Topological Quantum Gravity of the Ricci Flow

❀ Petr Hořava ❀

Berkeley Center for Theoretical Physics

Berkeley String-Math Seminar

February 22, 2021

Based on work with Alex Frenkel and Stephen Randall:

A. Frenkel, P. Hořava and S. Randall,
Topological Quantum Gravity of the Ricci Flow,
arXiv:2010.15369[hep-th],

A. Frenkel, P. Hořava and S. Randall,
The Geometry of Time in Topological Quantum Gravity of the Ricci Flow,
arXiv:2011.06230[hep-th],

A. Frenkel, P. Hořava and S. Randall,
Perelman's Ricci Flow in Topological Quantum Gravity,
arXiv:2011.11914[hep-th].

Main Idea

To connect three areas of physics and math:

- topological quantum field theory (of the cohomological type: cf. Witten's topological Yang-Mills in 4 dimensions [since 1988])
- mathematics of Ricci flows on Riemannian manifolds (of the Hamilton-Perelman type [since 1982])
- nonrelativistic gravity (of the Lifshitz type; [PH, since 2008])

Expected to be useful in both directions.

Ricci Flow: History

Hamilton's Ricci flow:
Eqn for $g_{ij}(t, x^k)$, a Riemannian metric on spatial manifold Σ^D,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

Ricci Flow: History

Hamilton's Ricci flow:
Eqn for $g_{ij}(t, x^k)$, a Riemannian metric on spatial manifold Σ^D,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

Perelman's Ricci flow:

$$\dot{g}_{ij} = -2R_{ij} - 2\nabla_i \partial_j \phi,$$
$$\dot{\phi} = -\Delta\phi - R\phi.$$

Ricci Flow: History

Hamilton's Ricci flow:
Eqn for $g_{ij}(t, x^k)$, a Riemannian metric on spatial manifold Σ^D,

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}.$$

Perelman's Ricci flow:

$$\dot{g}_{ij} = -2R_{ij} - 2\nabla_i \partial_j \phi,$$
$$\dot{\phi} = -\Delta\phi - R\phi.$$

DeTurck's trick: Apply diffeo generated by ξ^i, $\xi_i = \partial_i \phi$:

$$\dot{g}_{ij} = -2R_{ij},$$
$$\dot{\phi} = -\Delta\phi + (\partial\phi)^2 - R\phi.$$

Ricci Flow: History

RHS of Perelman's Ricci flow follows from a variational principle, Hamilton's doesn't.

Perelman's $\mathcal{F}$-functional:

$$\mathcal{F} = \int d^D x \sqrt{g} e^{-\phi} \left(R + g^{ij} \partial_i \phi \partial_j \phi \right) ,$$

with variations subjected to a fixed-volume condition:

$$\sqrt{g} e^{-\phi} d^D x = dm, \quad \text{fixed in time.}$$

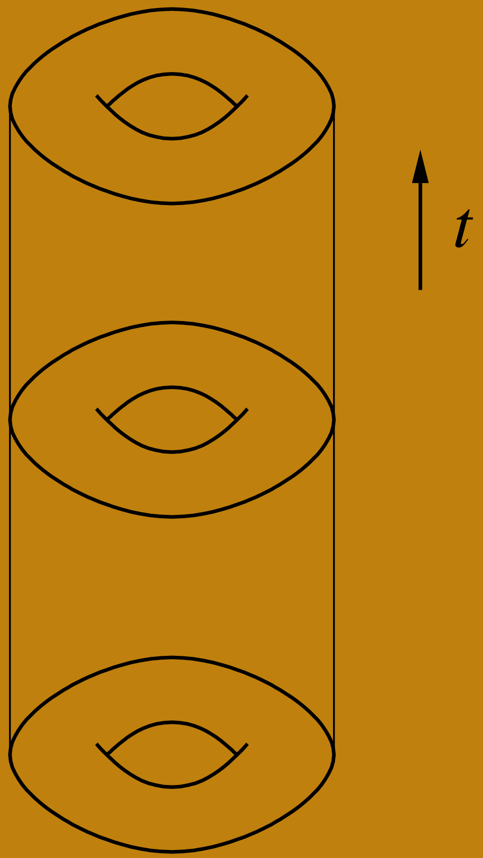
Ricci Flow: History

Importance for topology:

- Poincaré conjecture
- Thurston's geometrization conjecture for 3-manifolds
- New proof of uniformization theorem for 2-manifolds
- Generalized Smale conjecture

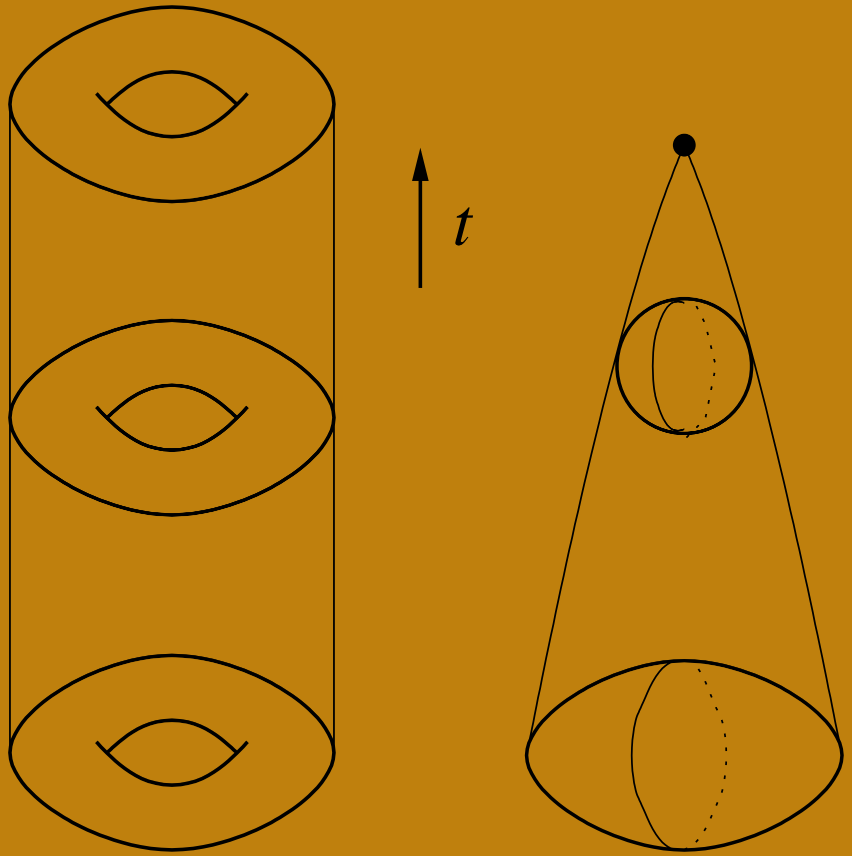
Interesting for physics: A theory of gravity, with central role played by concepts of entropy, leading to spacetime singularities with controllable topology change ("Ricci flows with surgery"), for general evolving 3-geometries.

