Maximal vectors in Hilbert space and quantum entanglement

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Overview

Quantum Information Theory is quantum mechanics in matrix algebras - the algebras $\mathcal{B}(H)$ with H finite dimensional. I'll stay in that context for this talk; but much of the following discussion generalizes naturally to infinite dimensional Hilbert spaces.

We discuss *separability* of states, *entanglement* of states, and propose a numerical measure of entanglement in an abstract context. Then we apply that to compute maximally entangled vectors and states of tensor products $H = H_1 \otimes \cdots \otimes H_N$.

Not discussed: the physics of entanglement, how/why it is a resource for quantum computing, the EPR paradox, Bell's inequalities, Alice and Bob, channels, qubits, philosophy.

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Separable states, entangled states

Consider states of a “composite" quantum system

$$\mathcal{B}(H_1 \otimes \cdots \otimes H_N) \cong \mathcal{B}(H_1) \otimes \cdots \otimes \mathcal{B}(H_N), \quad N = 2, 3, \ldots.$$

A state ρ of $\mathcal{B}(H_1 \otimes \cdots \otimes H_N)$ is said to be *separable* if it is a convex combination of product states $\sigma_1 \otimes \cdots \otimes \sigma_N$

$$\rho(A_1 \otimes \cdots \otimes A_N) = \sum_{k=1}^{s} t_k \cdot \sigma_1^k(A_1) \cdots \sigma_N^k(A_N),$$

with positive t_k summing to 1.

An *entangled* state is one that is not separable. We will see examples shortly.

Entanglement is a noncommutative phenomenon

For commutative tensor products

$$A = C(X_1) \otimes \cdots \otimes C(X_N) = C(X_1 \times \cdots \times X_n)$$

$X_1, \ldots, X_N$ being finite sets, every state of A is a convex combination of pure states, pure states correspond to points of $X_1 \times \cdots \times X_N$, and point masses are pure product states.

Hence every state is a convex combination of product states, and entangled states do not exist.

• The existence of entangled states reflects the fact that observables are operators, not functions, and operator multiplication is not commutative.

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Entangled pure states

- Factoid: for every unit vector $\xi \in H_1 \otimes \cdots \otimes H_N$, the pure state

$$\rho(A) = \langle A\xi, \xi \rangle, \qquad A \in \mathcal{B}(H_1 \otimes \cdots \otimes H_N)$$

is separable iff $\xi = \xi_1 \otimes \cdots \otimes \xi_N$, for some $\xi_k \in H_k$, $1 \leq k \leq N$.

So a vector in the unit sphere $S = \{\xi \in H : \|\xi\| = 1\}$ gives an entangled pure state iff it is *not* decomposable. Such vectors are generic in two senses: they are a dense open subset of S, and they are a set whose complement has measure zero.

- The situation for mixed states is not so simple. For example, entangled states are *not* generic; they are not even dense in the state space.

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Examples in the “bipartite" case $N = 2$

Choose a unit vector $\zeta \in H_1 \otimes H_2$ that does *not* decompose into a tensor product $\xi_1 \otimes \xi_2$, and define

$$\alpha = \sup_{\|\xi_1\|=\|\xi_2\|=1} |\langle \zeta, \xi_1 \otimes \xi_2 \rangle|^2.$$

Easy to see that the self-adjoint operator

$$A = \alpha \cdot \mathbf{1} - \zeta \otimes \bar{\zeta}$$

has the property $(\sigma_1 \otimes \sigma_2)(A) \geq 0$ for every product state $\sigma_1 \otimes \sigma_2$ and hence $\rho(A) \geq 0$ for every separable state ρ.

But the choice of ζ implies that $\alpha < 1$, hence the operator A is *not* positive.

- Conclusion: Every state ρ such that $\rho(A) < 0$ is entangled.

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What are maximally entangled pure states?

The term “maximally entangled pure state" occurs frequently in the physics literature, and several “measures of entanglement" have been proposed in the bipartite case $H = H_1 \otimes H_2$. For example, when $H_1 = H_2$, everyone agrees that

$$\frac{1}{\sqrt{n}}(e_1 \otimes f_1 + \cdots + e_n \otimes f_n)$$

is a maximally entangled unit vector (here, (e_k) and (f_k) are orthonormal bases for $H_1 = H_2$).

But despite the attention it receives, in the multipartite case $H = H_1 \otimes \cdots \otimes H_N$ with $N \geq 3$, there does not seem to be general agreement about what properties a maximally entangled vector should have.

One needs a *definition* to do mathematics....

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Aside: the case $N = 2$ is too special

• The case $H = H_1 \otimes H_2$ has special features that are not available for higher order tensor products.

