Quantum Cohomology

and

Slices of the Affine Grassmannian

What structures?

\[ H^*_T(X, \mathbb{C}) \quad \text{cohomology} \]
\[ \{ T \cap X \} \]

\[ H^*_T(X) \quad \text{T-equivariant cohomology} \]
\[ \text{algebra wrt } \cup \]
\[ \{ \text{deform } \cup \text{ to } * \} \]

\[ QH^*_T(X) \]
\[ \langle a, b \cup c \rangle = \langle a, b \cup c \rangle + \sum_{d \in \mathbb{C}_T} \langle a, b \cup c \rangle \cdot q^d \]

\[ \text{(as a space } H^*_T[X] \otimes \mathbb{C}[q^d] \text{)} \]
\[ \text{fibers coordinates on a base} \]

Quantum Connection
\[ \nabla^Q_\lambda = \frac{\partial}{\partial \lambda} + \lambda \star \]
\[ \lambda \in H^2_T(X) \quad \text{flat connection} \]

\[ H^2_T(\text{pt}) \subset H^2_T(X) \text{ is not interesting} \]
We'll focus on transversal directions.

We want to express \( \nabla^Q \) in terms of more well-known flat connections. (For a special family of \( X \))
Let's start with the answer:

\[ X = \hat{G}V \text{ in ADE types} \]

**Some Representation Theory**

Let \( G^v \) be a complex connected reductive group (e.g., \( SL_n, PSL_n, \ldots \))

\[ V_1, \ldots, V_e \] - \( G^v \)-reps.

- For \( X \in g^v \), we will write \( X^i \) for \( X \) acting on \( i^{th} \) position in \( V_1 \otimes \cdots \otimes V_e \).

- Similarly, for \( Y \in g^v \), we will write \( Y_{ij} \) for the operator acting by the first component of \( Y \) on \( i^{th} \) place, the second component on \( j^{th} \) place.

- \( g^v \otimes g^v \rightarrow \mathbb{C} \), symmetric invariant pairing \( \mathbb{C} \rightarrow g^v \otimes g^v \), the image of \( 1 \) is called the Casimir operator \( \Omega \).

Explicitly,

\[ \Omega = \sum x_i \otimes x_i + \sum \alpha \otimes e_\alpha \]
\[ \Omega = \sum_{i} x_{i} \otimes x_{i} + \sum_{\alpha \text{-root}} \epsilon_{\alpha} \otimes \epsilon_{-\alpha} \]

\[ x_{i}, x_{i} \text{ - dual bases in } \mathfrak{g}^{\vee} \text{ (Cartan)} \]

\[ \epsilon_{\alpha} \text{ - elements in root subspaces with root } \alpha. \]

We'll need "half" Casimir operators:

\[ \Omega_{\epsilon} = \frac{1}{2} \sum_{i} x_{i} \otimes x_{i} + \sum_{\alpha \text{-root } \delta_{i} \epsilon > 0} \epsilon_{\alpha} \otimes \epsilon_{-\alpha} \]

Here \( \epsilon \) is a Weyl chamber.

\[ \delta_{i} \epsilon > 0 \iff \alpha \text{ is positive w.r.t. } \epsilon. \]

Usually people take \( \epsilon \) to be dominant & antidominant.

The trigonometric Knizhnik-Zamolodchikov connection

\[ D_{i}^{kz} = Z_{i} \frac{\partial}{\partial z_{i}} + h \sum_{i \neq j} \frac{\epsilon_{j} \Omega_{\epsilon} + \epsilon_{i} \Omega_{-\epsilon}}{z_{i} - z_{j}} \]

\[ h_{i} \in \mathfrak{h}_{\epsilon}. \]

We will act on \( \bigotimes_{\lambda_{1}, \ldots, \lambda_{g}} C(z_{1}, \ldots, z_{g}) [t^{\lambda}] \)

simple reps with h.w. \( \lambda_{i} \)

\[ \mu \text{-weight space.} \]

To compute with power series in \( Q^{+} \), expand

\[ |z_{1}| \ll |z_{2}| \ll \ldots \ll |z_{g}| \]

\[ \nabla_{i}^{kz} = z_{i} \frac{\partial}{\partial z_{i}} - h_{i} - t \sum \Omega_{i}^{+} - \sum \Omega_{-i}^{-}. \]
\[ \nabla^2 k^2 = \sum_{i} \frac{\partial^2}{\partial x_i^2} - H^2 - \hbar \left[ \sum_{j \neq i} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right] \]

\[
{\text{classical}} \cup \]
\[ - \hbar \left[ \sum_{j \neq i} \left( \frac{\bar{z}_j / \bar{z}_i}{1 - \bar{z}_j / \bar{z}_i} \right) \frac{\partial}{\partial x_i} \right] - \sum_{j \neq i} \left( \frac{\bar{z}_j / \bar{z}_i}{1 - \bar{z}_j / \bar{z}_i} \right) \frac{\partial}{\partial x_j} \]

corrections in \( z \),
purely quantum part.

