Homological mirror symmetry

X symplectic mfld	Y alg. vty.
$\mathrm{Fuk}(X)$	$\mathrm{Coh}(Y)$
<u>objects</u> (triangulated envelope)	modules for functions on Y.
Lagrangians (equipped with local systems)	(derived)
<u>morphisms</u>	
intersection pts	
<u>composition</u>	
counting hol. disks	

<u>Examples</u>.

T^*S^1 $\qquad\qquad$ $\mathbb{C}^*$

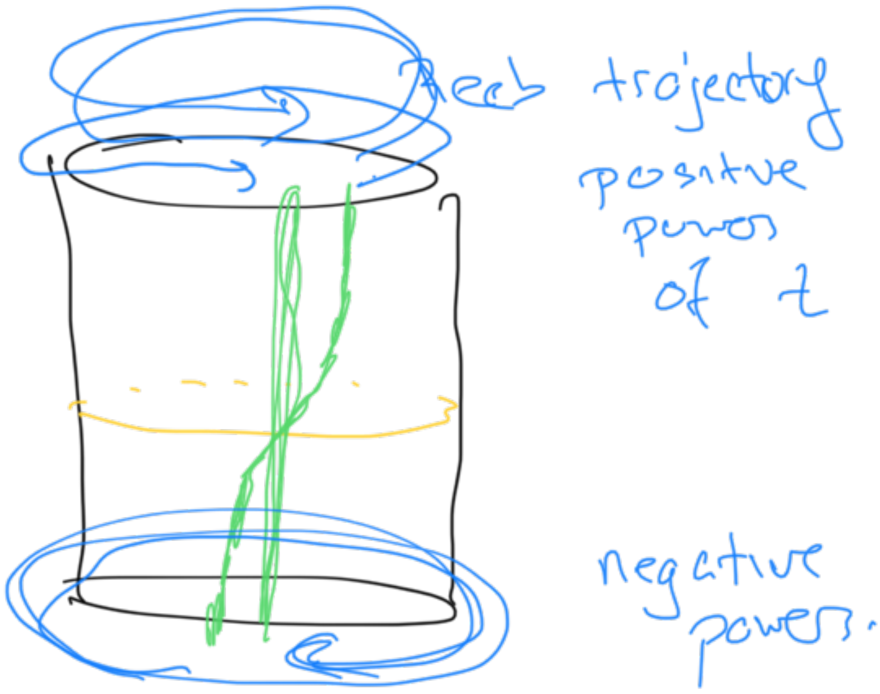
$(S^1, x) \longmapsto \mathcal{O}_x$

$x \in \mathbb{C}^*$

$\mathcal{O}$

cotangent fiber $\longmapsto$

$\mathrm{Hom}(\mathcal{O}, \mathcal{O}) \cong \mathbb{C}[t, t^{-1}]$



"wrapped Fukaya category"

$\mathbb{C}$

$(S^1, x) \longmapsto \mathcal{O}_x$

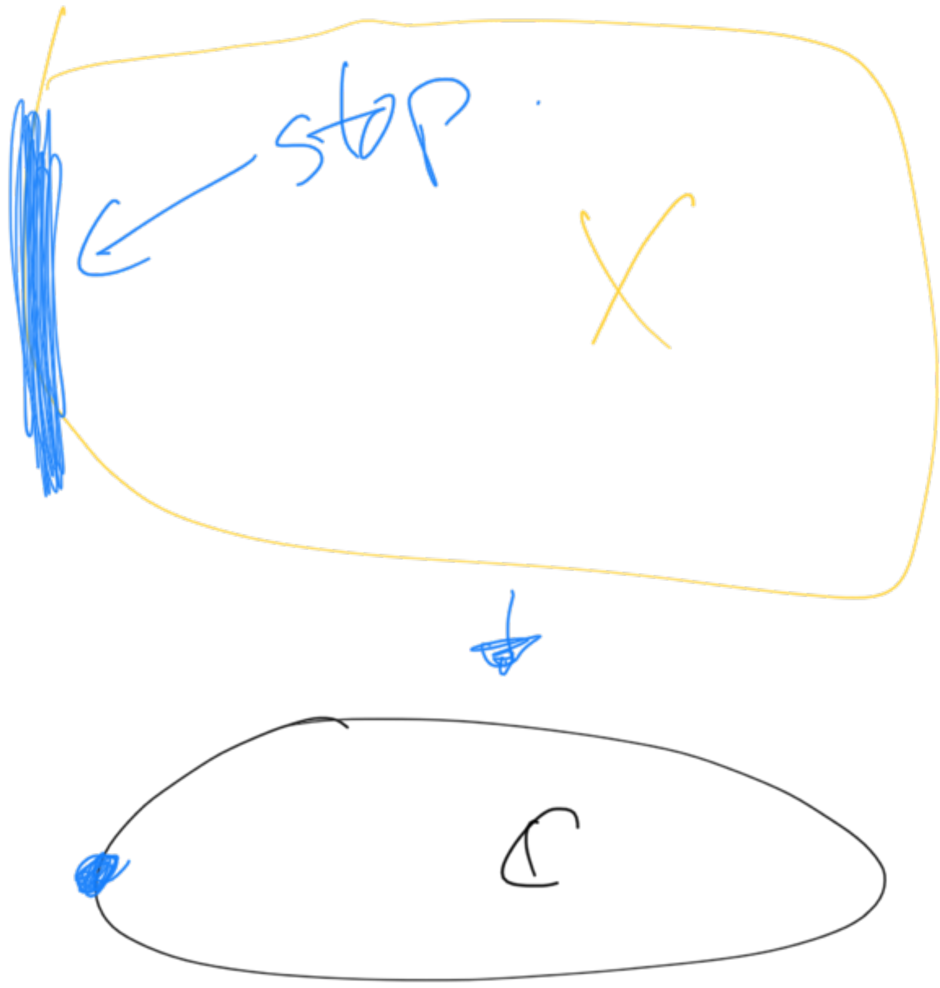
cotangent fiber $\longmapsto \mathcal{O}$

linking disk $\longmapsto \mathcal{O}_0$

$$\boxed{\mathbb{C}^* \xhookrightarrow[z]{} \mathbb{C}} \qquad \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbb{C}[t]$$

Hori-Vafa superpotential.