That is because vectors in $H_1 \otimes H_2$ can be identified with Hilbert Schmidt operators $A : H_1 \to H_2$, so *one can access operator-theoretic invariants to analyze vectors.*

Example: Using the singular value list of the operator that corresponds to a unit vector $\xi \in H_1 \otimes H_2$, it follows that there are orthonormal sets (e_k) in H_1 and (f_k) in H_2 and a set of nonnegative numbers $p_1, \dots, p_n$ with sum 1 such that

$$\xi = \sqrt{p_1} \cdot e_1 \otimes f_1 + \cdots + \sqrt{p_n} \cdot e_n \otimes f_n$$

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$N = 3$ is more typical

In the case $H = H_1 \otimes H_2 \otimes H_3$, one can stubbornly identify vectors in H with various Hilbert Schmidt operators, e.g.,

$$A : H_1 \to H_2 \otimes H_3, \text{ or}$$
$$B : H_2 \to H_1 \otimes H_3, \text{ or}$$
$$C : H_3 \to H_1 \otimes H_2.$$

Which one should we use? Maybe use the triple (A, B, C)? Unfortunately, triples don't have singular value lists.

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Entanglement pairs (H, V)

We will work with pairs (H, V) consisting of a Hilbert space H and a norm-closed set V of unit vectors in H such that:

V1: $\lambda \cdot V \subseteq V$ for every $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

V2: H is the closed linear span of V.

Motivating example: a Hilbert space $H = H_1 \otimes \cdots \otimes H_N$ presented as an N-fold tensor product of Hilbert spaces H_k, where V is the set of all decomposable unit vectors

$$V = \{\xi_1 \otimes \cdots \otimes \xi_N : \xi_k \in H_k,\ \|\xi_1\| = \cdots = \|\xi_N\| = 1\}.$$

Of course there are lots of other examples of entanglement pairs, many/most of which have nothing to do with physics.

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Maximal vectors

Fix an entanglement pair (H, V).

• By a *maximal vector* we mean a unit vector $\xi \in H$ whose distance to V is maximum:

$$d(\xi, V) = \max_{\|\eta\|=1} d(\eta, V),$$

$d(\xi, V)$ denoting the distance from ξ to V.

If H is finite dimensional, then maximal vectors exist; and they exist in many infinite dimensional examples as well.

Maximal vectors are at the opposite extreme from the “central vector" of V described in Jesse Peterson’s talk.

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The simplest examples

Take $H = \mathbb{C}^2$, choose two unit vectors $e_1, e_2 \in H$, and let

$$V = \{\lambda e_1 : |\lambda| = 1\} \cup \{\lambda e_2 : |\lambda| = 1\}.$$

Calculation shows that a unit vector $\xi \in \mathbb{C}^2$ is maximal iff

$$\max(|\langle \xi, e_1 \rangle|, |\langle \xi, e_2 \rangle|)$$

is as small as possible. So taking $e_1 = (1, 0), e_2 = (0, 1)$ to be the usual basis vectors, the maximal vectors turn out to be

$$\xi = (\frac{\lambda}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}}), \qquad |\lambda| = 1.$$

In general, $0 \leq d(\xi, V) \leq \sqrt{2}$. Note that since V is closed,

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Measuring the entanglement of vectors

For every $\xi \in H$ we define a preliminary norm $\|\cdot\|_V$ by

$$\|\xi\|_V = \sup_{v \in V} \Re\langle \xi, v\rangle = \sup_{v \in V} |\langle \xi, v\rangle|.$$

The "entanglement measuring" function from H to the extended interval $[0, +\infty]$ is defined as follows:

$$\|\xi\|^V = \sup_{\|\eta\|_V \leq 1} \Re\langle \xi, \eta\rangle = \sup_{\|\eta\|_V \leq 1} |\langle \xi, \eta\rangle|, \qquad \xi \in H.$$

It is possible for $\|\xi\|^V$ to be infinite (when $\dim H = \infty$); but otherwise, $\|\cdot\|^V$ behaves like a norm on H such that

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The inner radius $r(V)$

- The *inner radius* $r(V)$ is the largest $r \geq 0$ such that

$$\{\xi \in H : \|\xi\| \leq r\} \subseteq \overline{\text{convex hull} V}.$$

In general, $0 \leq r(V) \leq 1$, and $r(V) = 1 \iff V$ is the entire unit sphere of H. More significantly for our purposes:

- If $\dim H < \infty$ then $r(V) > 0$.

Theorem: Each of the three formulas characterizes $r(V)$:

(i) $\inf_{\|\xi\|=1} \|\xi\|_V = r(V)$.

(ii) $\sup_{\|\xi\|=1} \|\xi\|^V = r(V)^{-1}$.