Ricci Flow: Simple Examples



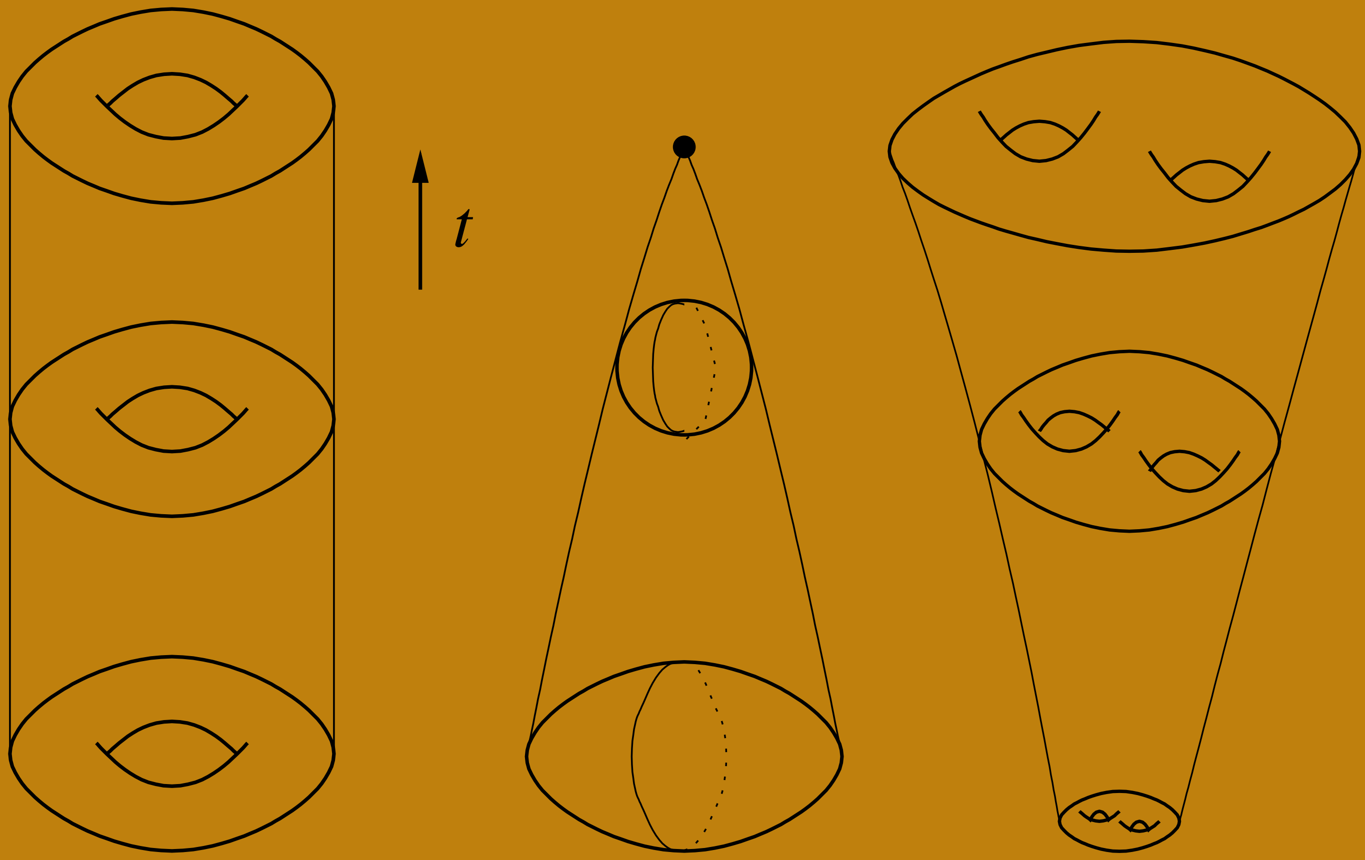
Ricci-flat Σ

Ricci Flow: Simple Examples



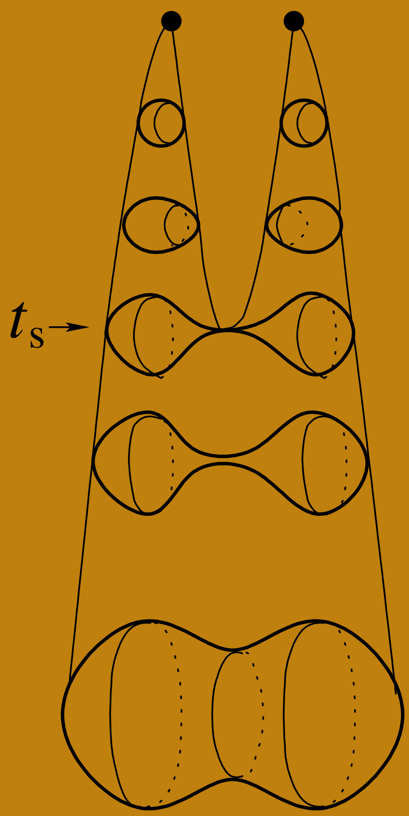
Ricci-flat Σ Σ of positive curvature

Ricci Flow: Simple Examples



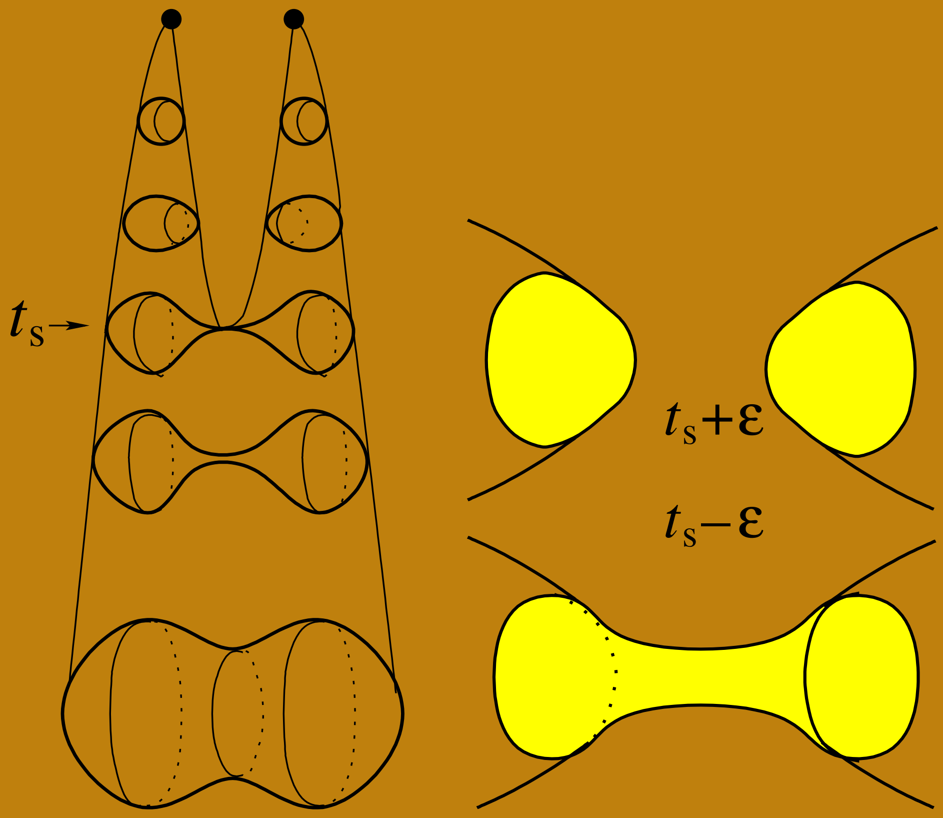
Ricci-flat Σ $\quad$ Σ of positive curvature $\quad$ hyperbolic Σ

Ricci Flow: The Neckpinch (in $D > 2$)