Thm (D’20) Under identification:

\[ V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \left[ \mathfrak{g}^m \right] \leftarrow H^k \left( \mathfrak{g}^n \left[ \mathfrak{g}^m \right] \right) \]

choice of basis

\[ V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_n} \left[ \mathfrak{g} \right] \leftarrow \text{Stab}_e \left[ \mathfrak{g} \right] \text{ - basis} \]

\( e \) in \( \mathfrak{g}_+ \left[ \mathfrak{g} \right] \leftarrow e \) is \( \text{Stab}_e \) in \( \mathfrak{g}_+ \)

\( \bar{e}_i \leftarrow \bar{g} \bar{e}_i \leftarrow \text{some homology classes} \)

\[ |z_i| \ll |z_j| \leftarrow e_i, e_j \text{ is effective} \]

\[ \nabla \leftarrow \nabla \left[ \mathfrak{g} \right] \]

\[ \lambda = C_i^T(\bar{e}_i) \]

\( \bar{e}_i \) - some \( T \)-equiv.

line bundle

The affine Grassmannian

Let \( G \) be a complex connected reductive group

\[ K = C(\mathfrak{g}), \quad \mathfrak{g} = C(\mathfrak{g}) \]
\[ \text{Spec } \mathcal{O} - \text{ formal disk} \]
\[ \text{Spec } k - \text{ formal punctured disk.} \]

Set theoretically the affine Grassmannian is

\[ \text{Gr}_\mathcal{O} = G(K)/G(O) \]

There is a moduli space formulation

\[ \text{Gr}_\mathcal{O} = \left\{ (P, \psi) \mid P - \text{G-principle bundle on Spec } \mathcal{O} \right\} \]

\[ \psi - \text{isomorphism of } P \text{ with the trivial G-principle bundle} \]

\[ \text{Gr}_\mathcal{O} \text{ is an ind-pro} \text{-scheme} \]

and has an ample line bundle \( \mathcal{O}(1) \)

which generates \( \text{Pic } \text{Gr}_\mathcal{O} \)

We'll fix \( \mathcal{O} \), so we omit it from notation.

**Actions on Gr:**

1) \( G(K) \triangleleft \text{Gr} \)

\[ V \quad \text{genus terms} \]

2) \( \mathbb{C}^x \triangleleft \text{Gr} \) by scaling \( z / \text{disk} \)

We'll need \( T = A \times \mathbb{C}^x \) action on \( \text{Gr} \).
We'll need \( T = A \times C^x \) action on \( G_r \).

\[ \text{Fixed Points} \]

Let \( \lambda \in \text{Hom}(C^x, A) \) be a cocharacter.

Then we can make an element of \( G_r(k) \):

\[ \text{Spec } k \to C^x \xrightarrow{\lambda} A \subset G \]

We call it \( \underline{z}^\lambda \).

This gives \( \{ \underline{z}^\lambda \} \subset G_r \).

\( \underline{z}^\lambda \) is a fixed

\( C^x \)-fixed

\[ \text{Prop } G_r^T = G_r^A = \bigsqcup_{\lambda \text{-dominant}} \{ \underline{z}^\lambda \} \]

\[ \text{Cell structure in } G_r : \]

\( G_r^\lambda = G(0) \cdot \{ \underline{z}^\lambda \} \) — as a space this is

a vector bundle over \( G/B \).

\( C^x \) scales the fibers

\( G_r = \bigsqcup_{\lambda \text{-dominant}} G_r^\lambda \) — similar to Bruhat cells

in \( G/B, G/P \).

\[ G_r^\lambda = \bigsqcup_{\mu \text{-dominant}} G_r^\mu \]

\( G_r^\lambda \subset G_r^\mu \) — smooth part

\[ \overline{G_r^\lambda} \subset \overline{G_r^\mu} \]
ur c ur smooth part

\[ \overline{Gr}^\lambda \text{ is smooth} \iff \overline{Gr}^\lambda = Gr^\lambda \iff \lambda \text{ is minuscule.} \]

\[ \overline{Gr}^\lambda_m = \overline{Gr}^\lambda \cap Gr^m \]

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Prop 1) \( Gr^\lambda_m \subset Gr^\lambda \) is \( T \)-invariant.

