$SU(2)$ Refined Chern-Simons Theory in Genus Two

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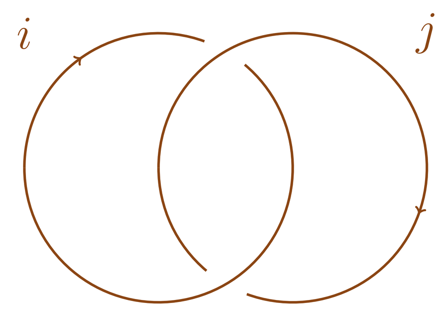
Abstract

We show that the action of the modular group of genus two surface on the Hilbert space of $SU(2)$ Chern-Simons Topological Quantum Field Theory admits a unique one parameter deformation (refinement). On the level of characters this deformation corresponds to passage from Schur functions to Macdonald polynomials. This generalizes the genus one construction suggested by M. Aganagic and Sh. Shakirov.

Introduction

Observables in topological quantum Chern-Simons theory correspond to link invariants known as $SU(N)$ colored Jones polynomials [Wit89]. One of the most efficient way of computing these polynomials was suggested by Reshetikhin and Turaev [RT90] using modular tensor categories. The key role in the underlying Modular Tensor Category is played by the action of the $SL(2,Z)$ - modular group of a torus:

$\widetilde{s}_{ij} =$ 

$\theta_i =$ 

Matrix s along with a matrix of balancing isomorphism $t_{ij} = \delta_{ij}\theta_i$ defines a projective representation of $SL(2,\mathbb{Z})$ [MS90, Lyu95] — mapping class group of a torus

$$(st)^3 \propto s^2 \propto 1,$$

Review of Refined Chern-Simons Theory in genus one

It was suggested by Aganagic and Shakirov [AS11] that this $SL(2,Z)$-action admits a one parameter deformation - refinement which on the level of characters corresponds to passage from Schur functions to Macdonald polynomials. This allows one to compute refined Jones polynomials of torus knots which agree with refined Jones polynomials defined by Cherednik.

Below we recall the key formulas for the $SU(2)$ case. Denote

$$q = \mathrm{e}^{\frac{2\pi i}{k+2\beta}}, \qquad t = q^\beta = \mathrm{e}^{\frac{2\pi i\beta}{k+2\beta}}.$$

Define one-parameter deformation $S_{ij}(\beta),\ S_{ij}(1) = s_{ij}$ and $T_{ij}(\beta),\ T_{ij}(1) = t_{ij}$ as

$$\begin{aligned} T_{ij} =& q^{-j^2/4}t^{-j/2}\delta_{ij},\\ S_{ij} =& S_{00}\, q^{-ij/2}g_i^{-1}M_i(t^{1/2},t^{-1/2})M_j(t^{1/2}q^i,t^{-1/2}). \end{aligned}$$

Here

$$g_i = \prod_{m=0}^{i-1}\frac{[i-m][m+2\beta]}{[i-m+\beta-1][m+\beta+1]}, \qquad [n] = \frac{q^{n/2}-q^{-n/2}}{q^{1/2}-q^{-1/2}}.$$

and M_j is an A_1 Macdonald polynomial:

$$M_j(x_1,x_2) = \sum_{l=0}^{j} x_1^{j-l}x_2^{l}\prod_{i=0}^{l-1}\frac{[j-i][i+\beta]}{[j-i+\beta-1][i+1]}.$$

Proposition 1. *[AS11] S and T provide a projective representation of $SL(2,\mathbb{Z})$*

$$(ST)^3 \propto S^2 \propto 1.$$

Using Verlinde formula one can write down the deformation of the structure constants of the Grothendieck ring of the underlying MTC:

$$N_{ijk} = \sum_{l=0}^{k}\frac{S_{il}S_{jl}S_{kl}}{g_l S_{0l}}.$$

Fourier Duality for $6j$-symbol and deformation

The square of the quantum $\mathfrak{sl}_2$ 6j-symbol satisfy the Fourier duality [Bar03, NRF07]:

$$\begin{Bmatrix} j_{12} & j_{13} & j_{23}\\ j_{34} & j_{24} & j_{14}\end{Bmatrix}_q^2 = \sum_{i_{12},i_{13},i_{23},i_{14},i_{24},i_{34}=0}^{k}\ \prod_{a<b} s_{j_{ab},i_{ab}} \begin{Bmatrix} i_{34} & i_{24} & i_{14}\\ i_{12} & i_{13} & i_{23}\end{Bmatrix}_q^2 \qquad (1)$$

Proposition 2. *[AS15] Solution for the following system of linear equations is unique at least up to $k \leqslant 8$*

$$\left\{\begin{Bmatrix} j_{12} & j_{13} & j_{23}\\ j_{34} & j_{24} & j_{14}\end{Bmatrix}\right\}_{q,t} = \sum_{i_{12},i_{13},i_{23},i_{14},i_{24},i_{34}=0}^{K}\ \prod_{a<b} S_{j_{ab},i_{ab}} \left\{\begin{Bmatrix} i_{34} & i_{24} & i_{14}\\ i_{12} & i_{13} & i_{23}\end{Bmatrix}\right\}_{q,t}$$

with S_{ij} — refined Foruier duality matrix.