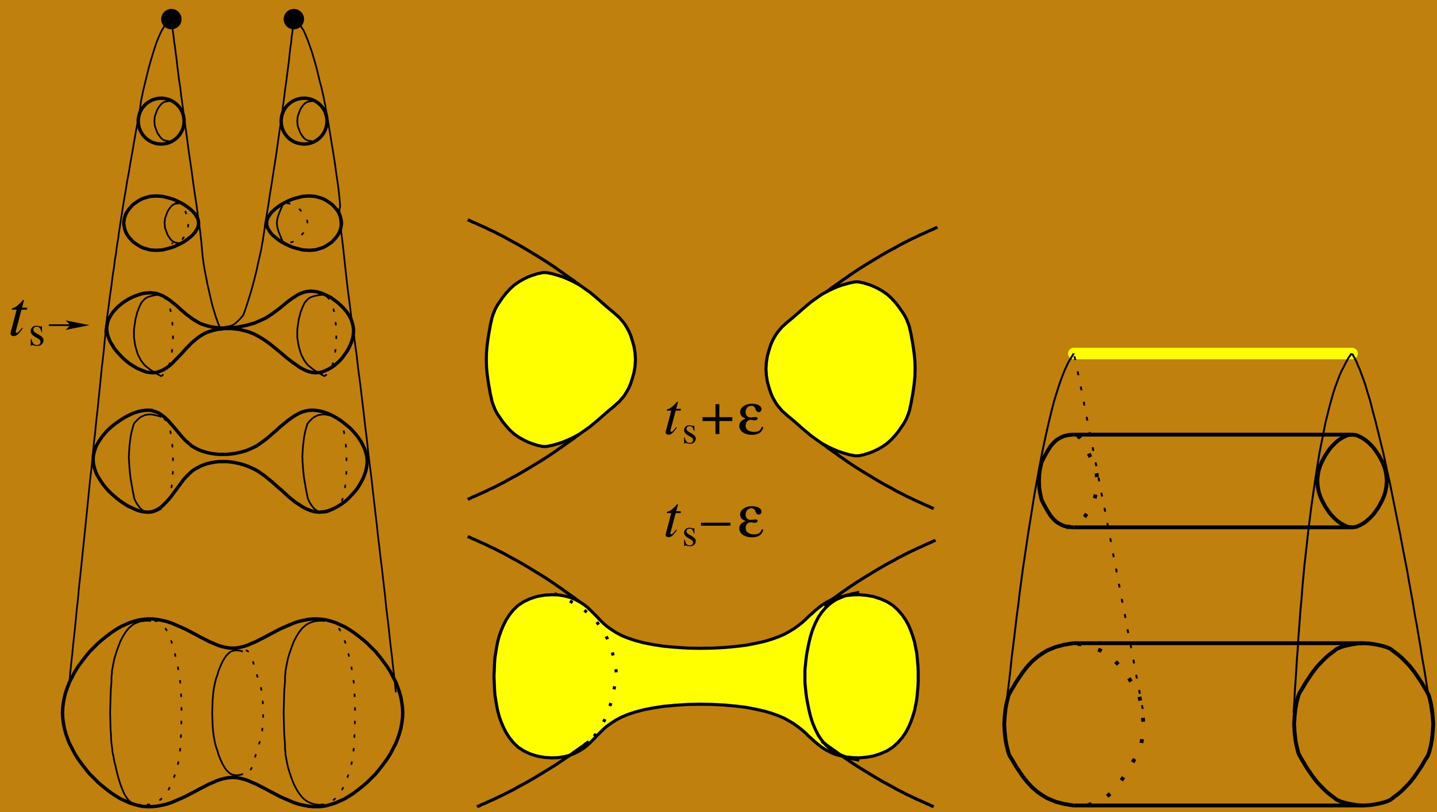
topology change

Ricci Flow: The Neckpinch (in $D > 2$)



topology change   Ricci flow with surgery

Ricci Flow: The Neckpinch (in $D > 2$)



topology change   Ricci flow with surgery   model of singularity

Gravity with anisotropic scaling

(also known as Hořava-Lifshitz gravity)

Field theory with anisotropic scaling ($\mathbf{x} = \{x^i, i = 1, \ldots D\}$):

$$\mathbf{x} \to \lambda \mathbf{x}, \quad t \to \lambda^z t.$$

z: dynamical critical exponent – characteristic of RG fixed point.

Many interesting examples: $z = 1$, 2, . . ., n, . . .
fractions: 3/2 (KPZ surface growth in $D = 1$), . . ., $1/n$, . . .
families with z varying continuously . . .

Condensed matter, dynamical critical phenomena, quantum critical systems, . . .

Goal: Extend to gravity, with propagating gravitons, formulated as a quantum field theory of the metric.

Example: Lifshitz scalar [Lifshitz, 1941]

Gaussian fixed point with $z = 2$ anisotropic scaling:

$$S = S_K - S_V = \frac{1}{2}\int dt\, d^D\mathbf{x}\left\{\dot{\Phi}^2 - (\Delta\Phi)^2\right\},$$

(Δ is the spatial Laplacian).

Compare with the Euclidean field theory

$$W = -\frac{1}{2}\int d^dx\,(\partial\phi)^2.$$

Shift in the (lower) critical dimension:

$$[\phi] = \frac{d-2}{2}, \quad [\Phi] = \frac{D-2}{2}.$$

Gravity at a Lifshitz point

Spacetime structure: Preferred foliation by leaves of constant time (avoids the "problem of time").

Fields: Start with the spacetime metric in ADM decomposition: the spatial metric g_{ij}, the lapse function N, the shift vector N_i.

Symmetries: foliation-preserving diffeomorphisms, $\mathrm{Diff}(M, \mathcal{F})$.

Action: $S = S_K - S_V$, with

