
A set \( A \) is said to be countable if it is either finite, or countably infinite in the sense that there is a bijection \( f : \{1, 2, 3, \ldots \} \to A \). Thus, the elements of any nonempty countable set can be enumerated \( A = \{x_1, \ldots, x_n\} \) for some finite positive integer \( n \), or else \( A = \{x_1, x_2, \ldots\} \), with \( x_i \neq x_j \) for all \( i \neq j \).

A \( \sigma \)-algebra is a family \( A \) of subsets of a fixed nonempty set \( X \) with the following properties:

(i) \( \emptyset \in A \).
(ii) \( E \in A \implies X \setminus E \in A \), where \( X \setminus E \) denotes the complement of \( E \).
(iii) If \( E_1, E_2, \ldots \) is a sequence of elements of \( A \) then \( \bigcup_n E_n \in A \).

We have pointed out in the lecture that every set \( X \) has a smallest \( \sigma \)-algebra \( A_0 = \{\emptyset, X\} \) and a largest one \( A_1 = 2^X \) = \{all subsets of \( X\}\}. In these problems you will look at other examples.

**Exercise 1.** Let \( X \) be a nonempty set and let \( A \) be the family of all subsets \( E \subseteq X \) which are either countable or co-countable (thus, a set \( E \) belongs to \( A \) iff \( E \) is countable or \( X \setminus E \) is countable). Show that \( A \) is a \( \sigma \)-algebra.

**Exercise 2.** Answer true or false, or yes or no, giving a brief reason for your reply. The following assertions/questions relate to the \( \sigma \)-algebra \( A \) of Exercise 1, for various sets \( X \).

(a) For a countable set \( X \), \( A = 2^X \).
(b) If \( X = [0,1] \) is the unit interval then the set of rational numbers in \([0,1]\) belongs to \( A \).
(c) If \( X = [0,1] \), then the set of irrational numbers in \([0,1]\) belongs to \( A \).
(d) If \( X = [-1,1] \) does the set of rational numbers in \([0,1]\) belong to \( A \)?
(e) Let \( X = [-1,1] \) be as in (c), let \( B \) be the set of rational numbers in \([0,1]\) and let \( C \) be the set of irrational numbers in \([-1,0]\). Does \( B \cup C \in A \)?

**Exercise 3.** Let \( X \) be a set and let \( \mathcal{F} \) be an arbitrary nonempty family of subsets of \( X \). Show that there is a smallest \( \sigma \)-algebra \( A \) that contains every set of \( \mathcal{F} \) in the sense that 1) \( A \) is a \( \sigma \)-algebra containing \( \mathcal{F} \), and 2) for every other \( \sigma \)-algebra \( B \) which contains \( \mathcal{F} \) one has \( B \supseteq A \).

The \( \sigma \)-algebra \( A \) associated with a family of sets \( \mathcal{F} \) as in Exercise 3 is called the \( \sigma \)-algebra generated by \( \mathcal{F} \). The remaining exercises relate to the real line \( X = \mathbb{R} \) and the \( \sigma \)-algebra \( B \) generated by the family \( \{(a,b) : -\infty < a < b < \infty\} \) of all open intervals in \( \mathbb{R} \). \( B \) is called the Borel \( \sigma \)-algebra of the real line, and subsets of \( \mathbb{R} \) that belong to \( B \) are called Borel sets.

**Exercise 4.**

(a) Show that every open subset of \( \mathbb{R} \) is a Borel set. Hint: show that every open set can be written as a union of open intervals with rational endpoints.
(b) Show that every closed subset of \( \mathbb{R} \) is a Borel set.
(c) Show that the set \( (0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\} \) is a Borel set.

**Exercise 5.** Can you exhibit a subset of \( \mathbb{R} \) that is not a Borel set? If your answer is “no”, then just say that; if your answer is “yes” please give an example.