into the JW*-algebra M, it makes sense to speak of its definite set B_α. Since B_α is a JW*-subalgebra of B and the restriction of α to B_α is a Jordan homomorphism, it follows that $\alpha(B_\alpha) \subseteq M_\infty$ is a JW*-subalgebra of M.

Being a JW*-algebra, $\alpha(B_\alpha)$ is generated by its projections. Conversely, we claim that every projection $e \in M_\infty$ belongs to $\alpha(B_\alpha)$. To see that, fix such an e and choose $f \in B$ such that $\alpha(f) = e$. Then $0 \le f \le \mathbf{1}$ because α is a UP order isomorphism, so by Kadison's Schwarz inequality

$$0 = e - e^2 = \alpha(f) - \alpha(f)^2 \ge \alpha(f - f^2) \ge 0,$$

hence $\alpha(f - f^2) = 0$ and finally $f = f^2$. Since $\alpha(f^2) = \alpha(f)^2$, f belongs to the definite set of α, hence $e = \alpha(f) \in \alpha(B_\alpha)$. □

We conclude the section by describing intrinsic conditions on a UP map which imply that its asymptotic lift is embeddable as the multiplicative core; note that the sufficient conditions of Theorem 4.5 are clearly necessary as well.

For a linear functional ρ on M and a set of operators $S \subseteq M$, we write $\rho \perp S$ whenever $\rho(S) = \{0\}$. We require the following result characterizing the equality $M_\infty = C_\phi$ in terms of the action of ϕ on the predual of M:

Lemma 4.4. *Let $\phi : M \to M$ be a UP map. Then $M_\infty = C_\phi$ iff for every $\rho \in M_*$ satisfying $\rho \perp C_\phi$, one has*

$$\lim_{n\to\infty} \|\rho \circ \phi^n\| = 0. \tag{4.2}$$

Proof. We claim first that $\rho \in M_*$ satisfies (4.2) iff $\rho \perp M_\infty$. Indeed, setting $M_\infty(r) = \cap_n \phi^n(\mathrm{ball}_r\, M)$ for $r > 0$, and noting that the compact convex sets $\phi^n(\mathrm{ball}_r\, M)$ decrease to $M_\infty(r)$ as $n \uparrow \infty$, we can apply Lemma 3.5 of [Arv06] to conclude that (4.2) is equivalent to the assertion

$$\begin{aligned}\sup\{|\rho(x)| : x \in M_\infty(r)\} &= \lim_{n\to\infty} \sup\{|\rho(x)| : x \in \phi^n(\mathrm{ball}_r\, M)\} \\ &= r \cdot \lim_{n\to\infty} \|\rho \circ \phi^n\| = 0,\end{aligned}$$

for every $r > 0$. Noting that $M_\infty = \cup_{r>0} M_\infty(r)$, the claim follows.

The preceding paragraph, together with a standard separation theorem, shows that the assertion

$$\rho \perp C_\phi \implies \lim_{n\to\infty} \|\rho \circ \phi^n\| = 0 \tag{4.3}$$

is equivalent to the assertion $\overline{M_\infty}^{\mathrm{w}*} \subseteq C_\phi$; and since in general we have $C_\phi \subseteq M_\infty \subseteq \overline{M_\infty}^{\mathrm{w}*}$, (4.3) is seen to be equivalent to $C_\phi = M_\infty$. □

Theorem 4.5. *Let $\phi : M \to M$ be a UP map such that*

(i) *The positive linear map obtained by restricting ϕ to C_ϕ is faithful.*
(ii) $\lim_{n\to\infty} \|\rho \circ \phi^n\| = 0$ *for every $\rho \in M_*$ satisfying $\rho \perp C_\phi$.*

Then $M_\infty = C_\phi$, the restriction of ϕ to C_ϕ is a Jordan automorphism of C_ϕ, and the asymptotic lift of ϕ is $(C_\phi, \phi\restriction_{C_\phi}, \iota)$.

Proof. Hypothesis (i) implies that the restriction of ϕ to C_ϕ is a Jordan automorphism. Thus, the triple $(C_\phi, \phi\restriction_{C_\phi}, \iota)$ is a nondegenerate reversible lift of ϕ. To show that it is the asymptotic lift, we must establish the following inequality for every $\rho \in M_*$:

$$\lim_{n\to\infty} \|\rho \circ \phi^n\| \leq \|\rho \restriction_{C_\phi} \|. \tag{4.4}$$

For that, fix ρ and, for every $n = 1, 2, \dots$, choose an element $x_n \in M$ satisfying $\|x_n\| = 1$ and $|\rho(\phi^n(x_n))| = \|\rho \circ \phi^n\|$. We can find a subsequence $n_1 < n_2 < \dots$ such that $\phi^{n_k}(x_{n_k})$ converges to $y \in M$ weak* as $k \to \infty$. Since the weak*-compact sets $\phi^n(\operatorname{ball} M)$ decrease with increasing n, y must belong to their intersection $\cap_n \phi^n(\operatorname{ball} M) = M_\infty$, and of course $\|y\| \leq 1$.

