
Exercise 1. Exercise 25, page 420, part a). You can assume that $M$ is a complete metric space.

b) Give a simple example of an upper semicontinuous function $f : [0, 1] \to \mathbb{R}$ that is not continuous.

c). Let $M$ have its natural Borel $\sigma$-algebra, namely, the $\sigma$-algebra generated by the family of open subsets of $M$. Show that every upper semicontinuous function $f : M \to \mathbb{R}$ is Borel-measurable.

Exercise 2. Let $(X, \mathcal{B})$ be a Borel space and let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of measurable functions. Show that the set of all points $x \in X$ for which the sequence $f_1(x), f_2(x), \ldots$ converges belongs to $\mathcal{B}$.

Exercise 3. Let $X : \Omega \to \mathbb{R}$ be a random variable. For every Borel set $E \subseteq \mathbb{R}$, let $m_X(E)$ be the number $P\{\omega \in \Omega : X(\omega) \in E\}$. $m_X(E)$ represents the probability of the event “$X$ belongs to $E$”. Show that $m_X$ is a $\sigma$-additive probability measure on the $\sigma$-algebra of all Borel sets of $\mathbb{R}$. The measure $m_X$ is called the probability distribution of the random variable $X$.

Exercise 4. Let $X, Y : \Omega \to \mathbb{R}$ be two random variables.

a) Show that for every Borel set $E \subseteq \mathbb{R}^2$, the set $\{\omega \in \Omega : (X(\omega), Y(\omega)) \in E\}$ belongs to $\mathcal{E}$. Thus we can consider its probability

$$m_{X,Y}(E) = P\{\omega \in \Omega : (X(\omega), Y(\omega)) \in E\}.$$ 

b) Show that $m_{X,Y}$ is a probability measure on $\mathbb{R}^2$. $m_{X,Y}$ is called the joint distribution of the pair $(X, Y)$.

c) Show that both $m_X$ and $m_Y$ are uniquely determined by the joint distribution by writing down an explicit formula for $m_X$ and $m_Y$ in terms of $m_{X,Y}$.

Exercise 5. Let $(\Omega, \mathcal{B}, P)$ be the probability space $\Omega = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, $\mathcal{B} = 2^\Omega$, with probabilities $P\{(i, j)\} = 1/4$, $1 \leq i, j \leq 2$. Show that the converse of Exercise 4 c) is false by giving examples of two pairs of random variables $X, Y$, and $X', Y'$, such that $m_X = m_Y = m_{X'} = m_{Y'}$, but $m_{X,Y} \neq m_{X',Y'}$. 

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