NOTES ON THE LIFTING THEOREM

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We have seen that the proof of existence of inverses for elements of Ext(X) can be based on a lifting theorem for (completely) positive maps of $C(X)$ into a quotient $C^*$-algebra of the form $E/K$, where $E \subseteq B(H)$ is a $C^*$-algebra containing the compact operators $K$. That argument works equally well for arbitrary $C^*$-algebras in place of $C(X)$ whenever a completely positive lifting exists. Thus we are led to ask if every completely positive linear map $\phi$ of an arbitrary $C^*$-algebra $A$ into a quotient $C^*$-algebra $B/K$ has a completely positive lifting $\phi_0 : A \to B$. The answer is yes if $A$ is nuclear by a theorem of Choi and Effros [CE76], but no in general. We will sketch a proof of the Choi-Effros theorem that is based on the existence of quasicentral approximate units; full details can be found in [Arv77]. Throughout this lecture, all Hilbert spaces are assumed to be separable.

1. Quasicentral Approximate Units

An operator $T \in B(H)$ is said to be quasidiagonal if there is a sequence $F_n$ of finite rank projections such that $F_n \uparrow 1$ and $\|F_nT - TF_n\| \to 0$ as $n \to \infty$. It is not hard to see that this is equivalent to the existence of a sequence of mutually orthogonal finite-dimensional projections $E_1, E_2, \ldots$ in $B(H)$ such that $\sum_n E_n = 1$ and $T = \sum_{n=1}^{\infty} E_n TE_n + K$ where $K$ is a compact operator. Equivalently, $T$ is quasidiagonal if and only if it is a compact perturbation of a block diagonal operator – a countable direct sum of finite dimensional operators.

Not all operators are quasidiagonal. Indeed, it is an instructive exercise to show that if a Fredholm operator $T$ is quasidiagonal then its index satisfies $\text{ind} T = 0$. Thus the simple unilateral shift is not quasidiagonal. Nevertheless, in this section we will show that it is always possible to find a sequence of positive finite rank operators $F_n$ such that $F_n \uparrow 1$ and $\|F_nT - TF_n\| \to 0$ as $n \to \infty$. Given that result, it is not hard to deduce that there is a sequence of positive finite rank operators $E_1, E_2, \ldots$ such that $\sum_n E_n^2 = 1$ and $\sum_n E_n TE_n$ is a compact perturbation of $T$. Of course, $\sum_n E_n TE_n$ is not necessarily block diagonal or even quasidiagonal, but one can show that it (and $T$ itself) is always a direct summand of a quasidiagonal operator.

Let $K$ be a two-sided ideal in a $C^*$-algebra $A$, not necessarily closed. Recall that an approximate unit for $K$ is an increasing net $u_\lambda$ of positive elements of $K$ such that $\|u_\lambda\| \leq 1$ and $\lim_\lambda \|u_\lambda k - k\| = 0$. If $u_\lambda$ also satisfies $\lim_\lambda \|u_\lambda a - au_\lambda\| = 0$ for all $a \in A$ then $u_\lambda$ will be called quasicentral.

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Theorem 1.1. Every ideal $K$ in a $C^*$-algebra $A$ has a quasicentral approximate unit. If $A$ is separable, the approximate unit can be chosen to be a sequence $u_1 \leq u_2 \leq \cdots$.

We sketch the main idea of the proof, which requires two observations.

First, if $u_\lambda$ is any approximate unit for $K$ and $f$ is any bounded linear functional on $A$, then we have

$$\lim_{\lambda} f(u_\lambda a - au_\lambda) = 0, \quad a \in A. \quad (1.1)$$

Indeed, since every bounded linear functional on $A$ is a linear combination of four positive linear functionals of norm 1, it suffices to prove (1.1) for the states $f$; and in that case the proof is a straightforward argument using the GNS representation for $f$.

Second, given any approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ for $K$, we point out that one can view its convex hull $\Lambda'$ as an approximate unit. Indeed, by definition $\Lambda'$ consists of all finite convex combinations

$$\Lambda' = \{ \theta_1 u_{\lambda_1} + \cdots + \theta_n u_{\lambda_n} : \lambda_j \in \Lambda, \quad \theta_j \geq 0, \quad \theta_1 + \cdots + \theta_n = 1 \}. \quad (1.2)$$

Using the fact that the original net $u_\lambda$ is directed increasing one finds that $\Lambda'$ is an increasing directed set with respect to the operator ordering of $A$.

