EXTENSIONS OF \( \mathcal{K} \) BY \( C(X) \)

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An operator \( A \in \mathcal{B}(H) \) whose self-commutator \( A^*A - AA^* \) is compact is called \textit{essentially normal}. Two operators \( A \in \mathcal{B}(H) \) and \( B \in \mathcal{B}(K) \) are said to be \textit{approximately equivalent} if they are unitarily equivalent modulo compact operators; more precisely, if there is a unitary operator \( U : H \rightarrow K \) such that \( B - UAU^* \) is compact. This relation is written \( A \sim B \), whereas the stronger relation of unitary equivalence will be written \( A \sim = B \). Roughly speaking, \( A \sim B \) means that \( A \) and \( B \) have the same geometric properties, while \( A \sim = B \) means that \( A \) and \( B \) have the same \textit{asymptotic} properties (see Chapter 3 of [Arv01]). We begin by discussing the classification of essentially normal operators, and its generalization to the computation of \( \text{Ext}(X) \), originating in work of Brown, Douglas and Fillmore during the mid seventies [BDF77]. In a subsequent lecture we will describe the connection between those results, quasicentral approximate units and the lifting theorem for nuclear \( C^* \)-algebras.

1. Essentially Normal Operators and Extensions

Every operator \( A \in \mathcal{B}(H) \) has an \textit{essential spectrum} \( \sigma_e(A) \), defined as the spectrum of the image of \( A \) in the Calkin algebra \( \mathcal{B}(H)/\mathcal{K} \). The essential spectrum of \( A \) is a nonvoid compact subset of the complex plane, and it provides an invariant for approximate equivalence:

\[
A \sim B \implies \sigma_e(A) = \sigma_e(B).
\]

On the surface of it, one might guess that the essential spectrum is a \textit{complete} invariant for essentially normal operators. But the unilateral shift \( S \) and its adjoint \( S^* \) provide a simple example of two essentially normal operators having the same essential spectrum which are not approximately equivalent. Indeed, elementary computations show that both \( S \) and \( S^* \) are essentially normal operators with essential spectrum the unit circle \( \mathbb{T} \). It follows that both \( S \) and \( S^* \) are Fredholm operators, and one observes that

\[
\text{ind } S = -1, \quad \text{ind } S^* = +1.
\]

So if \( S^* \sim S \) then \( S^* \) would be unitarily equivalent to a compact perturbation of \( S \); but that would imply \( \text{ind } S^* = \text{ind } S \) because the Fredholm index is stable under unitary equivalence and compact perturbations (see [Arv01]).

Given a compact subset \( X \) of the complex plane, one may consider the class \( \mathcal{E}N(X) \) of all essentially normal operators \( A \) that act on a separable

\textit{Date: Lecture given 15 September, 2003.}
Hilbert space and have essential spectrum $\sigma_e(A) = X$. Strictly speaking, one must be careful in order to avoid set-theoretic anomalies in forming $\mathcal{EN}(X)$, and the easiest way to do that is to select a particular infinite-dimensional separable Hilbert space $H_0$ and define $\mathcal{EN}(X)$ to be the indicated subset of operators on $H_0$. With this convention, the direct sum of two operators $A, B \in \mathcal{B}(H_0)$ should be defined as $W^*(A \oplus B)W$, where $W : H_0 \to H_0 \oplus H_0$ is a unitary operator that is, once and for all, fixed. We will systematically ignore such issues, and instead we treat the proper class $\mathcal{EN}(X)$ as if it were a set, and treat the direct sum $A \oplus B$ in the usual way. Such set-theoretically naive conventions will not cause trouble so long as we limit ourselves to countable operations.

For a nonvoid compact subset $X \subseteq \mathbb{C}$, we define $\text{Ext}(X)$ to be the set of equivalence classes $\mathcal{EN}(X)/\sim$. $\text{Ext}(X)$ is in general an honest set, and it carries a natural binary operation $+$ defined by $[A] + [B] = [A \oplus B]$, where $[A]$ denotes the equivalence class of an operator $A \in \mathcal{EN}(X)$. This addition is commutative and associative, making $\text{Ext}(X)$ into a commutative semigroup. The problem of classifying essentially normal operators having essential spectrum $X$ becomes that of a) determining any additional structure that may exist on $\text{Ext}(X)$ and b) describing a set of concrete invariants for distinguishing between the elements of $\text{Ext}(X)$.

We have already alluded to the fact that the Fredholm index provides a nontrivial invariant. More precisely, choose any operator $A \in \mathcal{B}(H)$ having essential spectrum $X$. For every complex number $\lambda$ in the complement of $X$, the operator $A - \lambda = A - \lambda \mathbf{1}$ is a Fredholm operator whose index $\text{ind}(A - \lambda)$ is a function of $\lambda$ that is constant throughout each connected component of $\mathbb{C} \setminus X$, which vanishes identically on the unbounded component of $\mathbb{C} \setminus X$, and which is stable under compact perturbations. Thus we have defined an integer-valued function from the set of bounded components of $\mathbb{C} \setminus X$ that provides a concrete invariant for approximate equivalence in $\mathcal{EN}(X)$. The theorem of Brown, Douglas and Fillmore for subsets of the plane implies that this is a complete invariant:

**Theorem 1.1 (BDF Theorem).** Let $A$ and $B$ be essentially normal operators having the same essential spectrum $X \subseteq \mathbb{C}$. Then $A \sim B$ iff

$$\text{ind}(A - \lambda) = \text{ind}(B - \lambda), \quad \lambda \notin X.$$  

Moreover, with respect to the operation defined by $[A] + [B] = [A \oplus B]$, $\text{Ext}(X)$ is an abelian group and the map $X \to \text{Ext}(X)$ defines a homotopy-invariant functor from compact subsets of $\mathbb{C}$ to abelian groups.