$W : X \longrightarrow \mathbb{C}$



(Seidel school: hope W is a Lefschetz fibration. use vanishing cycle methods etc.)

$$(S^1, x) \longmapsto \mathcal{O}_x$$

$$D_0 \longmapsto \mathcal{O}_0$$

$$D_\infty \longmapsto \mathcal{O}_\infty$$

General theory.	for computing fukaya category for Weinstein symplectic mflds.
[GPS 2]	

Weinstein

⇕

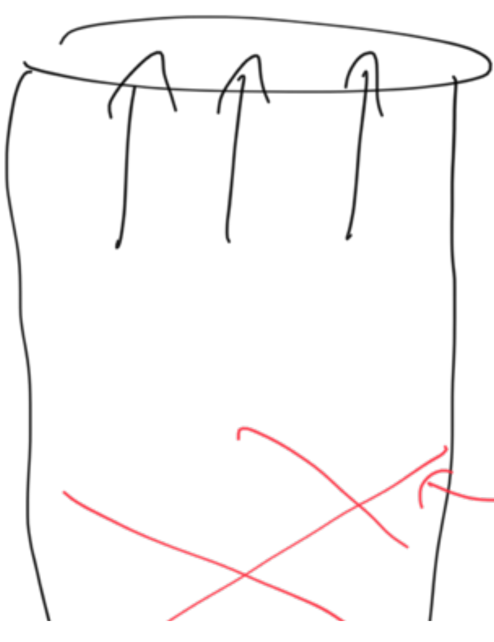
core is isotropic

$$(X, \omega = d\lambda)$$

$\updownarrow$ $\omega(Z, \cdot) = \lambda$

$\exists Z$ $\quad \mathcal{L}_Z \omega = \omega$

Liouville v.f.

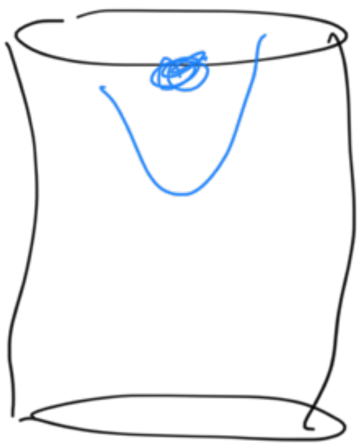


what does

not escape

Geometry $\longrightarrow$ Categories

① stop removal $\longrightarrow$ quotients



$$\simeq \quad Ch(A')$$

$$\downarrow$$

$$Ch(A')/\langle \text{linking dist} \rangle$$

$$\|$$

$$\simeq \quad Ch(\mathbb{C}^*)$$

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② products of spaces $\longrightarrow$ tensor product of categories