Thus, we may regard $\Lambda'$ as an increasing directed net of positive operators, relative to the operator ordering, that indexes itself. One can now show that $\Lambda'$ is also an approximate unit for $K$ as on page 330 of [Arv77].

Here is the key observation.

Lemma 1.2. Let $\Lambda$ be a convex approximate unit for an ideal $K$ in a $C^*$-algebra $A$. Then for every finite set of elements $a_1, \ldots, a_n \in A$ and every $\epsilon > 0$, there is an element $u \in \Lambda$ such that

$$\|ua_k - a_k u\| \leq \epsilon, \quad 1 \leq k \leq n. \quad (1.2)$$

Proof of Lemma 1.2. One can immediately reduce to the case $n = 1$ by replacing $A$ with the $n$-fold direct sum $n \cdot A = A \oplus \cdots \oplus A$ of copies of $A$, $K$ with $n \cdot K$, $\Lambda$ with $\{ u \oplus \cdots \oplus u : u \in \Lambda \}$ (a convex approximate unit for $n \cdot K$), and then considering the single element $a_1 \oplus \cdots \oplus a_n \in A \oplus \cdots \oplus A$.

For the case of a single element $a \in A$, (1.2) simply asserts that 0 belongs to the norm closure of the set $C = \{ ua - au : u \in \Lambda \}$. But if $0 \notin C$ then, since $C$ is convex, a standard separation theorem implies that there is a bounded linear functional $f$ on $A$ such that $|f(ua - au)| \geq \epsilon > 0$ for every $u \in \Lambda$, and that contradicts (1.1) above. □

Proof of Theorem 1.1. Choose an arbitrary approximate unit $\{u_\lambda : \lambda \in \Lambda\}$ for $K$ and let $\Lambda'$ be its convex hull. Choose elements $a_1, \ldots, a_n \in A$, $v \in \Lambda'$ and $\epsilon > 0$. Since $\{ u \in \Lambda' : u \geq v \}$ is a cofinal convex subnet of $\Lambda'$, it is also a convex approximate unit for $K$. Thus, Lemma 1.2 implies that there is an element $w \geq v$ in $\Lambda'$ such that $\|wa_k - a_k w\| \leq \epsilon$ for $k = 1, \ldots, n$.

That assertion is clearly enough to allow us to extract a subnet $v_\lambda$ of $\Lambda'$ with the property that $\lim_{\lambda} \|v_\lambda a - av_\lambda\| = 0$ for every $a \in A$, and such a
subnet \{v_\lambda\} is a quasicentral approximate unit. The proof that \{v_\lambda\} can be chosen as a sequence when \(A\) is separable is a straightforward argument that we omit.

\[\square\]

Remark 1.3. Theorem 1.1 was discovered during the writing of [Arv77]; it was discovered independently by Charles Akemann and Gert Pedersen in their work on ideal perturbations of elements of \(C^*\)-algebras, at about the same time.

2. LIFTABLE MAPS

Let \(A\) be and \(B\) be unital \(C^*\)-algebras. We consider completely positive linear maps \(\phi : A \to B\) which preserve units in the sense that \(\phi(1_A) = 1_B\), and we refer to such a \(\phi\) as a UCP map. Given a closed two-sided ideal \(K \subseteq B\) in a unital \(C^*\)-algebra \(B\), we want to know if there is a UCP map \(\phi : B/K \to B\) that provides a lifting for the projection \(b \in B \mapsto b \in B/K\). More generally, given a UCP map from a given \(C^*\)-algebra \(A\) into a quotient \(B/K\), the lifting problem for \(\phi\) is the the problem of finding a UCP map \(\psi : A \to B\) such that \(\psi(a) = \phi(a)\), \(a \in A\). If such a map \(\psi\) exists then \(\phi\) is said to be liftable and we write \(\phi = \dot{\psi}\).