Making use of hypothesis (ii) and Lemma 4.4, we conclude that $M_\infty = C_\phi$, so that $y \in \operatorname{ball} C_\phi$. Hence

$$\lim_{n\to\infty} \|\rho \circ \phi^n\| = \lim_{k\to\infty} \|\rho \circ \phi^{n_k}\| = \lim_{k\to\infty} |\rho(\phi^{n_k}(x_{n_k}))| = |\rho(y)| \leq \|\rho \restriction_{C_\phi} \|,$$

and the desired inequality (4.4) follows. □

Remark 4.6. The conclusion of Theorem 4.5 can be significantly strengthened whenever there is a normal positive projection E of M onto C_ϕ - in particular, whenever M is a finite von Neumann algebra. In such cases, a simple argument (that we omit) shows that the automorphism $\alpha = \phi \restriction_{C_\phi}$ of C_ϕ satisfies

$$\lim_{n\to\infty} \|\rho \circ \phi^n - \rho \circ \alpha^n \circ E\| = 0, \qquad \rho \in M_*.$$

5. UP maps on finite-dimensional algebras

In this section we show that the asymptotic lift of every UP map acting on a finite-dimensional algebra M embeds in M. We identify M_∞ with the multiplicative core whenever there is a faithful ϕ-invariant state, and more generally, we identify the multiplicative core when ϕ is faithful. We conclude with an elementary example exhibiting nontrivial asymptotic dynamics, for which M_∞ is not closed under the Jordan product of M and hence differs from the multiplicative core.

Theorem 5.1. *Let $\phi : M \to M$ be a UP map on a finite-dimensional von Neumann algebra. Then $(M_\infty, \phi \restriction_{M_\infty}, \iota)$ is the asymptotic lift of ϕ. If, in addition, for every positive operator $x \in M$ one has*

$$\lim_{n\to\infty} \|\phi^n(x)\| = 0 \implies x = 0, \tag{5.1}$$

then M_∞ is the multiplicative core C_ϕ. Condition (5.1) is satisfied whenever there is a faithful state ρ of M satisfying $\rho \circ \phi = \rho$.

The proof requires a known elementary result (see [ES79]):

Lemma 5.2. *Let M be a unital C^*-algebra and let $E : M \to M$ be an idempotent UP map that is faithful: $x \in M^+$, $E(x) = 0 \implies x = 0$. Then $E(M)$ is a Jordan subalgebra of M.*

Proof. Choose a self-adjoint element $x \in E(M)$. Then $E(E(x^2) - x^2) = E(x^2) - E(x^2) = 0$. By Kadison's Schwarz inequality, $x^2 = E(x)^2 \leq E(x^2)$, so that $E(x^2) - x^2 \geq 0$. Since E is faithful, $E(x^2) = x^2 \in E(M)$. This shows that $E(M)^{\text{sa}}$ is closed under the Jordan product, hence $E(M)$ is a Jordan subalgebra of M. □

Proof of Theorem 5.1. There is a sequence $n_1 < n_2 < \cdots$ of positive integers such that ϕ^{n_k} converges to a unique idempotent E (this is a result of Kuperberg, see Theorem 4.1 of [Arv04] et. seq.). Note that $E(M) = M_\infty$. Indeed, for every $x \in M$,

$$E(x) = \lim_k \phi^{n_k}(x) \in \cap_n \overline{\{\phi^n(x), \phi^{n+1}(x) \dots\}} \subseteq \cap_n \phi^n(\text{ball}_{\|x\|} M),$$

hence $E(x) \subseteq M_\infty$. For the opposite inclusion, choose $y \in M_\infty$. Then there is a bounded sequence $x_k \in M$ such that $y = \phi^{n_k}(x_k)$ for every k. Let k' be a subsequence of k such that $x_{k'}$ converges to $x \in M$. Then

$$\|y - \phi^{n_{k'}}(x)\| \leq \|y - \phi^{n_{k'}}(x_{k'})\| + \|\phi^{n_{k'}}(x_{k'}) - \phi^{n_{k'}}(x)\| \leq \|x_{k'} - x\|,$$

and the right side tends to zero as $k' \to \infty$.

It was also shown in [Arv04] that ϕ restricts to a surjective isometry on M_∞ and that the powers of ϕ tend to zero on $\ker E$, so that

$$\lim_{n \to \infty} \|\phi^n \circ E - \phi^n\| = 0.$$

It follows that for every bounded linear functional ρ on M, we have

$$\begin{aligned}\limsup_{n \to \infty} |\, \|\rho \restriction_{M_\infty}\| - \|\rho \circ \phi^n\|\,| &= \limsup_{n \to \infty} |\, \|\rho \circ \phi^n \circ E\| - \|\rho \circ \phi^n\|\,| \\ &\leq \lim_{n \to \infty} \|\rho \circ \phi^n \circ E - \rho \circ \phi^n\| = 0.\end{aligned}$$

The latter implies that $(M_\infty, \phi \restriction_{M_\infty}, \iota)$ is the asymptotic lift of ϕ.