Having enough solutions for the fixed K one can reconstruct the full rational function in q and t.

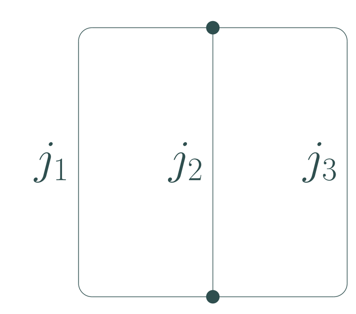
Example 3.

$$\left\{\begin{Bmatrix} 1 & 1 & 2\\ 2 & 2 & 1\end{Bmatrix}\right\}_{q,t} = \frac{t^{3/2}(1-t)(1-qt)^3(1-q^2t^2)}{(1+t^5)(1-q)^3(1-qt^2)^2}$$

Using recursion for knot operators one gets explicit conjectural formulas for the Macdonald deformation of the square of the $6j$-symbol.

Mapping Class group action

For $g = 2$ we have the following basis [MV94, BHMV95]

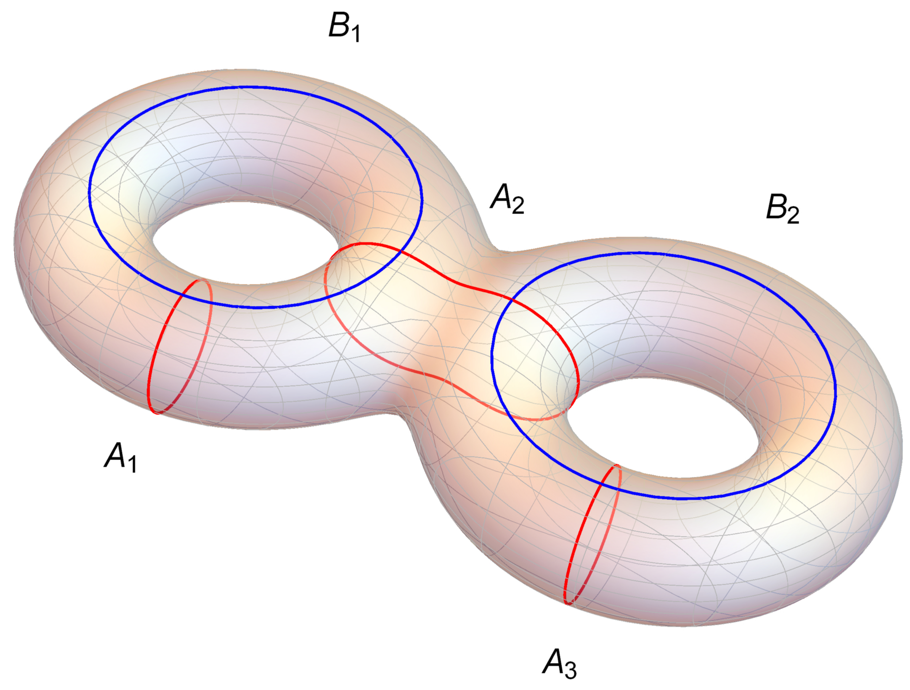


$$0 \leqslant j_1,j_2,j_3 \leqslant k, \qquad \forall \text{ distinct } a,b,c \quad j_a+j_b \geqslant j_c,$$
$$j_a+j_b+j_c \equiv 0 \bmod 2, \qquad j_a+j_b+j_c \leqslant 2k.$$

Proposition 4. *[AS15] Let*

$$\begin{aligned} \langle i_1,i_2,i_3|\, A_\alpha\, |j_1,j_2,j_3\rangle =& T_{j_\alpha}^{-1}\,\delta_{i_1j_1}\,\delta_{i_2j_2}\,\delta_{i_3j_3}, \quad \alpha = 1,2,3\\ \langle i_1,i_2,i_3|\, B_1\, |j_1,j_2,j_3\rangle =& \delta_{i_3j_3}\frac{d_{i_1}d_{i_2}}{N_{i_1i_2i_3}}\sum_{s=0}^{K} T_s\, d_s \left\{\begin{Bmatrix} i_3 & i_2 & i_1\\ s & j_1 & j_2\end{Bmatrix}\right\}_{q,t}\\ \langle i_1,i_2,i_3|\, B_2\, |j_1,j_2,j_3\rangle =& \delta_{i_1j_1}\frac{d_{i_2}d_{i_3}}{N_{i_1i_2i_3}}\sum_{s=0}^{K} T_s\, d_s \left\{\begin{Bmatrix} i_3 & i_2 & i_1\\ j_2 & j_3 & s\end{Bmatrix}\right\}_{q,t} \end{aligned}$$