In this section we discuss some results about liftable maps in general, most of which depend strongly on quasicentral approximate units. Throughout this section and the next, \(A\) will denote a separable \(C^*\)-algebra with unit. Let us fix an ideal \(K\) in another unital \(C^*\)-algebra \(B\) (\(B\) need not be separable), and consider the set \(\text{UCP}(A, B/K)\) of all UCP maps \(\phi : A \to B/K\) as a topological space in its point-norm topology. Thus, a net \(\phi_\lambda : A \to B/K\) of linear maps converges to a map \(\phi : A \to B/K\) iff

\[\lim_\lambda \|\phi_\lambda(a) - \phi(a)\| = 0, \quad a \in A.\]

The first key fact is that in general, the set of liftable maps is closed:

**Theorem 2.1.** Let \(A\) be a separable unital \(C^*\)-algebra. Then the set of all liftable UCP maps from \(A\) to a quotient \(B/K\) is closed in the point-norm topology of \(\text{UCP}(A, B/K)\).

We will say something about what goes into the proof of Theorem 2.1, skipping over the technicalities. The first observation is that the point-norm topology is metrizable when \(A\) is separable. Indeed, if we fix a sequence \(a_1, a_2, \ldots\) of elements that is dense in the unit ball of \(A\) then

\[d(\phi, \psi) = \sum_{n=1}^{\infty} 2^{-n} \|\phi(a_n) - \psi(a_n)\|\]

is a metric with the stated property. Let us fix a sequence \(a_1, a_2, \ldots\) throughout the discussion, thereby fixing a single metric \(d(\cdot, \cdot)\) on \(\text{UCP}(A, B/K)\) that makes it into a complete metric space.
Suppose that we are given a pair of UCP maps $\phi, \psi : A \to B$. Then we may compose both maps with the natural projection $b \in B \mapsto \hat{b} \in B/K$ to obtain UCP maps $\hat{\phi}, \hat{\psi} : A \to B/K$ into the quotient. Obviously,

$$d(\hat{\phi}, \hat{\psi}) \leq d(\phi, \psi).$$

What is important here is that the left side of this inequality can almost be realized by perturbing one of the two maps $\phi, \psi : A \to B$.

**Lemma 2.2.** For any two UCP maps $\phi, \psi : A \to B$ and every $\epsilon > 0$, there is a UCP map $\psi' : A \to B$ such that $d(\phi, \psi') \leq d(\phi, \psi) + \epsilon$.

**Sketch of proof.** Let $u_\lambda$ be an approximate unit for $K$ that is quasicentral in $B$. For every $\lambda$ we can define a UCP map $\psi_\lambda : A \to B$ by

$$\psi_\lambda(a) = u_\lambda^{1/2} \phi(a) u_\lambda^{1/2} + (1 - u_\lambda)^{1/2} \psi(a)(1 - u_\lambda)^{1/2}, \quad a \in A.$$ 

Obviously, $\psi' = \hat{\psi}$. The main property of these perturbations $\psi_\lambda$ is:

$$\limsup_{\lambda} d(\phi, \psi_\lambda) \leq d(\phi, \hat{\psi}). \tag{2.1}$$

The estimates required for the proof of (2.1) can be found on pp. 346–347 of [Arv77]. Once one has (2.1), one obtains the assertion of Lemma 2.2 by choosing an appropriately large $\lambda$. \qed

**Proof of Theorem 2.1.** To prove Theorem 2.1, let $\phi_1, \phi_2, \ldots$ be a sequence of liftable maps in $\text{UCP}(A, B/K)$ that converges to a UCP map $\phi_\infty$. By passing to a subsequence if necessary, we can also arrange that $d(\phi_n, \phi_\infty) < 1/2^n$.

We claim that there is a sequence $\psi_1, \psi_2, \ldots$ in $\text{UCP}(A, B/K)$ satisfying

$$\psi_n = \phi_n \text{ and } d(\psi_n, \psi_{n+1}) < 1/2^n, \quad n = 1, 2, \ldots.$$ 

Indeed, let $\psi_1$ be any UCP lifting of $\phi_1$. Assuming that $\psi_1, \ldots, \psi_n$ have been defined and satisfy the stated conditions, choose any lifting $\lambda$ of $\phi_{n+1}$. Noting that $d(\psi_n, \lambda) = d(\phi_n, \phi_{n+1}) < 1/2^n$, Lemma 1.2 implies that there is a UCP map $\psi_{n+1}$ satisfying

$$\psi_{n+1} = \lambda = \phi_{n+1} \text{ and } d(\psi_n, \psi_{n+1}) < 1/2^n.$$ 