③ "sectorial" gluing $\longrightarrow$ pushout

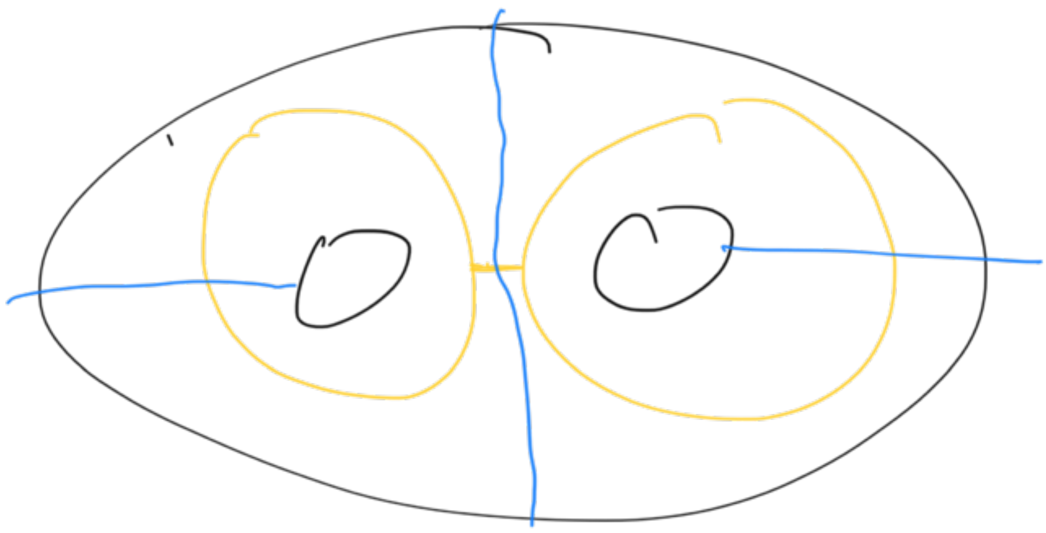
SS

along

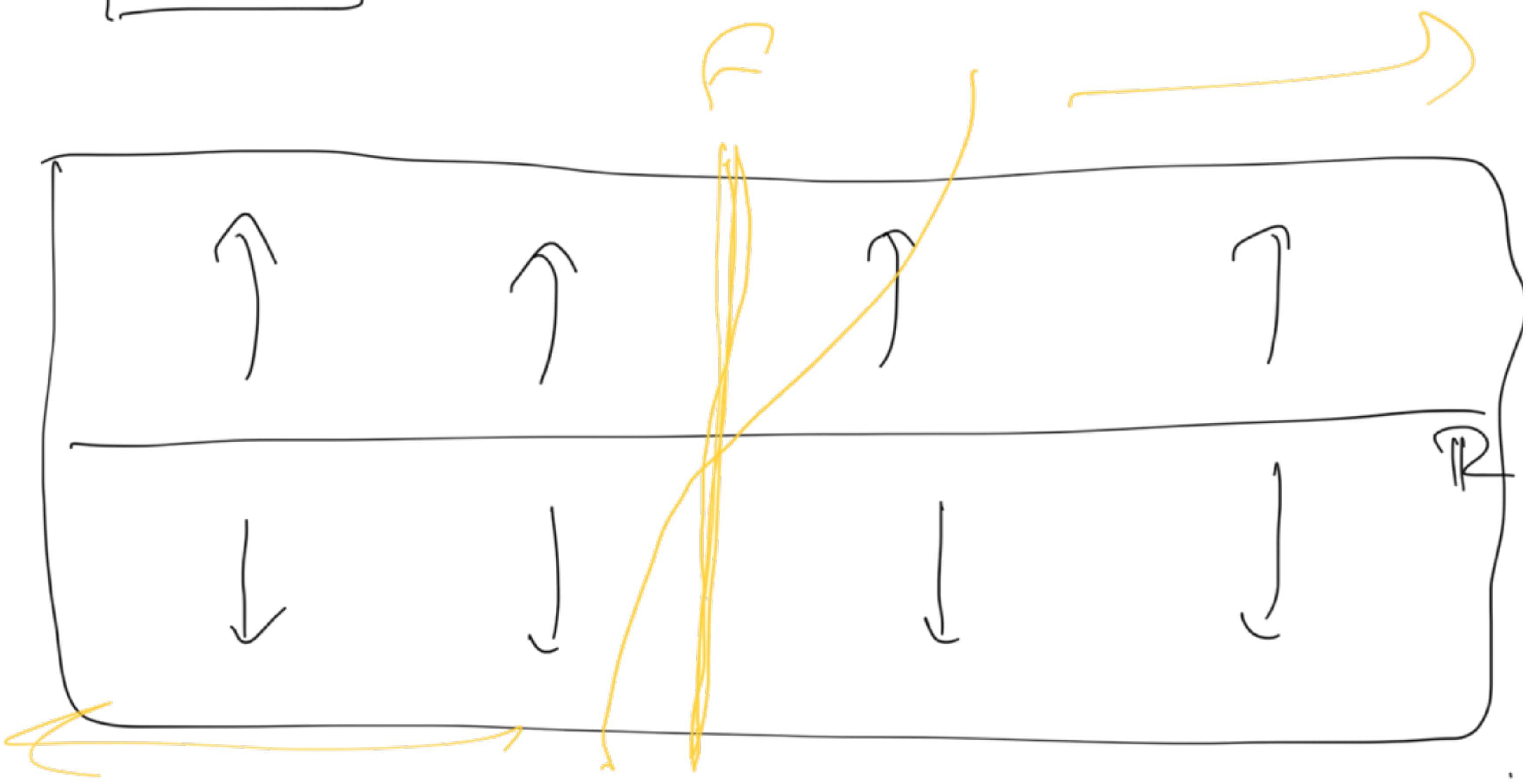
gluing [illegible]
stops

(4) collection of disks transverse to each conn. cpt. of smooth Lag. locus of skeleton → generators

"cocores"

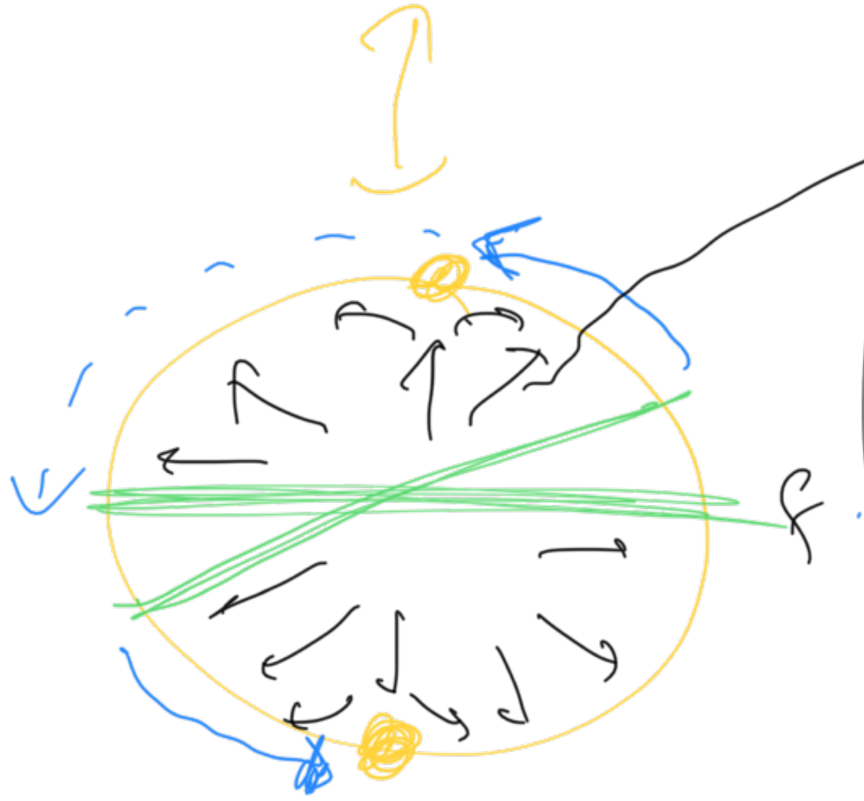


$\boxed{T^*\mathbb{R}}$



$\mathrm{Hom}(F, F) = \mathbb{Z}.$

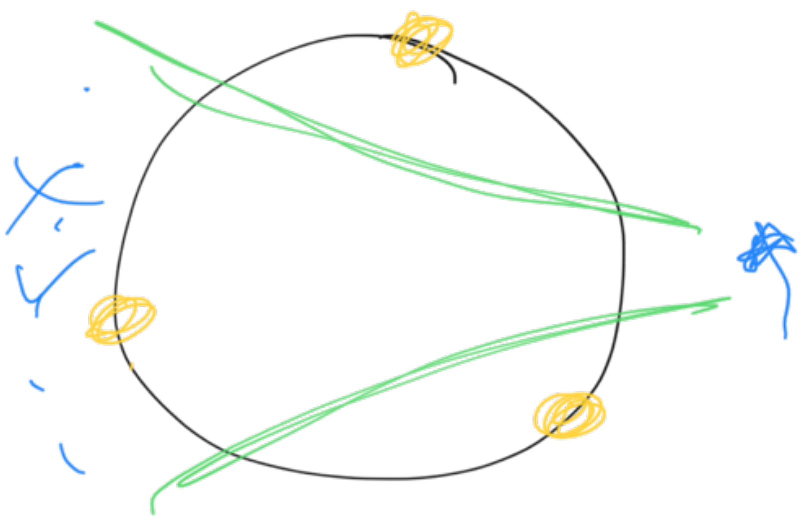
$\mathrm{Fuk}(T^*\mathbb{R}) = \mathrm{Mod}\ \mathbb{Z}.$



GPS 1
section 2.