(iii) $\sup_{\|\xi\|=1} d(\xi, V) = \sqrt{2 - 2 \cdot r(V)}$.

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Characterization of maximal vectors

Theorem: For every unit vector $\xi \in H$, the following are equivalent:

(i) $\|\xi\|_V = r(V)$ is minimum.

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More significantly, $\|\cdot\|^V$ measures "degree of entanglement":

Theorem: If $\dim H < \infty$, then $\|\cdot\|^V$ is a norm on H whose restriction to the unit sphere $S = \{\xi \in H : \|\xi\| = 1\}$ has the following properties:

(i) Range of values: $1 \leq \|\xi\|^V \leq r(V)^{-1}$.

(ii) Membership in V: $\xi \in V \iff \|\xi\|^V = 1$.

(iii) Maximal vectors: ξ is maximal $\iff \|\xi\|^V = r(V)^{-1}$.

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Entanglement of mixed states

Fix (H, V). A state ρ of $\mathcal{B}(H)$ is said to be V-correlated if it is a convex combination of vector states of the form

$$\omega(A) = \langle A\xi, \xi\rangle, \qquad \xi \in V.$$

A state that is not V-correlated is said to be V-entangled, or simply entangled.

We introduce a numerical measure of entanglement of *states* as follows. Consider the convex subset of $\mathcal{B}(H)$

$$\mathcal{B}_V = \{A \in \mathcal{B}(H) : |\langle A\xi, \eta\rangle| \leq 1, \forall \xi, \eta \in V\}.$$

$\mathcal{B}_V$ contains the unit ball of $\mathcal{B}(H)$. For every $\rho \in \mathcal{B}(H)'$ define

$$E(\rho) = \sup_{A \in \mathcal{B}_V} |\rho(A)|.$$

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Basic properties of the function E

According to the following result, the function $E(\cdot)$ faithfully detects entanglement of states. Moreover, it recaptures the entanglement norm $\|\xi\|^V$ of unit vectors $\xi \in H$.

Theorem: When $r(V) > 0$, E is a norm on $\mathcal{B}(H)'$ whose restriction to the state space behaves as follows:

(i) $1 \leq E(\rho) \leq r(V)^{-2}$, for every state ρ.

(ii) $E(\rho) = 1$ iff ρ is V-correlated.

(iii) $E(\rho) > 1$ iff ρ is entangled.

(iv) For every pure state $\omega_\xi(A) = \langle A\xi, \xi\rangle$, $A \in \mathcal{B}(H)$,

$$E(\omega_\xi) = (\|\xi\|^V)^2.$$

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Maximally entangled mixed states

So the maximum possible value of $E(\cdot)$ on states is $r(V)^{-2}$.

A state ρ of $\mathcal{B}(H)$ is said to be maximally entangled if

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Theorem: The maximally entangled pure states are the vector states ω_ξ where ξ is a maximal vector.

Every maximally entangled state is a convex combination of maximally entangled pure states.

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Back to earth: Identification of $\|\cdot\|^V$ and $E(\cdot)$

Back to the formative examples (H, V), in which

$$H = H_1 \otimes \cdots \otimes H_N,$$
$$V = \{\xi_1 \otimes \cdots \otimes \xi_N : \xi_k \in H_k,\ \|\xi_k\| = 1\}.$$

Identify the dual of $\mathcal{B}(H)$ with the Banach space $\mathcal{L}^1(H)$ of all trace class operators $A \in \mathcal{B}(H)$ in the usual way

$$\rho(X) = \text{trace}(AX), \qquad X \in \mathcal{B}(H).$$

• **Theorem:** $\|\cdot\|^V$ is the greatest cross norm of the projective tensor product of Hilbert spaces $H_1 \hat{\otimes} \cdots \hat{\otimes} H_N$.

$E(\cdot)$ is the greatest cross norm of the projective tensor product of Banach spaces $\mathcal{L}^1(H_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{L}^1(H_N)$.

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The inner radius

Continuing with the cases

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whereas if $n_N < n_1 n_2 \cdots n_{N-1}$ then all I know is:

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The inner radius

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Identification of maximal vectors

We continue to assume that $n_N \geq n_1 n_2 \cdots n_{N-1}$.

Theorem: A unit vector $\xi \in H_1 \otimes \cdots \otimes H_N$ is maximal iff it purifies the tracial state τ of $\mathcal{A} = \mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1})$:

$$\langle (A \otimes \mathbf{1}_{H_N})\xi, \xi \rangle = \tau(A), \qquad A \in \mathcal{A}.$$

Corollary: The maximal vectors of $H_1 \otimes \cdots \otimes H_N$ are:

$$\xi = \frac{1}{\sqrt{n_1 n_2 \cdots n_{N-1}}}(e_1 \otimes f_1 + \cdots + e_{n_1 n_2 \cdots n_{N-1}} \otimes f_{n_1 n_2 \cdots n_{N_1}}),$$

where (e_K) is an orthonormal basis for $H_1 \otimes \cdots \otimes H_{N-1}$ and (f_k) is an orthonormal set in H_N.