$$S_K = \frac{1}{\kappa^2} \int dt\, d^D\mathbf{x} \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 \right)$$

where $K_{ij} = \dfrac{1}{N} (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ the extrinsic curvature,

$$\text{and} \quad S_V = \frac{1}{\kappa_V^2} \int dt\, d^D\mathbf{x} \sqrt{g} N\, \mathcal{V}(R_{ijk\ell}, \nabla_i).$$

Projectable and nonprojectable theory

N, N_i are the gauge fields for the $\mathrm{Diff}(M, \mathcal{F})$ symmetries generated by $\delta t = f(t)$, $\delta x^i = \xi^i(t, \mathbf{x})$. Hence:

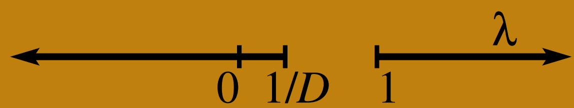
(1) we can restrict $N(t)$ to be a function of time only: projectable theory.

(2) or, we allow $N(t, \mathbf{x})$ to be a spacetime field. New terms, containing $\nabla_i N/N$, are then allowed in S by symmetries: nonprojectable theory.

Spectrum: Tensor graviton polarizations, plus an extra scalar graviton. Three options for the scalar: Live with it, gap it, or eliminate it by an extended gauge symmetry.

Dispersion relation: Nonrelativistic, $\omega^2 \sim k^{2z}$, around this Gaussian fixed point.

Allowed range of λ:



RG flows

Assume $z > 1$ UV fixed point. Relevant deformations trigger RG flow to lower values of z. Example: Lifshitz scalar.

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\left\{\dot{\Phi}^2 - (\Delta\Phi)^2 - \mu^2\partial_i\Phi\partial_i\Phi - m^4\Phi^2\right\},$$

Multicriticality. New phases: modulated.

Similarly for gravity:

$$S = \frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\sqrt{g}N\left\{K_{ij}K^{ij} - \lambda K^2 - \ldots - \mu^{2z-2}R - M^{2z}\right\}.$$

Flows in IR to $z = 1$ scaling. In the IR regime, S_V is dominated by the spatial part of Einstein-Hilbert.