$\mathrm{Hom}(F, F) = \mathbb{Z}$

$(\mathbb{C}, z^2)$

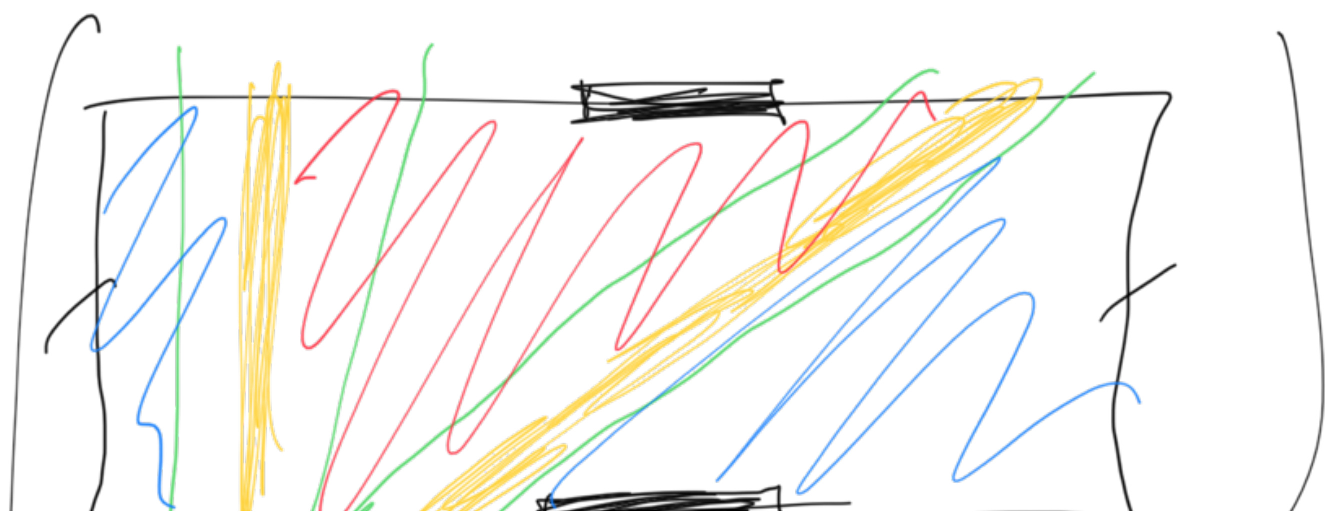


$\mathrm{Fuk}(\circ) =$

$\mathrm{Mod}(\circ \to \circ)$

Ex. TP^1's mirror.

Fuk



$$\cong$$

$$\mathrm{Colim}\left(\begin{tikzcd} & \mathrm{Fuk}(\circ) \arrow[dl] \arrow[dr] & \\ \mathrm{Fuk}(\circ) & & \mathrm{Fuk}(\circ) \\ & \mathrm{Fuk}(\circ) \arrow[ul] \arrow[ur] & \end{tikzcd} \right)$$

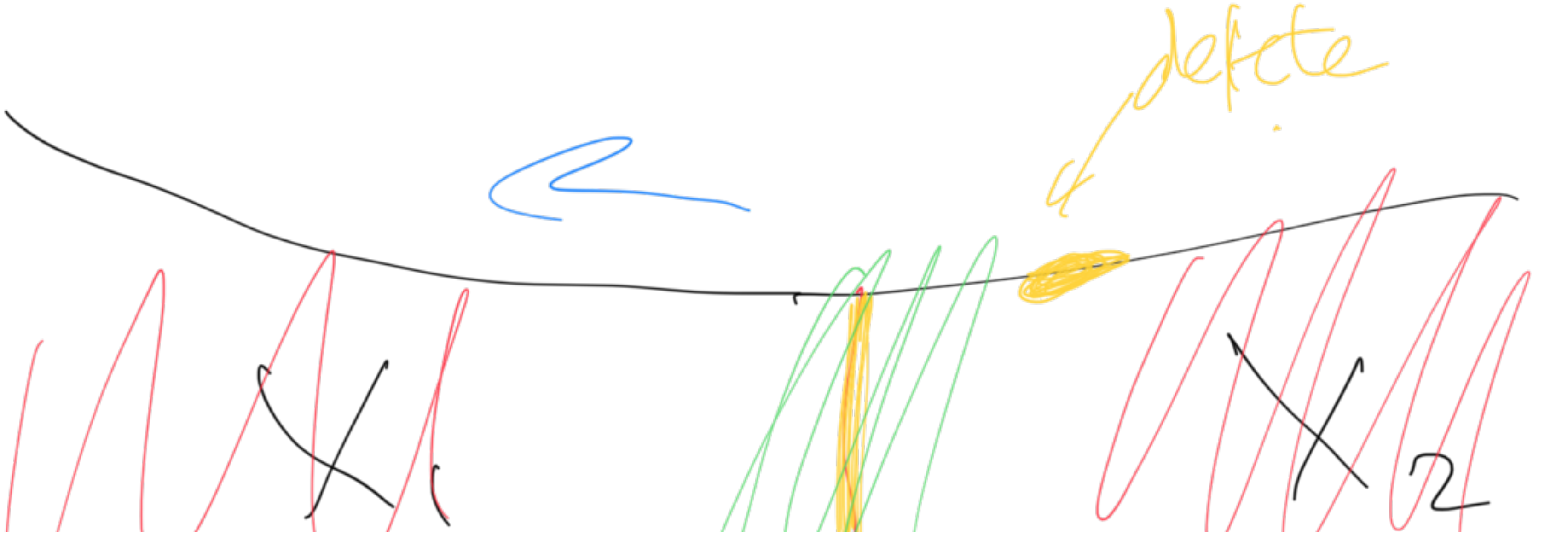
$$\mathrm{Colim}\left(\begin{tikzcd} \mathrm{Mod}(\circ \to \circ) \arrow[rr, bend left, "\mathrm{Mod}(\circ)"] \arrow[rr, bend right, "\mathrm{Mod}(\circ)"'] & & \mathrm{Mod}(\circ \to \circ) \end{tikzcd} \right)$$