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Unexpected stability of maximal vectors

In more physical terms, consider a tensor product $H \otimes K$ with $n = \dim H \leq m = \dim K < \infty$. The maximal vectors are

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Now assume H is a composite of several subsystems, so that $H = H_1 \otimes \cdots \otimes H_r$. The inner radius of $H_1 \otimes \cdots \otimes H_r \otimes K$ does not change, but the norms $\|\cdot\|^V$ and $E(\cdot)$ *do* change. They depend strongly on the relative sizes of $\dim H_1, \ldots, \dim H_r$.

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Significant problems remain unsolved

We have much less information about N-fold tensor products

$$H = H_1 \otimes \cdots \otimes H_N$$

in cases where $n_N < n_1 n_2 \cdots n_{N-1}$.

Example: $H = (\mathbb{C}^2)^{\otimes N} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$.

- What is the inner radius?
- What are the maximal vectors?
- Which states ρ of $\mathcal{B}(H_1 \otimes \cdots \otimes H_{N-1})$ have maximal vectors as "purifications"? i.e., which ρ can be written in the form

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The case $N = 3$ (in progress)

Let H, K be Hilbert spaces of dimensions p, q. Here is an "operator space" formula for the inner radius $r(p, q, n)$ of $H \otimes K \otimes \mathbb{C}^n$ in the critical cases $n \leq pq$.

Let M_{pq} be the operator space of $p \times q$ complex matrices, $M_{pq} \cong \mathcal{B}(K, H)$. We consider the following two norms on the space of linear maps $\phi : M_{pq} \to M_{pq}$:

$$\|\phi\|_{HS} = (\sum_{i,j=1}^{p,q} \text{trace}\, |\phi(E_{ij})|^2)^{1/2}$$

(the Hilbert Schmidt norm of $\phi : \mathcal{L}^2(K, H) \to \mathcal{L}^2(K, H)$), and

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Formula for the inner radius

The *rank* of ϕ is the dimension of its range $\dim \phi(M_{pq})$.

Theorem: For $n \leq pq$, the inner radius of $\mathbb{C}^p \otimes \mathbb{C}^q \otimes \mathbb{C}^n$ is determined by linear maps $\phi : M_{pq} \to M_{pq}$ as follows:

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Let's save notation by fixing p, q and writing $r_n = r(p, q, n)$ for $n = 1, 2, \ldots, pq$. We can prove that

$$r_1 = \frac{1}{\sqrt{\min(p, q)}} \geq r_2 \geq \cdots \geq r_{pq} = \frac{1}{\sqrt{pq}}.$$

Conjecture: $r(p, q, n) > r(p, q, n+1)$ for $n < pq$.

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Three qubits: $p = q = n = 2$

$$H = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \quad V = \{\xi \otimes \eta \otimes \zeta : \|\xi\| = \|\eta\| = \|\zeta\| = 1\}.$$

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This has significant consequences. For example, maximal vectors must have “unequal weights" (and entropy less than the expected value $\log 2$), in the sense that

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There is compelling numerical evidence (thanks to Michael Lamoureux and Geoff Price) indicating that

$$r(V) \leq 0.68 < \frac{1}{\sqrt{2}} \cong 0.71.$$

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Two days ago, I received an email from Geoff Price in which he seems to prove that $r(2,2,2) \leq \frac{2}{3} \cong 0.68$.

More precisely, for the unit vector

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and with some trickery, he hand-calculates

$$\|\xi\|_V = \sup_{\|v_k\|=1} |\langle \xi, v_1 \otimes v_2 \otimes v_3 \rangle| = \frac{2}{3},$$

which implies $r(2,2,2) \leq 2/3 < 1/\sqrt{2}$.

It is conceivable that $r(2,2,2) = 2/3$, but numerical evidence suggests $r(2,2,2) \leq 0.65 < 2/3$.

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Connects with the local theory of Banach spaces

Let $H_1, \ldots, H_N$ be finite dimensional Hilbert spaces, consider the two Banach spaces

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$$E = H_1 \hat{\otimes} \cdots \hat{\otimes} H_N,$$

and let c be the smallest constant that relates the two norms $\|\xi\|_E \leq c \cdot \|\xi\|_H$. The Banach space folks want to calculate or estimate the value of c, and they have many results.

Our calculations provide the following new result: Arrange that n_N is is the largest of $n_1, \ldots, n_N$. Then

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