Since $\sum_n d(\psi_n, \psi_{n+1}) < \infty$, $\{\psi_n\}$ is a Cauchy sequence relative to the $d$-metric, and we can define a UCP map $\psi_\infty$ as the limit $\lim_n \psi_n$. Since $\psi_n = \phi_n$ converges to $\phi_\infty$, $\psi_\infty$ is a lifting of $\phi_\infty$. \qed

The second key fact that we require is that the lifting problem can always be solved for matrix algebras $M_n = M_n(\mathbb{C})$, $n = 1, 2, \ldots$:

**Proposition 2.3** (M.-D. Choi). Every UCP map $\phi : M_n \to B/K$ is liftable.

**Sketch of Proof.** Let $\{e_{pq} : 1 \leq p, q \leq n\}$ be a system of matrix units for $M_n$. Thus, $e_{pq} e_{rs} = \delta_{qr} e_{ps}$, $e_{pq}^* = e_{qp}$, and $M_n$ is spanned by $\{e_{pq}\}$. Define an array of elements $f_{pq} \in B/K$ by $f_{pq} = \phi(e_{pq})$. The $n \times n$ matrix $(f_{pq})$ can be considered an element of $M_n \otimes (B/K) \cong (M_n \otimes B)/(M_n \otimes K)$. It is positive because $\phi$ is $n$-positive. An elementary exercise with the functional calculus shows that every positive element of a quotient of $C^*$-algebras can be lifted to a positive element of the ambient $C^*$-algebra. Applying this to the ideal
M_n \otimes K in M_n \otimes B we obtain a positive n \times n matrix (F_{pq}) of elements of B such that F_{pq} projects} to f_{pq}, via the map B \to B/K, 1 \leq p, q \leq n.

Now let \phi_0 : M_n \to B be the unique linear map satisfying \phi_0(e_{pq}) = F_{pq}, 1 \leq p, q \leq n. Since the n \times n matrix (\phi_0(e_{pq})) = (F_{pq}) is positive, it follows that \phi_0 must be completely positive (see Lemma 3.2 of [Arv77]). Obviously \phi_0 is a lifting of \phi. \phi_0 need not carry unit to unit, but a simple argument shows that it can be perturbed into another completely positive lifting that does (see p. 350 of [Arv77]).

3. LIFTINGS AND NUCLEARITY

We now show that the results of the preceding section imply the lifting theorem for nuclear C*-algebras. Let A and B be unital C*-algebras. A UCP map \phi : A \to B is called factorable if it can be factored through some matrix algebra M_n, n = 1, 2, \ldots in the sense that there are UCP maps \sigma : A \to M_n and \tau : M_n \to B such that \phi = \tau \circ \sigma. \phi is called a nuclear map if it belongs to the point-norm closure of the set of all factorable maps in UCP(A, B). Finally, a C*-algebra A is called nuclear if the identity map of A is a nuclear map. The notion of nuclearity is equivalent to several natural and useful properties when A is separable, including:

(i) For any C*-algebra B, the natural *-homomorphism of A \otimes_{\max} B onto A \otimes_{\min} B is an isomorphism.
(ii) A is amenable.
(iii) The weak closure of A in any representation is injective.
(iv) The weak closure of A in any factor representation is hyperfinite.

The following result was originally proved in [CE76] by a rather different method.

**Theorem 3.1** (Choi-Effros). Every nuclear UCP map from a separable C*-algebra A into a quotient B/K is liftable.

**Proof.** Let \phi : A \to B/K be a nuclear UCP map. By Theorem 2.1, the set of liftable maps in UCP(A, B/K) is closed in the point-norm topology. Since \phi is the point-norm limit of a set of factorable UCP maps, it suffices to show that every factorable UCP map is liftable.

But if \phi is a composition \tau \circ \sigma where \sigma : A \to M_n and \tau : M_n \to B/K are UCP maps, then Proposition 2.3 implies that \tau can be lifted to a UCP map \tau_0 : M_n \to B, and clearly \tau_0 \circ \sigma is a lifting of \phi = \tau \circ \sigma. □

**Corollary 3.2.** Every UCP map of a separable nuclear C*-algebra A into a quotient B/K is liftable.

**References**