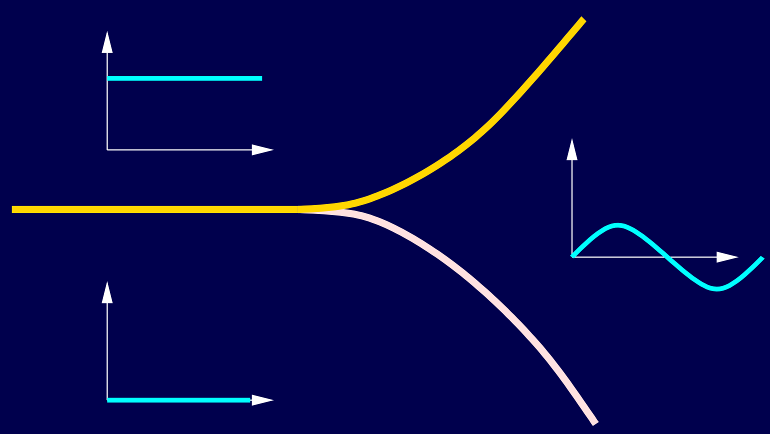
(The $z > 1$ Gaussian gravity fixed points also emerge in IR in condensed matter lattice models, [Cenke Xu & PH].)

Relevant deformations, RG flows, phases

The Lifshitz scalar can be deformed by relevant terms:

$$S = \frac{1}{2}\int dt\, d^D\mathbf{x}\, \left\{\dot{\phi}^2 - (\Delta\phi)^2 - \mu^2\partial_i\phi\partial_i\phi + m^4\phi^2 - \phi^4\right\}$$

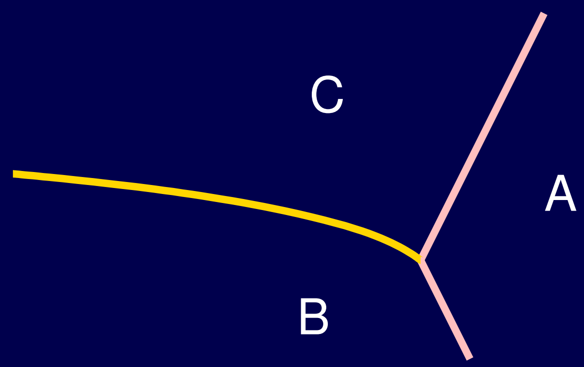
The undeformed $z = 2$ theory describes a tricritical point, connecting three phases – disordered, ordered, spatially modulated (“striped”) [A. Michelson, 1976]:



Phase structure in the CDT approach

Compare the phase diagram in the causal dynamical triangulations:

[Ambjørn et al, 1002.3298]



Note: $z = 2$ is sufficient to explain three phases.
Possibility of a nontrivial $z \approx 2$ fixed point in $3 + 1$ dimensions?

RG flows in gravity: $z=1$ in IR

Theories with $z>1$ represent candidates for the UV description. Under relevant deformations, the theory will flow in the IR. Relevant terms in the potential:

$$\Delta S_V = \int dt\, d^D\mathbf{x}\sqrt{g}N\left\{\ldots + \mu^2(R-2\Lambda)\right\}.$$

the dispersion relation changes in IR to $\omega^2 \sim k^2 + \ldots$
the IR speed of light is given by a combination of the couplings
μ^2 combines with $\kappa, \ldots$ to give an effective G_N.

Sign of k^2 in dispersion relation is opposite for the scalar and the tensor modes! Can we classify the phases of gravity? Can gravity be in a modulated phase?

Preliminaries: Structure of spacetime

Goal: Topological quantum gravity, localization to Ricci flows.
Expect M a foliation, by leaves Σ of constant t. Take

$$M^{D+1} = I \times \Sigma^D, \qquad I \subset \mathbf{R}.$$

Topological BRST charge Q:

$$Qg_{ij} = \psi_{ij}$$

Antighosts and auxiliary:

$$Q\chi_{ij} = B_{ij}.$$

Balanced theory – natural to formulate in $\mathcal{N} = 2$ superspace:

$$G_{ij}(t, x^k, \theta, \bar{\theta}) = g_{ij} + \theta\psi_{ij} + \bar{\theta}\chi_{ij} + \theta\bar{\theta}B_{ij}.$$

Primitive Topological Gravity of Ricci Flow

Supercharges and superderivatives:

$$Q = \partial_\theta, \qquad \bar{Q} = \partial_{\bar\theta} + \theta\partial_t,$$
$$\bar{D} = \partial_{\bar\theta}, \qquad D = \partial_\theta - \bar\theta\partial_t.$$

Superalgebra: $\{Q, \bar{Q}\} = \partial_t, \qquad \{D, \bar{D}\} = -\partial_t.$

Action: $S = \frac{1}{\kappa^2}(S_K - S_W)$, with

$$S_K = \int d^D x\, dt\, d^2\theta \sqrt{G}\left(G^{ik}G^{j\ell} - \lambda G^{ij}G^{k\ell}\right)\bar{D}G_{ij}DG_{k\ell}$$

$$S_W = \int d^D x\, dt\, d^2\theta \sqrt{G}\left(\ldots + \alpha_R R^{(G)} + \alpha_\Lambda\right).$$

Localization and Hamilton's Ricci flow

Action in bosonic components:

$$S_K = -\int d^D x \, dt \sqrt{g} \left(g^{ik} g^{j\ell} - \lambda g^{ij} g^{k\ell} \right) \left(B_{ij} - \dot{g}_{ij} \right) B_{k\ell}$$

$$S_W = \int d^D x \, dt \sqrt{g} \, B_{ij} \left\{ \ldots + \alpha_R \left(\frac{1}{2} g^{ij} R - R^{ij} \right) + \alpha_\Lambda \frac{1}{2} g^{ij} \right\}.$$

Localization to solutions of $B_{ij} = 0$:

$$\dot{g}_{ij} = (g_{ik} g_{j\ell} - \tilde{\lambda} g_{ij} g_{k\ell}) \frac{\delta W}{\delta g_{k\ell}}.$$

This is Hamilton's Ricci flow when we set

$$\alpha_R = 2, \qquad \alpha_\Lambda = 0, \qquad \tilde{\lambda} = \frac{1}{D-2}.$$

Gauge Theory I: Spatial Diffeomorphisms

Physicist's instinct: Symmetries, in particular gauge symmetries.

Gauging spatial diffeomorphisms: The shift vector n^i.