Assuming that (5.1) is satisfied, we claim that $M_\infty = C_\phi$. By Theorem 4.2, it is enough to show that M_∞ is closed under the Jordan multiplication of M. Note that E is must be faithful. Indeed, if $x \geq 0$ and $E(x) = 0$, then

$$\lim_k \|\phi^{n_k}(x)\| = \|E(x)\| = 0,$$

and since the sequence of norms $\|\phi_n(x)\|$ is decreasing, (5.1) implies that $x = 0$. By Lemma 5.2, M_∞ is a Jordan subalgebra of M.

Finally, assuming that there is a faithful ϕ-invariant state ρ, we claim that (5.1) holds. Indeed, if x is a positive operator satisfying $\|\phi^n(x)\| \to 0$ as $n \to \infty$, then

$$|\rho(x)| = |\rho \circ \phi^n(x))| \leq \|\phi^n(x)\| \to 0$$

as $n \to \infty$. Hence $\rho(x) = 0$, and $x = 0$ follows because ρ is faithful. □

We conclude with the following result, which identifies the multiplicative core C_ϕ in many cases where $C_\phi \neq M_\infty$.

Corollary 5.3. *Let $\phi : M \to M$ be a faithful UP map on a finite-dimensional von Neumann algebra. Then the multiplicative core of ϕ is the linear space spanned by the projections in M_∞.*

Proof. Let A be the linear span of all projections in M_∞. We claim that $\phi(A) = A$. Indeed, if e is a projection in A then since ϕ is an order automorphism of M_∞ there is an operator $f \in M_\infty$ such that $0 \leq f \leq \mathbf{1}$ and $\phi(f) = e$. As in the proof of Proposition 4.3, this implies $\phi(f - f^2) = 0$, hence $f = f^2$ is a projection because ϕ is faithful. This implies that $A \subseteq \phi(A)$ and, since A is finite-dimensional, $\phi(A) = A$.

Proposition 4.3 implies that A is the largest Jordan algebra in M_∞. The multiplicative core C_ϕ is a Jordan algebra in M_∞ by Theorem 4.2, hence $C_\phi \subseteq A$. Lemma 4.1 now implies that $C_\phi = A$. □

We conclude by describing an example of a UP map on the 3-dimensional commutative C^*-algebra $M = \mathbb{C}^3$ for which M_∞ is not closed under the ambient Jordan multiplication of M. While there are simpler examples with that specific property, this one exhibits nontrivial asymptotic dynamics that are not detected by the multiplicative core. Viewing the elements of M as column vectors, the map ϕ is multiplication by the stochastic matrix

$$\phi = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The even and odd powers of ϕ are

$$\phi^{2n} = \begin{pmatrix} \frac{1}{9^n} & \frac{1}{2} - \frac{1}{2\cdot 9^n} & \frac{1}{2} - \frac{1}{2\cdot 9^n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \phi^{2n+1} = \begin{pmatrix} \frac{1}{3\cdot 9^n} & \frac{1}{2} - \frac{1}{6\cdot 9^n} & \frac{1}{2} - \frac{1}{6\cdot 9^n} \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and the unique *idempotent* limit point of $\{\phi, \phi^2, \phi^3, \dots\}$ is given by

$$E = \lim_{n\to\infty} \phi^{2n} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The range of E is the two-dimensional space (written as row vectors)

$$M_\infty = E(M) = \{(\frac{a+b}{2}, a, b) : a, b \in \mathbb{C}\}.$$

We summarize the basic properties of this example without proof: *Relative to the intrinsic (Jordan) multiplication defined by $x \circ y = E(xy)$, M_∞ is isomorphic to the two-dimensional commutative C^*-algebra $\mathbb{C}^2$. This identification implements a conjugacy of $\phi \restriction_{M_\infty}$ with the order 2 automorphism $(a, b) \mapsto (b, a)$, $a, b \in \mathbb{C}$. The multiplicative core of ϕ is the one-dimensional C^*-algebra $C_\phi = \mathbb{C} \cdot 1$.*

Remark 5.4 (Invariant states of ϕ). Perhaps it is worth pointing out that the above UP map $\phi : \mathbb{C}^3 \to \mathbb{C}^3$ has a unique invariant state, namely

$$\rho(a, b, c) = \frac{b + c}{2}, \qquad (a, b, c) \in \mathbb{C}^3.$$

This state is not faithful of course, so there is no conflict with Theorem 5.1.

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