Then, A_1, A_2, A_3, B_1, B_2 define a representation of mapping class group of genus 2 surface at least for $k \leqslant 8$.



Explicit conjectural formulas for general k are given in [AS15]. One should expect propositions above to hold for $k \geqslant 9$ as well.

Refined Jones polynomials

The operators of insertion the unknot have the following form

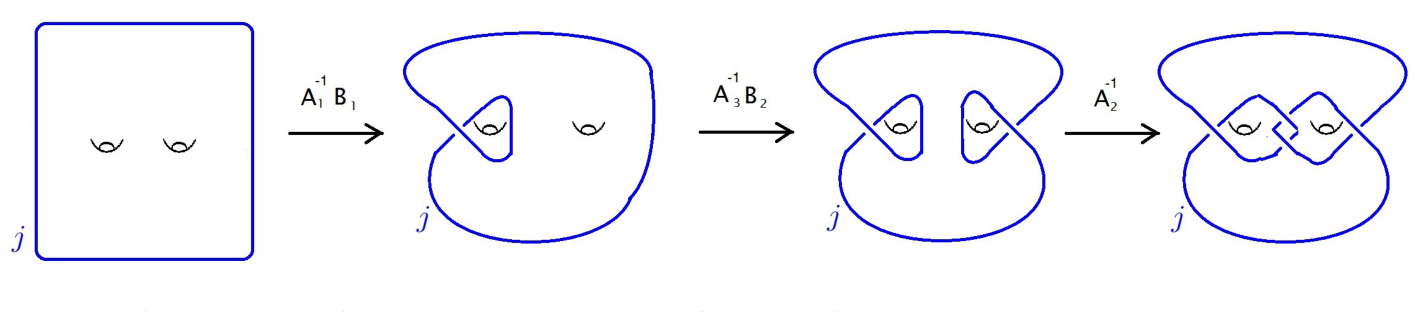
$$\begin{aligned} \langle i_1,i_2,i_3|\mathcal{O}_j^{(1)}|j_1,j_2,j_3\rangle =& \delta_{i_1j_1}\, g_j\frac{d_{i_2}d_{i_3}}{N_{i_1i_2i_3}}\left\{\begin{Bmatrix} i_3 & i_2 & i_1\\ j_2 & j_3 & j\end{Bmatrix}\right\}_{q,t}\\ \langle i_1,i_2,i_3|\mathcal{O}_j^{(2)}|j_1,j_2,j_3\rangle =& \delta_{i_2j_2}\, g_j\frac{d_{i_1}d_{i_3}}{N_{i_1i_2i_3}}\left\{\begin{Bmatrix} i_3 & i_2 & i_1\\ j_1 & j & j_3\end{Bmatrix}\right\}_{q,t}\\ \langle i_1,i_2,i_3|\mathcal{O}_j^{(3)}|j_1,j_2,j_3\rangle =& \delta_{i_3j_3}\, g_j\frac{d_{i_1}d_{i_2}}{N_{i_1i_2i_3}}\left\{\begin{Bmatrix} i_3 & i_2 & i_1\\ j & j_1 & j_2\end{Bmatrix}\right\}_{q,t} \end{aligned}$$

Mapping class group action allows one to express any genus two knot $\mathcal{K}$ insertion operator $\mathcal{O}_j(\mathcal{K})$ in terms of the corresponding operator for the unknot $\mathcal{O}_j$. The corresponding knot invariants are then given by

$$\mathcal{Z}_j(\mathcal{K}) = \left\langle 0,0,0|A_1B_1A_2B_2A_3\,U\mathcal{O}_jU^{-1}|0,0,0\right\rangle.$$

Example 5 (Figure eight knot)**.**

$$\mathcal{O}_j(4_1) = U\mathcal{O}_j^{(2)}U^{-1}, \qquad U = A_2^{-1}A_3^{-1}B_2A_1^{-1}B_1.$$



This gives a refinement of Jones polynomial for the figure eight knot:

$$\mathcal{P}_{4_1} = 1 - t + tq - t^2q + t^3q$$

References

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