$$\cong$$

$$\mathrm{Mod}\left(\begin{tikzcd} \circ \arrow[r, shift left] \arrow[r, shift right] & \circ \end{tikzcd} \right)$$

$$\cong$$

$$\mathrm{Coh}(\mathbb{P}^1)$$



Equivariance.

$$G = \mathrm{Hom}(\pi_1(X), \mathbb{G}_m)$$

acts on $\mathrm{Fuk}(X)$

by tensor with local system

$$\mathrm{Fuk}(X)^G = \mathrm{Fuk}(X^{ab}).$$

Q. ~~what~~ in precisely what sense is this true?

tensored are Mod(k)

given a sheaf of categories on Λ, same question for global sections of this sheaf.

versus $G = \mathrm{Hom}(\pi_1(\Lambda), \mathbb{G}_m)$

$\mathrm{Coh}(\mathbb{C})$ $\qquad$ $\mathrm{Mod}\ k[t]$

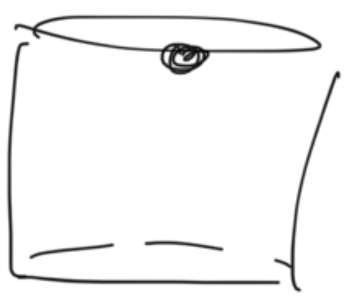
$$\mathrm{Coh}(\mathbb{C}/\mathbb{C}^*)$$

$$\|$$

$$\mathrm{Coh}(\mathbb{C}) \quad \mathbb{C}^*$$

$$\mathbb{C}^* \curvearrowright \mathrm{Coh}(\mathbb{C})$$

$$\mathrm{Coh}(\mathbb{A}^1) \simeq \mathrm{Fuk}$$

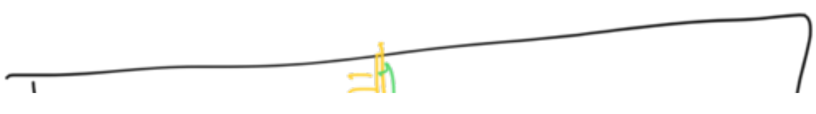


toric variety

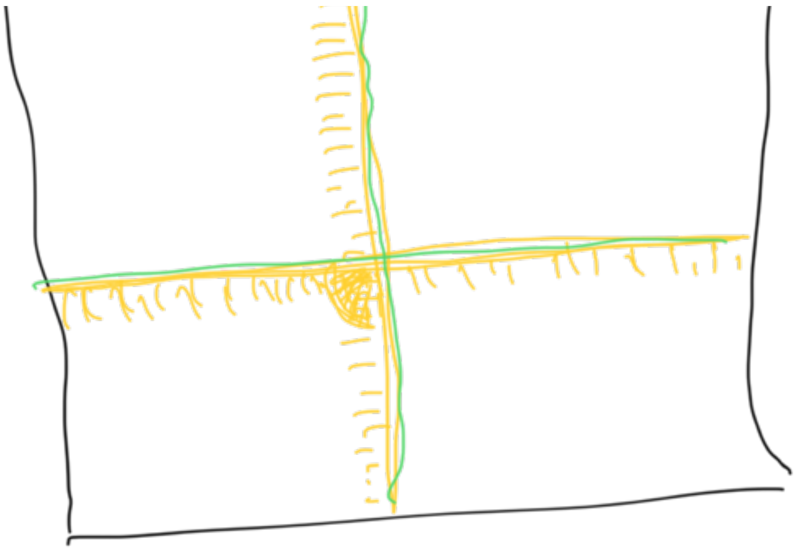
$$\mathrm{Coh}\big((\mathbb{A}^n \setminus Z)/G\big)$$