To say that $\text{Ext}(X)$ is an abelian group involves two concrete assertions:

(i) There is an essentially normal operator $N$ having essential spectrum $X$ which acts as a neutral element in that $N \oplus A \sim A$ for every $A \in \mathcal{EN}(X)$.

(ii) For every $A \in \mathcal{EN}(X)$ there is a $B \in \mathcal{EN}(X)$ such that $A \oplus B \sim N$, where $N$ is the “neutral” operator of (i).
Indeed, the main results of [BDF77] addressed a more general problem, in which $X \to \text{Ext}(X)$ was shown to be a homotopy-invariant functor from the category of compact metric spaces $X$ to abelian groups, that in fact gives a concrete realization of $K$-homology by way of the theory of extensions of commutative $C^*$-algebras by the compact operators. It is significant that in the broader category of compact metric spaces $X$ (or even compact $C^\infty$ manifolds), there are invariants for $\text{Ext}(X)$ that are more subtle than those associated with the Fredholm index. For example, the group $\text{Ext}(X)$ can have torsion - while on the other hand, any invariant associated with the Fredholm index cannot detect torsion elements of $\text{Ext}(X)$. Thus, the realization of $\text{Ext}(X)$ as the $K$-homology of $X$ provided essentially new information about almost commuting sets of operators on Hilbert spaces.

We now describe some of the key issues in the more general BDF theorem. Let $X$ be a compact metrizable space and let $C(X)$ be the commutative $C^*$-algebra of complex-valued continuous functions on $X$. We will write $\mathcal{K}$ for the ideal of all compact operators on a given separable infinite-dimensional Hilbert space $H$. By an extension of $\mathcal{K}$ by $C(X)$ we mean a $\ast$-monomorphism $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ such that $\sigma(1) = 1$. Given two Hilbert spaces $H_1$, $H_2$ and extensions $\sigma_k : C(X) \to \mathcal{B}(H_k)/\mathcal{K}$ we write $\sigma_1 \sim \sigma_2$ if there is a unitary operator $U : H_1 \to H_2$ such that

$$\theta_U(\sigma_1(f)) = \sigma_2(f), \quad f \in C(X)$$

where $\theta_U$ is the $\ast$-isomorphism of Calkin algebras induced by the spatial $\ast$-isomorphism $T \in \mathcal{B}(H_1) \to UTU^* \in \mathcal{B}(H_2)$. $\text{Ext}(X)$ is defined as the set of equivalence classes of such maps $\sigma$. It is a good exercise to show that when $X$ is a compact subset of the complex plane $\mathbb{C}$, a) extensions of $C(X)$ by $\mathcal{K}$ correspond to essentially normal operators with essential spectrum $X$, b) two operators determine the same extension iff they differ by a compact operator, and c) equivalence of extensions corresponds to approximate equivalence of operators.

It is useful to view extensions as short exact sequences of $C^*$-algebras in the following way. Given a $\ast$-monomorphism $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ as above, let $T \in \mathcal{B}(H) \mapsto \hat{T} \in \mathcal{B}(H)/\mathcal{K}$ be the natural projection onto the Calkin algebra, and consider the associated short exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \pi C(X) \to 0,$$

where $\mathcal{E} = \{T \in \mathcal{B}(H) : \hat{T} \in \sigma(C(X))\}$, the map of $\mathcal{K}$ to $\mathcal{E}$ is inclusion, and $\pi : \mathcal{E} \to C(X)$ is given by composing the natural map of $\mathcal{E}$ to the Calkin algebra with the inverse of $\sigma$, $\pi(T) = \sigma^{-1}(\hat{T})$. Conversely, every exact sequence of the form (1.1) arises from a uniquely determined extension $\sigma : C(X) \to \mathcal{B}(H)/\mathcal{K}$ as defined in the preceding paragraph.

Notice that an exact sequence of the form (1.1) is defined uniquely by specifying a pair $(\mathcal{E}, \pi)$ consisting of a $C^*$-algebra $\mathcal{E}$ of operators on $H$ that contains $\mathcal{K}$ together with a surjective $\ast$-homomorphism $\pi : \mathcal{E} \to C(X)$ that satisfies $\ker \pi = \mathcal{K}$. Two sequences such as (1.1) are said to be equivalent if
their associated pairs \((\mathcal{E}_k, \pi_k)\) are related as follows: there is a \(*\)-isomorphism \(\theta : \mathcal{E}_1 \to \mathcal{E}_2\) such that \(\pi_2 \circ \theta = \pi_1\). This equivalence relation can also be viewed as an equivalence relation existing between short exact sequences of the form (1.1), and it is denoted \(\sim\). Since both \(\mathcal{E}_1\) and \(\mathcal{E}_2\) contain the compact operators and \(\text{ker}\ \pi_k = \mathcal{K}\), it is a straightforward exercise to show that both the equivalence map \(\theta\) and its inverse must carry compact operators to compact operators, and is therefore implemented by a unitary operator \(U : H_1 \to H_2\) by way of \(\theta(T) = U T U^*,\ T \in \mathcal{E}_1\). In this way one sees that two short exact sequences of the form (1.1) with pairs \((\mathcal{E}_1, \pi_1)\) and \((\mathcal{E}_2, \pi_2)\) are equivalent iff their associated extensions \(\sigma_1\) and \(\sigma_2\) are equivalent.

References