Under $\xi^i(t,x^k)$:

$$\delta n^i = \dot{\xi}^i + \xi^k \partial_k n^i - \partial_k \xi^i n^k.$$

Morally speaking, n^i plays the role of the gauge field for spatial diffeomorphisms in bosonic gravity (relativistic or not).

In the supersymmetric case, ξ^i becomes a superfield,

$$\Xi^i(t,\theta,\bar{\theta},x^k) = \xi^i + \ldots$$

Type C, A, B: Chiral, antichiral, balanced.

Shift Superfields

In our $\mathcal{N} = 2$ supersymmetric theory, we must introduce several "shift superfields":

$$N^i = n^i + \ldots,$$

but also S^i, $\bar{S}^i$, to covariantize supertime derivatives,

$$\begin{aligned}
\dot{G}_{ij} &\rightarrow \nabla_t G_{ij} = \dot{G}_{ij} - N^k \partial_k G_{ij} - \partial_i N^k G_{kj} - \partial_j N^k G_{ik}, \\
DG_{ij} &\rightarrow \mathcal{D} G_{ij} = DG_{ij} - S^k \partial_k G_{ij} - \partial_i S^k G_{kj} - \partial_j S^k G_{ik}, \\
\bar{D}G_{ij} &\rightarrow \bar{\mathcal{D}} G_{ij} = \bar{D}G_{ij} - \bar{S}^k \partial_k G_{ij} - \partial_i \bar{S}^k G_{kj} - \partial_j \bar{S}^k G_{ik},
\end{aligned}$$

followed by constraints: $DS^i = S^k \partial_k S^i, \quad \bar{D}\bar{S}^i = \bar{S}^k \partial_k \bar{S}^i,$

$$N^i = -\bar{D}S^i - D\bar{S}^i + \bar{S}^k \partial_k S^i + S^k \partial_k \bar{S}^i.$$

Geometric Interpretation I: Flat Connection on Supertime

Turns out that in retrospect, one can interpret these constraints precisely as equivalent to the condition of vanishing curvatures

$$W = 0$$

where the W's are defined as obstructions against the covariant derivatives

$$\nabla_t, \qquad \mathcal{D}, \qquad \bar{\mathcal{D}}$$

satisfying the same algebra as the original ∂_t, D and $\bar{D}$:

$$\{D, \bar{D}\} = -\partial_t, \qquad \text{zero otherwise.}$$

Geometric Interpretation II: Super Yang-Mills with $\mathscr{G} = \mathbf{Diff}(\Sigma)$

Even more surprisingly, the formulation is identical to the supersymmetric Yang-Mills construction, with:

- The role of spacetime played by the supertime $(t, \theta, \bar{\theta})$,
- The role of the internal gauge group played by the inifinite-dimensional Diff(Σ), generated by the Lie algebra elements $\xi^i(x^k)$,
- The role of adjoint index A played by the multi-index (i, x^k)

Then $N^i(t, \theta, \bar{\theta}, x^k)$ is $\mathcal{A}_t^A(t, \theta, \bar{\theta})$, and $S^i = \mathcal{A}_\theta^A$, $\bar{S}^i = \mathcal{A}_{\bar{\theta}}^A$.

Constraints in superspace:
Exactly the "conventional constraints" of SYM!

Useful for BCJ?

Action

is again given by

$$S = \frac{1}{\kappa^2}(S_K - S_W),$$

with

$$S_K = \int d^D x \, dt \, d^2\theta \sqrt{G} \left(G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \bar{\mathcal{D}} G_{ij} \mathcal{D} G_{k\ell}$$

$$S_W = \int d^D x \, dt \, d^2\theta \sqrt{G} \left(\ldots + \alpha_R R^{(G)} + \alpha_\Lambda \right).$$

Localization:

The LHS of the flow equation replaces $\dot{g}_{ij}$ with $\nabla_t g_{ij}$.

Bonus: We can now perform DeTurck's trick, if needed.

Gauge Theory II: Time Translations

Now we wish to extend the gauge symmetry to full Diff$(M, \mathcal{F})$, foliation-preserving diffeos.

To gauge time translations in the bosonic theory, one introduces the lapse function n:

$$\delta n = f\dot{n} + \dot{f}n.$$

The simplest case: $n(t)$, projectable theory.

To supersymmetrize, we promote $f(t)$ to a superfield,

$$F(t, \theta, \bar{\theta}) = f + \theta\varphi + \bar{\theta}\bar{\varphi} + \theta\bar{\theta}\alpha.$$

Lapse Superfields: Projectable

Covariantize the derivatives. First,

$$\nabla_t G_{ij} \to \mathscr{D}_t G_{ij} \equiv E \nabla_t G_{ij}.$$

More importantly, the superderivatives are covariantized:

$$\begin{aligned} \mathcal{D} G_{ij} &\to \mathscr{D}_\theta G_{ij} \equiv \mathcal{E} \mathcal{D} G_{ij} + \Theta \nabla_t G_{ij}, \\ \bar{\mathcal{D}} G_{ij} &\to \mathscr{D}_{\bar\theta} G_{ij} \equiv \bar{\mathcal{E}} \bar{\mathcal{D}} G_{ij} + \bar\Theta \nabla_t G_{ij}, \end{aligned}$$

followed by constraints:

$$\mathcal{D}\Theta = -\Theta\dot\Theta, \quad \bar{\mathcal{D}}\bar\Theta = -\bar\Theta\dot{\bar\Theta}, \quad \mathcal{E} = \bar{\mathcal{E}} = 1,$$

and

$$E = 1 - \bar{D}\Theta - D\bar\Theta - \Theta\dot{\bar\Theta} - \bar\Theta\dot\Theta.$$

The Nonprojectable Theory

Importantly, the construction extends naturally to the case where the lapse superfields E, $\mathcal{E}$, $\bar{\mathcal{E}}$, Θ and $\bar{\Theta}$ are nonprojectable, i.e., functions of not only supertime coordinates $(t, \theta, \bar{\theta})$ but also of x^i.