$$\|$$

$$\big(\mathrm{Coh}(\mathbb{A}^n)/\mathrm{Coh}(Z)\big)^G$$



mirror



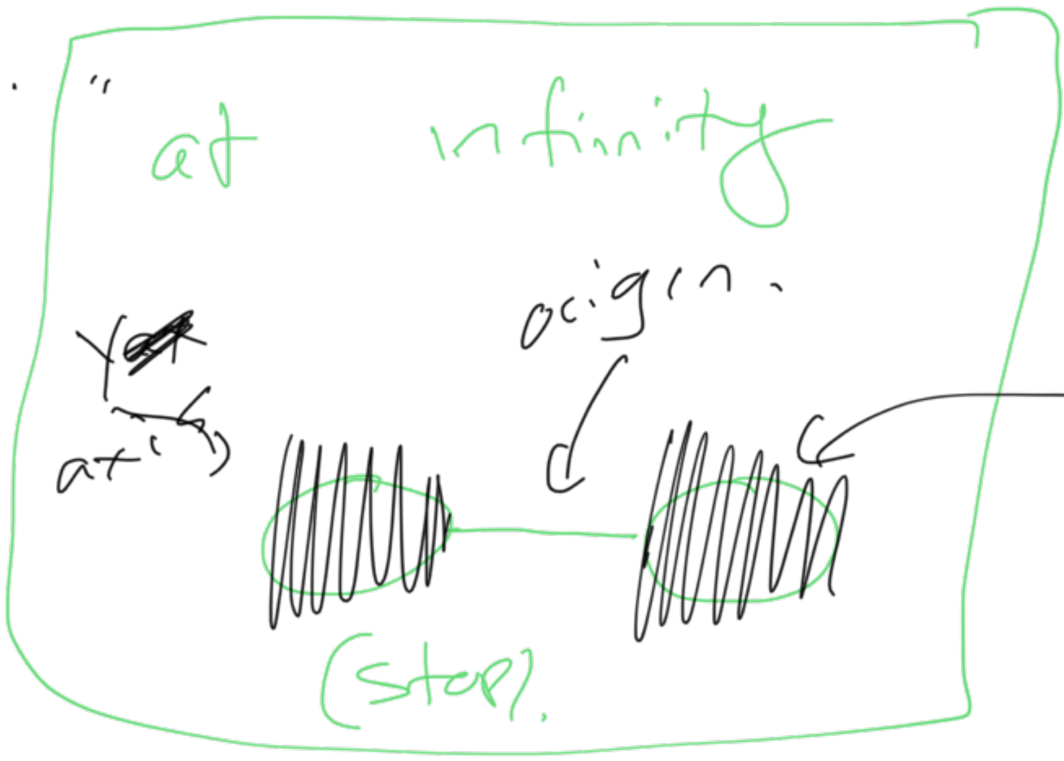
to $\mathbb{A}^2$

=

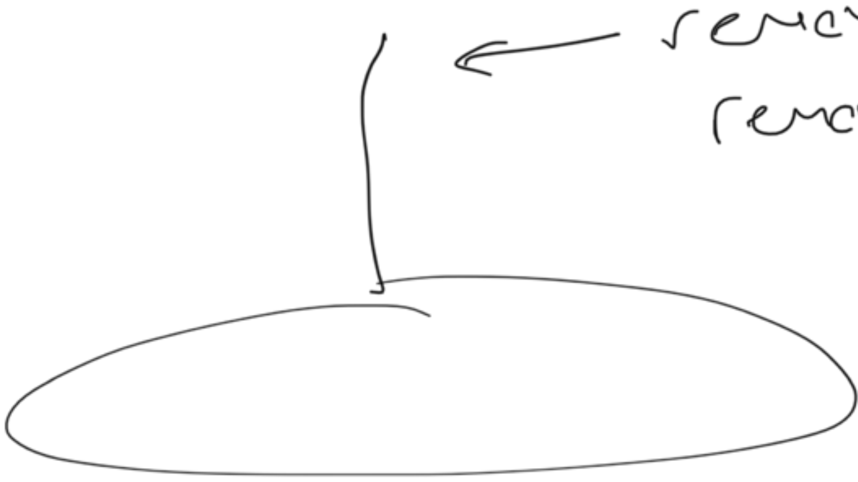
$\left(\text{mirror to } \mathbb{A}^1\right)^2$

$T^*((S^1)^2)$

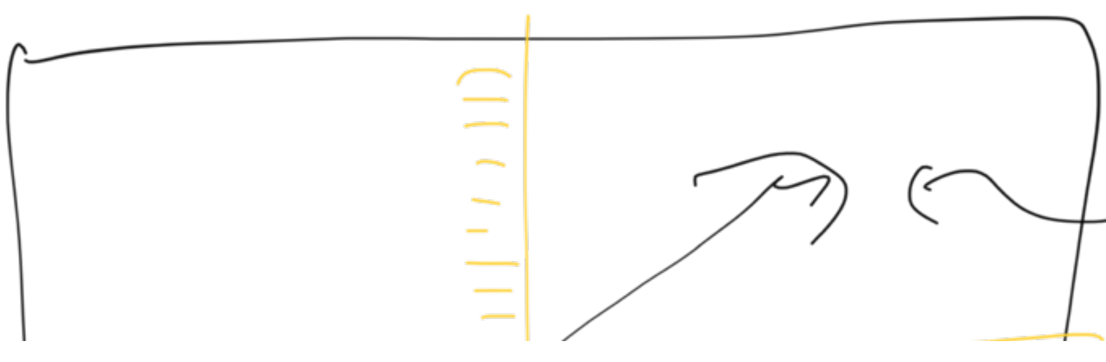
zero section union positive cotangent fiber



removing this removes X axis in mirror



removing this removes origin in mirror.



