Relation of the Riemann integral to the Legesgue integral.

The purpose of these notes is to review the basic properties of the Riemann integral of a real-valued function and to relate it to the Lebesgue integral. The results described below can be used freely in the problem sets due next Monday, 28 April. I will ask you to think more about these results in next week's problem set.

We begin by recalling the definition of the Riemann integral of a bounded real-valued function $f : [a, b] \to \mathbb{R}$. f can take on positive and/or negative values, but it is essential that a) f be bounded, and b) the domain of f be a compact interval. For every finite partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$, let m_k and M_k be the lower and upper bounds of the restriction of f to the kth subinterval

$$m_k = \inf_{x_{k-1} \le x \le x_k} f(x), \quad M_k = \sup_{x_{k-1} \le x \le x_k} f(x), \qquad 1 \le k \le n.$$

The corresponding lower and upper Riemann sums are defined by

$$\underline{I}(f, \mathcal{P}) = \sum_{k=1}^{n} m_k \Delta_k, \quad \overline{I}(f, \mathcal{P}) = \sum_{k=1}^{n} M_k \Delta_k,$$

where $\Delta_k = x_k - x_{k-1}$, $1 \le k \le n$. The lower and upper Riemann integrals are defined by

$$\underline{I}(f) = \sup_{\mathcal{P}} \underline{I}(f, \mathcal{P}), \quad \overline{I}(f) = \inf_{\mathcal{P}} \overline{I}(f, \mathcal{P}).$$

Obviously, $\underline{I}(f) \le \overline{I}(f)$, and f is said to be *Riemann integrable* if equality holds. In this case, we will write the common value as

$$I(f) = \underline{I}(f) = \overline{I}(f),$$

rather than $\int_a^b f(x)\,dx$, so as not to confuse this integral with the Lebesgue integral. We reserve the notation $\int_a^b f(x)\,dx$ for the Lebesgue integral of f.

The first important result on the existence of the Riemann integral is the following, whose proof is usually sketched in math 1A and is carried out in detail in math 104:

Theorem A. *Every continuous function $f \in C[a, b]$ is Riemann integrable.*

We have seen that every function $f \in C[a, b]$ is Lebesgue integrable (i.e., f is Borel-measurable and $\int_a^b |f(x)|\,dx < \infty$), but we have not examined the relation between the two integrals for continuous functions. The key result is:

Theorem B. *For every $f \in C[a, b]$ the two integrals agree:*

$$\int_a^b f(x)\,dx = \underline{I}(f) = \overline{I}(f).$$

To summarize: *The Riemann integral makes sense only for functions f that are defined on a compact interval, and which are bounded there. Continuous functions are Riemann integrable, and their Riemann and Lebesgue integrals coincide.*

This is a convenient result for computing the Lebesgue integrals of continuous functions, since it implies that for such functions you can use all the calculus tools you learned, such as antiderivatives, for computing $\int_a^b f(x)\,dx$.

In elementary calculus, various "improper" Riemann integrals are introduced in order to relax the two requirements (compact domain, boundedness). These improper integrals make the Riemann integral more useful and flexible; for example, improper integrals were there whenever you used the integral test to check an infinite series for absolute convergence. We now discuss two kinds of improper integrals, and show that they, too, can be interpreted as Lebesgue integrals in a very natural way. For simplicity, we consider only improper integrals of continous nonnegative functions, though it would not be very hard to extend the discussion so as to include others.

Suppose we are given a continuous function f defined on a semi-infinite interval $[a,\infty)$, satisfying $f(x)\geq 0$, $x\geq a$. For every $T>a$, f restricts to a nonnegative function in $C[a,T]$ and we can form its Riemann integral. These integrals are numbers that depend on T and we denote them by $I_T(f)$. Since f is nonnegative, $I_T(f)$ is an increasing function of T, so that either $I_T(f)$ increases to $+\infty$, or it converges to an ordinary limit. By definition, f is said to be *improperly integrable* if the partial integrals $I_T(f)$ remain bounded, and in that case the improper integral of f is defined by

$$I(f)=\lim_{T\to\infty} I_T(f).$$

Note that for a nonnegative continuous function f, the improper Riemann integral $I(f)$ exists if and only if the partial integrals $I_T(f)$ are bounded, $\sup_{T\geq a} I_T(f)<\infty$.

Improper integrals of this type are "ordinary" Lebesgue integrals in the following sense:

Theorem C. *Let f be a nonnegative function in $C[a,\infty)$. If f is improperly Riemann integrable then f belongs to the Lebesgue space $L^1[a,\infty)$ and we have*

$$\int_a^\infty f(x)\,dx=\lim_{T\to\infty} I_T(f).$$

The second type of improper integral that one encounters in elementary calculus courses involves functions that are defined on bounded intervals like $(a,b]$ or $[a,b)$, or (a,b), but which are unbounded on their domain. Example: the function

$$f(x)=\frac{1}{\sqrt{x}},\qquad 0<x\leq 1.$$

For definiteness, we consider a continuous nonnegative function f defined on an interval $(a,b]$, but which is perhaps unbounded near the left endpoint. In this case, the improper Riemann integral of f is defined as follows. For every positive number ϵ satisfying $0<\epsilon<b-a$, consider the restriction of f to the compact subinterval $[a+\epsilon,b]$. This function is Riemann integrable, and we denote its Riemann integral by $I^\epsilon(f)$. Again, one sees that as ϵ *decreases* to a, the integrals $I^\epsilon(f)$ increase. f is said to be improperly Riemann integrable if these partial integrals $I^\epsilon(f)$ remain bounded, and in that case the improper integral of f is defined by taking the limit of the monotonic partial integrals

$$I(f)=\lim_{\epsilon\to a+} I^\epsilon(f).$$

One would expect that improper integrals of this kind can also be incorporated into the Lebesgue theory, and that is correct:

Theorem D. *Let $f : (a, b] \to [0, \infty)$ be a nonnegative continuous function. If f is improperly Riemann integrable then it belongs to the Lebesgue space $L^1(a, b]$ and we have*

$$\int_{(a,b]} f(x)\,dx = \lim_{\epsilon \to a+} I^{\epsilon}(f).$$

We will return to these issues later in the course, when we discuss Lebesgue's characterization of Riemann integrable functions:

Lebesgue's characterization or Riemann integrable functions. *Let f be a bounded real-valued function defined on a compact interval $[a, b]$. Then f is Riemann integrable if, and only if, the set*

$$D = \{x \in [a, b] : f \textit{ is not continuous at } x\}$$

of all discontinuity points of f is a set of Lebesgue measure zero.

More briefly, this theorem asserts that a bounded function is Riemann integrable iff it is continuous almost everywhere. It is a fact that such a function can be modified on a set of Lebesgue measure zero so as to make it Borel-measurable, and once that is done, the Lebesgue integral of f and the Riemann integral of f agree. This is the precise sense in which the Lebesgue integral generalizes the Riemann integral: *Every bounded Riemann integrable function defined on $[a, b]$ is Lebesgue integrable, and the two integrals are the same.*