The constraints just become awfully more complicated; for example,

$$E = \mathcal{E}\bar{\mathcal{E}} - \bar{D}\Theta + \bar{S}^k\partial_k\Theta - D\bar{\Theta} + S^k\partial_k\bar{\Theta} \\ - \Theta\left(\dot{\bar{\Theta}} - N^k\partial_k\bar{\Theta}\right) - \bar{\Theta}\left(\dot{\Theta} - N^k\partial_k\Theta\right),$$

$\ldots$

Now we are ready to write down the action.

Action

is again given by

$$S = \frac{1}{\kappa^2}(S_K - S_W),$$

where now

$$S_K = \int d^D x\, dt\, d^2\theta \sqrt{G} N \left(G^{ik} G^{j\ell} - \lambda G^{ij} G^{k\ell} \right) \mathscr{D}_{\bar{\theta}} G_{ij} \mathscr{D}_{\theta} G_{k\ell},$$

$$S_W = \int d^D x\, dt\, d^2\theta \sqrt{G} N \left(\ldots + \alpha_R R^{(G)} + \alpha_\Phi G^{ij} \partial_i \Phi \partial_j \Phi + \alpha_\Lambda \right).$$

(Here we have used $N = 1/E$ and $\Phi = -\log N$.)

Perelman's $\mathcal{F}$-functional is our superpotential, for $\alpha_R = \alpha_\Phi = 2$ and $\alpha_\Lambda = 0$.

Perelman's "dilaton" is (minus the log of) the lapse function!

Perelman's Ricci Flow from Topological Quantum Gravity

Localization in our nonprojectable theory:

$$e^{\phi}\nabla_t g_{ij} = -\alpha_R R_{ij} + \frac{1}{2}\alpha_R[1+(2-D)\tilde{\lambda}]g_{ij}R - \alpha_R\nabla_i\partial_j\phi$$
$$+\alpha_R[1+(1-D)\tilde{\lambda}]g_{ij}\Delta\phi + (\alpha_R-\alpha_\Phi)\partial_i\phi\partial_j\phi$$
$$+\left\{\frac{1}{2}\alpha_\Phi[1+(2-D)\tilde{\lambda}] - \alpha_R[1+(1-D)\tilde{\lambda}]\right\}g_{ij}(\partial\phi)^2$$
$$+\frac{1}{2}\alpha_\Lambda(1-\tilde{\lambda}D)g_{ij}.$$

Lots of junk, which does not look like Perelman's equation.

First, reframe:

$$e^{\phi}g_{ij} = \tilde{g}_{ij}, \qquad \frac{D}{2}\phi = \tilde{\phi}$$

Perelman's Fixed-Volume Condition

Recall that Perelman holds a volume element fixed,

$$e^{-\tilde{\phi}}\sqrt{\tilde{g}} \qquad \text{measure fixed in time.}$$

In our frame, this simply becomes:

$$\nabla_t \sqrt{g} = 0!$$

This suggests to take the limit of

$$\lambda \to \pm\infty, \qquad \text{or } \tilde{\lambda} \to \frac{1}{D}.$$

The fixed-volume condition is realized dynamically, not as a gauge-fixing choice!

Perelman's Equations

Rewrite theory in Perelman's variables $\tilde{g}_{ij}$, $\tilde{\phi}$.

Set $\tilde{\alpha}_R = \tilde{\alpha}_\Phi = 2$, $\lambda = \pm\infty$. Then:

$$\tilde{\nabla}_t \tilde{g}_{ij} - \frac{2}{D}\tilde{g}_{ij}\tilde{\nabla}_t\tilde{\phi} = -2\tilde{R}_{ij} - 2\nabla_i\partial_j\tilde{\phi} + \frac{2}{D}\tilde{g}_{ij}\tilde{R} + \frac{2}{D}\tilde{g}_{ij}\tilde{\Delta}\tilde{\phi}.$$

This is just the sum of the two Perelman equations!

$$\left(\tilde{\nabla}_t\tilde{g}_{ij} + 2\tilde{R}_{ij} + 2\nabla_i\partial_j\tilde{\phi}\right) - \frac{2}{D}\tilde{g}_{ij}\left(\tilde{\nabla}_t\tilde{\phi} + \tilde{R} + \tilde{\Delta}\tilde{\phi}\right) = 0.$$

One can match Perelman's equations exactly, by performing an alternate gauge-fixing which also fixes time diffeomorphisms.

Perelman's $\mathcal{W}$-Functional

For shrinking Ricci solitons, Perelman introduces an even more useful $\mathcal{W}$-functional:

$$\mathcal{W} = \int d^D x \sqrt{\tilde{g}} e^{-\tilde{\phi}} \left\{ \tau \left(\tilde{R} - \tilde{g}^{ij} \partial_i \tilde{\phi} \partial_j \tilde{\phi} \right) + \tilde{\phi} - D \right\},$$

and fixes the following volume:

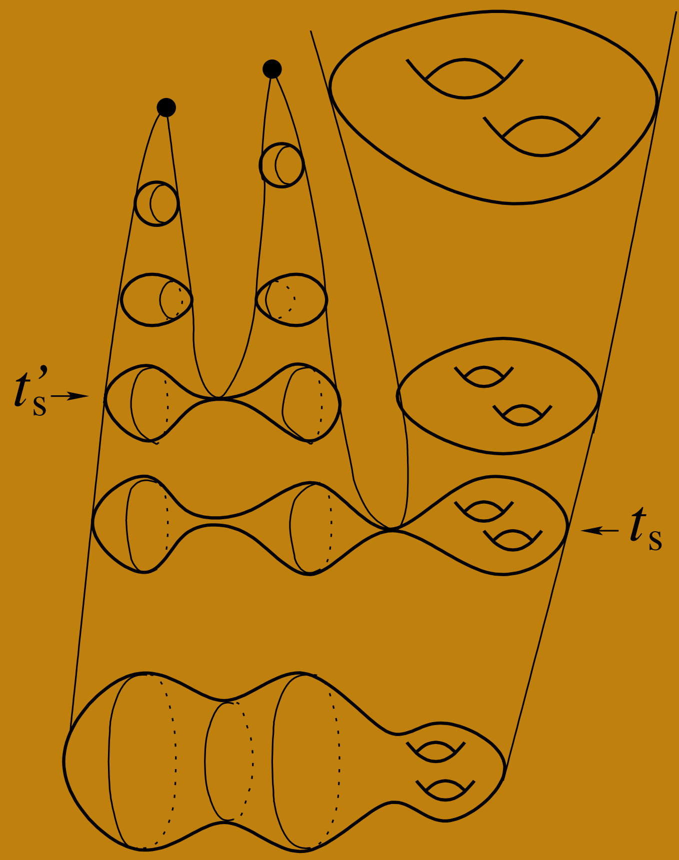
$$\frac{1}{(4\pi\tau)^{D/2}} e^{-\tilde{\phi}} \sqrt{\tilde{g}}.$$

We reproduce that by changing our variables to

$$\tilde{g}_{ij} = e^{\phi} g_{ij}, \quad \tilde{\phi} = \frac{D}{2}[\phi - \log(4\pi\tau)].$$

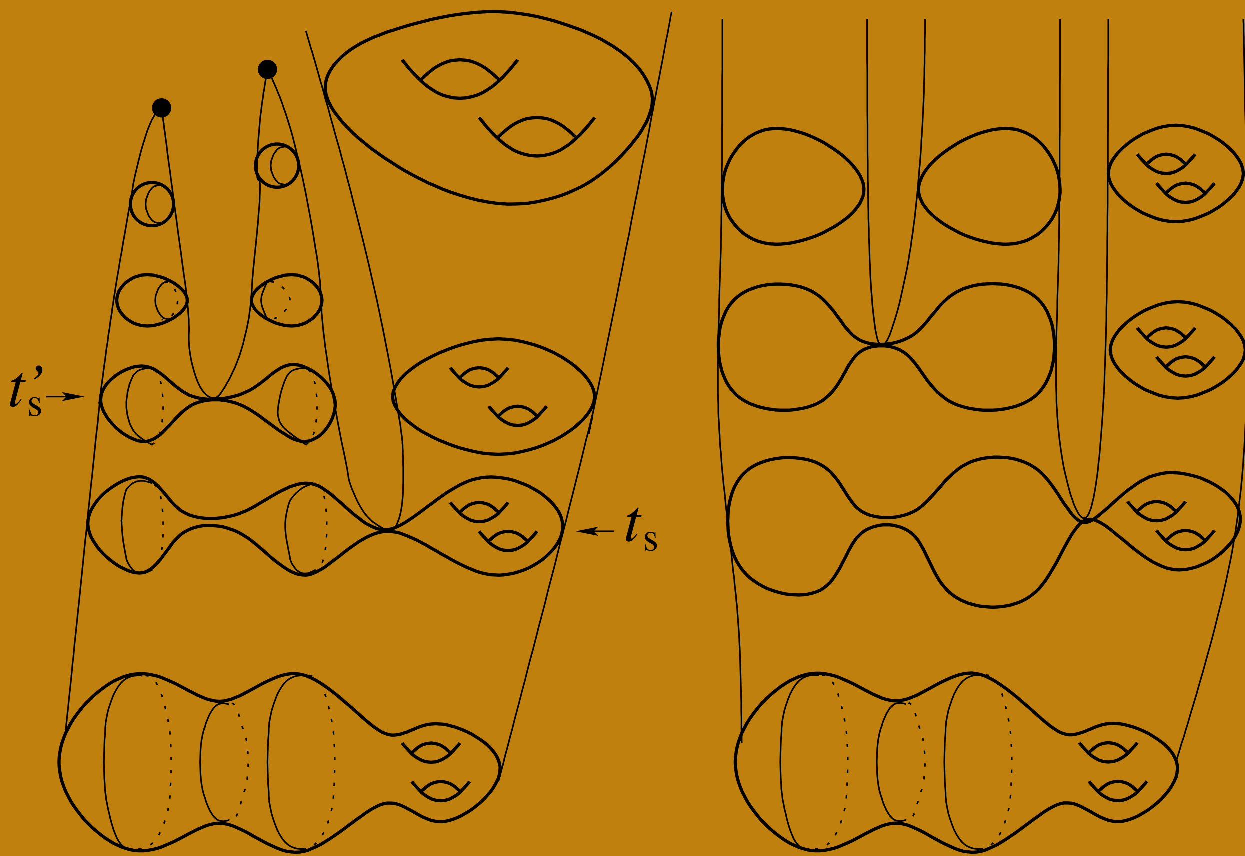
Similarly for $\mathcal{W}_+$-functional for expanding solitons, introduced by Feldman, Ilmanen and Ni.

Generic Flows



Perelman's frame

Generic Flows



Perelman's frame

our frame

Outlook

Exciting connection of three previously disconnected areas:

Topological QFT (of the cohomological type), mathematical theory of Ricci flow, nonrelativistic quantum gravity.

Sets the stage for many intriguing questions, both in physics and in math. Partial list:

- observables and correlation functions,
- probes: branes/strings, Perelman's $\mathcal{L}$-volume and $\mathcal{L}$-length, ...
- Hartle-Hawking wavefunction and initial value problem,
- quantum topology change and Ricci flows with surgery,
- short-distance completeness in $D = 3$ at $z = 2$?
- renormalization group properties, perturbative and not,
- dependence on spatial dimension D,
- quantum gravity out of equilibrium, theory in real time ...