unwind this

directions

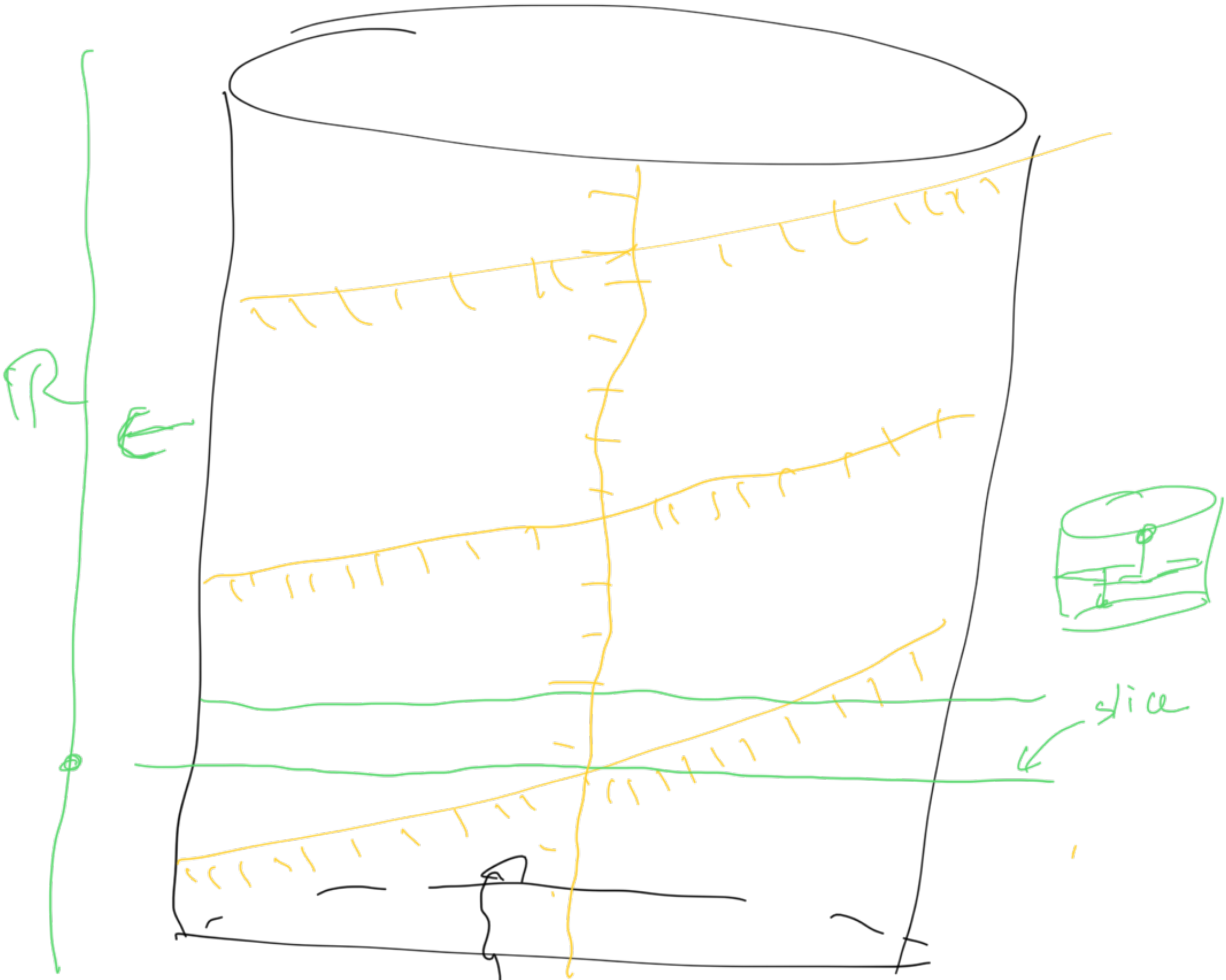
mirror to $\mathbb{A}^2 \setminus 0$.

$$\mathbb{P}^1 = (\mathbb{A}^2 \setminus 0)/\mathbb{G}_m$$

T^*M "at infinity" $= S^*M$

for any closed subset $S \subset S^*M$
$Fuk(T^*M, S)$
is defined.

I have a chance of computing this when S retracts to a isotropic subset (usually Legendrian).



$$\mathrm{Fuk}\left(T^*(S^1)^2, \,|\,\right) \simeq \mathrm{Coh}\left(\mathbb{A}^2 \setminus 0 \,/\, \mathbb{G}_m\right)$$

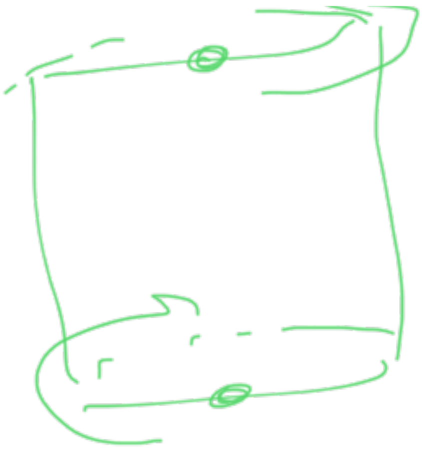
$$\downarrow \qquad\qquad\qquad\qquad \wr\wr$$

$$\mathrm{Fuk}\left(T^*S^1, \ominus\right) \qquad\qquad \mathrm{Coh}\left(\mathbb{P}^1\right)$$

① symplectic reduction is

$(T^*S^1, \ominus)$

② category of symplectic reduction is locally constant along $\mathbb{R}$.



<u>Claim</u>. this works for smooth DM toric stacks.

because exactly in this case, the fan of $T^\vee$ is simplicial & projects by a smooth submersion to the extra directions.

$$\boxed{(\mathbb{A}^n \setminus Z)/T}$$

$\big\downarrow$ coarse moduli space

$$\boxed{\text{singular toric variety.}}$$

no ~~superpote~~ superpotential description

$$\boxed{\lim(\cdots \to C \times \underbrace{\mathrm{Ch}(G)} \to C) = C^G}$$

Thm (FLTZ, Kuwagaki, ...)

$$\mathrm{Coh}(T_\Sigma) = \mathrm{Fuk}(T^*(S^1)^n, \Lambda_\Sigma)$$

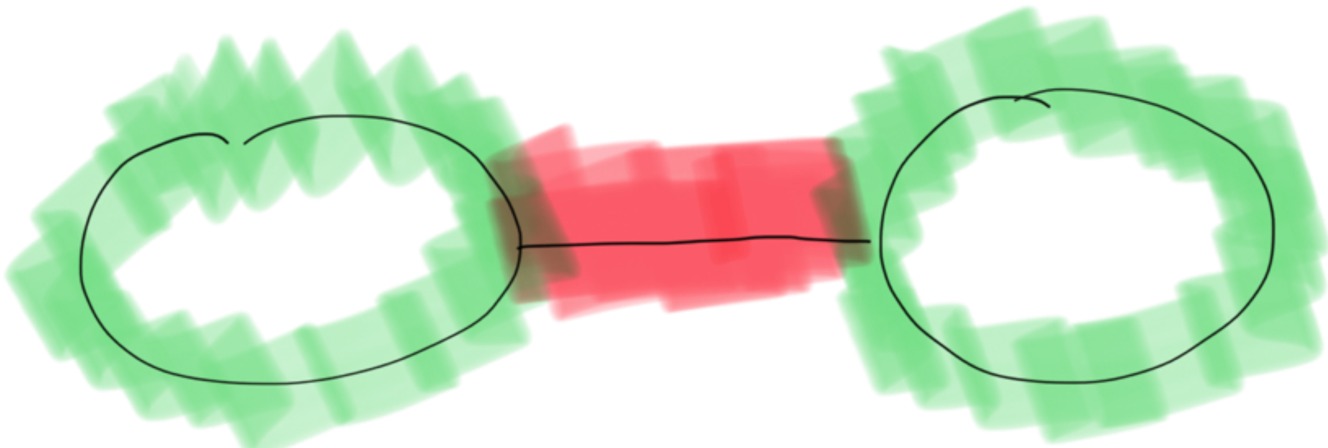
$$\uparrow$$

fan

$$\Lambda_\Sigma = \bigcup_{\sigma \in \Sigma} \sigma^\perp \times \sigma$$

Pf.