2) \( (Gr^\lambda_m)^T = (Gr^\lambda_m)^A = [\mu] \)

3) \( G^\lambda_A \) contracts \( Gr^\lambda_m \) to \( [\mu] \).

4) \( Gr^\lambda_m \) is normal affine.

Poisson variety \( \langle G^\lambda, \lambda-m \rangle \)

It is known when \( Gr^\lambda_m \)

has a symplectic resolution \([KMW]\)

if \( \lambda = \sum \lambda_i \) of minuscule coweights

\[ \text{smooth symplectic form} \]

\[ \text{depends on order} \]

\[ \overline{Gr}^\lambda_{\lambda_1 \ldots \lambda_n} \subset Gr^\lambda \]

\[ \overline{Gr}^\lambda_m \]

\[ \text{symplectic structure} \]

on the smooth part is the same.

Examples: \( \lim (X, \omega) \)

\[ \overline{Gr}^\lambda_{\mu + (\lambda, \omega)} \]

\( T^x G/B \) in type \( A \).
The $T \times A$ fixed loci are the same for $Gr^{\Lambda_1, \ldots, \Lambda_r}$:

\[
\left\{ \left[ z_1, \ldots, z_r \right] \right\}
\]

$\Sigma_i$ are coweights

$\Sigma_1 - \Sigma_\ell$ are in $W \Lambda_i \subset \text{coweights}$ in $V_{\Lambda_i} - G r_{\Lambda_i}$

$\Sigma_\ell = m$

They can be drawn as paths from $0$ to $m$

This is in bijection with a basis in $V_{\Lambda_1} \otimes \ldots \otimes V_{\Lambda_r}$

**Natural line bundles:**

$Gr_{\Lambda_1, \ldots, \Lambda_r} \subset Gr \times \ell$

This gives $\ell$ projection maps $p_1, \ldots, p_{\ell}$.

$Z_i = p_i^* O(1)$

$E_i = Z_i / Z_{i-1}$, behave like $e_i$'s in the root system.
\[ \sum c_i^T(\xi_i) = 0 \iff \sum \xi e_i = 0 \]

\[ \frac{c_i^T(\xi_i)}{\text{span} \ H^2(X)} \]

\[ \uparrow \]

\[ \emptyset \]

**Basis**

**Localization:** \( \varpi : X^T \to X \)

\[ \varpi^* : H^{-\ast}(X)_{\text{loc}} \xrightarrow{\cong} H^{-\ast}(X^T)_{\text{loc}} \rightarrow \text{a vector space over } \text{Frac}(H^0_{\text{T}}(pt)) \text{ of dim } \#X^T \]

\[ M_{\text{loc}} = M \otimes_{H^0_{\text{T}}(pt)} \text{Frac}(H^0_{\text{T}}(pt)) \]

This gives two bases:

\[ (\varpi^*)^{-1}(1_p) \leftarrow \text{only localized classes. little control} \]

\[ (\varpi)_{1_p} \rightarrow \text{non-localized, but } \in H^2_{\text{dim}X}(X) \]

**Maulik-Okounkov Stable Envelopes**

- \( \text{Stab}(p) \in H^{\text{dim}X}_{\text{T}}(X) \)
  - a refined version of \( P.D[\text{Art}^\text{\theta}(p)] \)

- \( \text{Stab}(p)_{\mid_q} \rightarrow \text{triangular} \)

- \( \text{Stab}(p)_{\mid_q} = 0 \mod \frac{k}{2} \iff p \neq q. \)
Classical Part

\[ D | \text{Stab}_e(p) | \quad D \in H^2_+(X) \]

\[ \langle \text{Stab}_e(p), D | \text{Stab}_e(p) \rangle \in H^2_+(p^1) \]

If \( p \neq q \) this is divisible by \( h \), so end.

If \( p = q \) this is a straightforward computation.

The only data you need is \( \text{Stab}(p) |_q \mod h^2 \).

1) Find \( \alpha \) in \( A_{\alpha} \)-type

There are explicit recursive relations.

2) For \( G \) of general type:

\( p \neq q \); If \( \text{Stab}_e(p) |_q \neq 0 \Rightarrow \text{Stab}_e(p) |_q = 0 \).

In type \( \alpha \), the \( \text{Stab}(p) |_\alpha \neq 0 \mod h \)

if \( p \) and \( q \) are related by “face”