Local constancy comes from: stop for ambient space is skeleton of some Weinstein mfld.

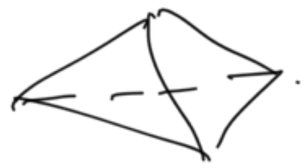


such that the handle attachment data is a bundle over the auxillary vector space.

Gluing the mirror [with Ben Gammage]

Large complex structure limit

Y = union of toric varieties glued along toric strata.

K3 ⇝ 

since we understand ~~toric~~ mirrors to toric varieties & we understand mirror to gluing along toric varieties, glue the mirror.

Eg.

should be skeleton for

mirror to

more generally, works for hypersurfaces in $(\mathbb{C}^*)^n$.

General case

B manifold

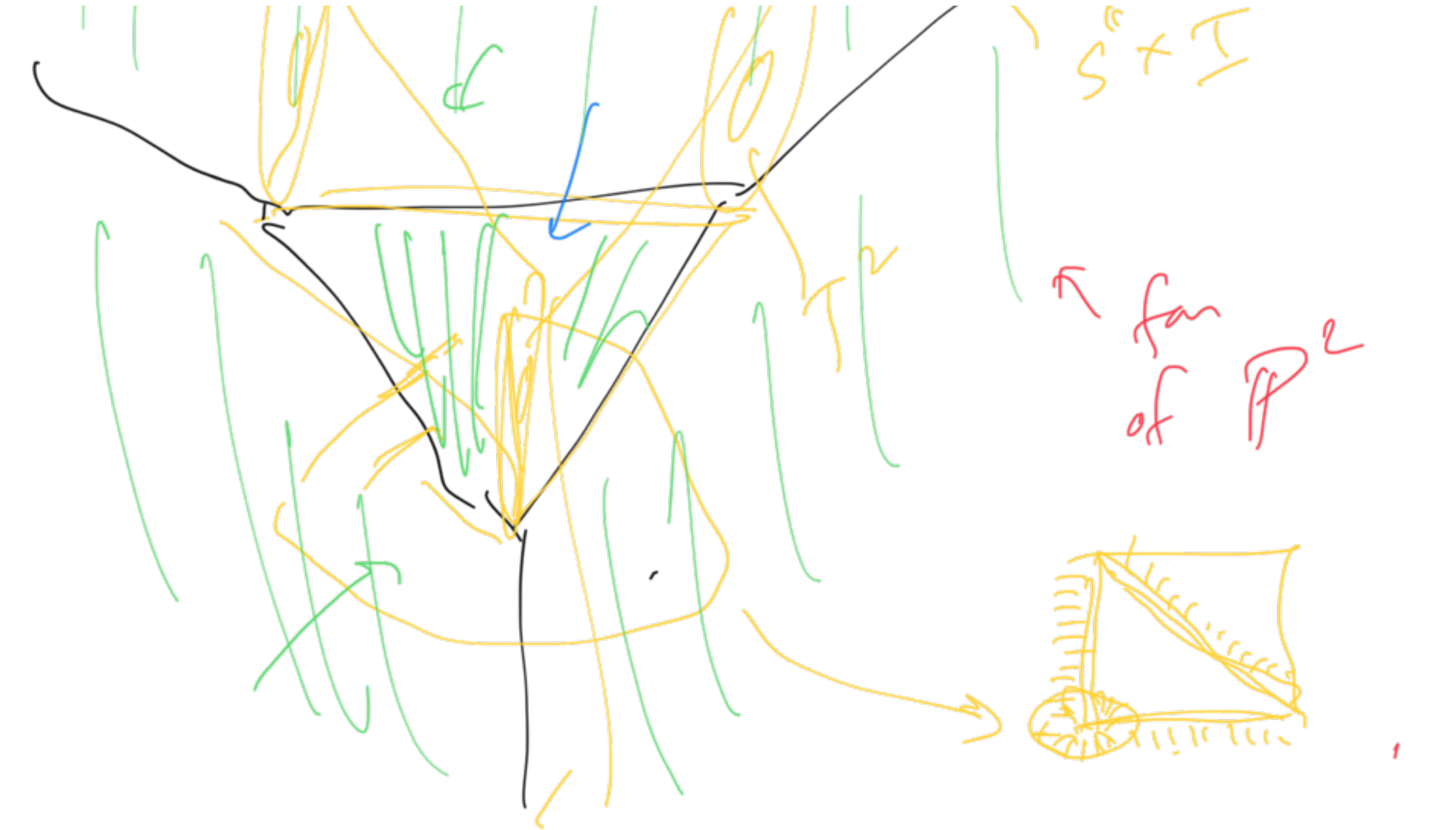
cell decomposition

~~normals~~ normal bundle to each cell

$\Updownarrow$

fan of toric vty.

$$\mathrm{Fuk}\left(\text{symplectic mfld.}\right) = \mathrm{Coh}\left(\begin{array}{c}\text{union}\\ \text{of toric}\\ \text{varieties}\end{array}\right)$$



this is the skeleton of a symplectic mfld.

<u>Thm</u> (Gammage – S)

$\exists$ a construction of symplectic mfld mirror to ~~$\exists$~~ appropriate unions of toric varieties such that HMS holds.

want now to deform away from