Mathematical hints of 3-d mirror symmetry

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reasonably competent mathematician

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I'll do my best to do a bit of both, but apologies to essentially everyone.

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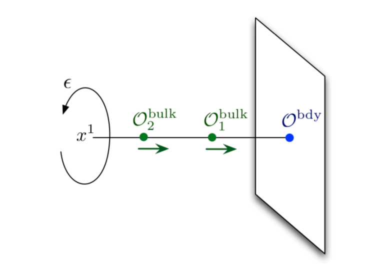
- For mathematicians: this gives a new perspective on categories we already know, such as representations of $\mathfrak{gl}_n$.
- For physicists: features of these categories hopefully suggest interesting things about the physics. (Unfortunately, all I'm able to do at the moment is suggest. Regard this talk as a cry for help.)

My talk is going to be about 3-dimensional $\mathcal{N} = 4$ supersymmetric quantum field theory.

Why $d = 3, \mathcal{N} = 4$?

Briefly: lots of room to have different topological twists whose interaction we can think about. In each of these twists, $d = 3 \supset d = 1$ is also a natural context in TQFT for commutative algebras deforming to associative ones ("almost commutative algebras"). A $d = 2$ boundary condition corresponds to a module.



Any $d = 3, \mathcal{N} = 4$ supersymmetric theory has an action of $\mathrm{Spin}_4(\mathbb{C}) = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$.

There are two twists that privilege these two different factors, attached to the terms "Higgs" and "Coulomb" which give us two almost commutative algebras.

We'll be interested in one specific class of $3d$ $\mathcal{N} = 4$ supersymmetric quantum field theories: the gauge theory attached to a compact group G and representation N over $\mathbb{C}$.

In physics-speak, we couple an adjoint vectormultiplet for G with a hypermultiplet tranforming in the representation N.

I'll only embarass myself when I try to explain what this means precisely, but as a classical theory, there is no question about the answer, in whatever framework you like to use for classical field theories.

The quantum theory is a different story. I think physicists are reasonably happy just saying that you do a path integral...this still leaves them with questions they cannot answer.

One of the key invariants of a QFT is the **moduli space of vacua** which captures the topological part of the **algebra of local operators**. This acts on everything in the picture (naturally on boundary conditions, etc.).

For a 3d $\mathcal{N} = 4$ supersymmetric theory, the moduli space of vacua is a (singular) hyperkähler manifold; a choice of a 2d $\mathcal{N} = (2, 2)$ supersymmetric boundary condition fixes a preferred complex structure.

Classically, the moduli space of vacua is given by the equations:

$$[\vec{\phi}, \vec{\phi}] = 0 \qquad (\vec{\phi} + \vec{m}) \cdot (X, Y) \qquad \vec{\mu}(X, Y) + \vec{t} = 0$$

The Higgs branch is when $\vec{\phi} = \vec{m} = 0$, so the result is a hyperkähler quotient:

$$\mathfrak{M}_H = \{(X, Y) \mid \vec{\mu}(X, Y) + \vec{t} = 0\}/G$$

The Coulomb branch is when $X = Y = 0$, so

$$\mathfrak{M}_C \approx T^*T^\vee_{\mathbb{C}}/W$$

For $T \subset G$ a maximal torus, $T^\vee$ its Langlands dual.

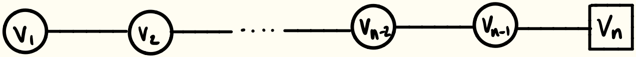
The $\approx$ is because this is the classical answer, and it will be "corrected" when we quantize. The Higgs branch does not have this issue.

The Coulomb branch had no precise mathematical description until one was given in this case by Braverman, Finkelberg and Nakajima.

Why as a mathematician am I interested in this story?

The varieties $\mathfrak{M}_H$ and $\mathfrak{M}_C$ are symplectic and have quantizations A_H and A_C over $\mathbb{C}[h]$. Some very interesting algebras show up as A_C:

- If $G = U(1)^k$, the Coulomb branch is a hypertoric enveloping algebra.
- If $G = GL_n$ and $N = \mathfrak{gl}_n \oplus (\mathbb{C}^n)^{\oplus \ell}$, the Coulomb branch is a spherical Cherednik algebra of the group $G(\ell, 1, n)$.
- For the quiver gauge theory

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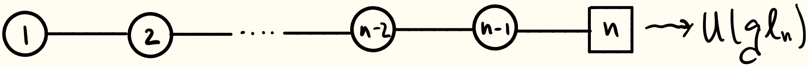

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From the physical perspective, there are two natural ways of getting our hands on elements of A_C:

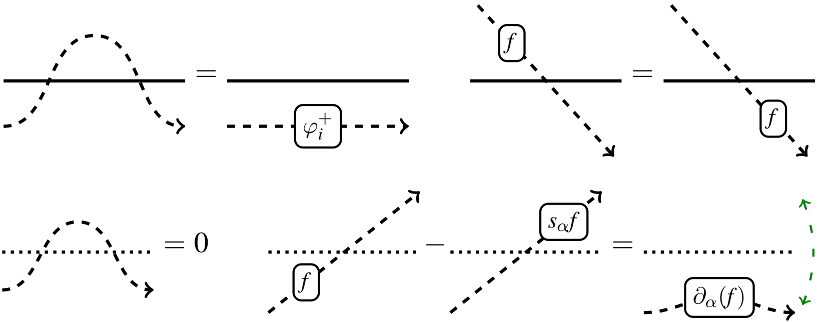
- We have a copy of $\mathbb{C}[\mathfrak{t}_{\mathbb{C}}/W] \cong H^*_G(pt)$ corresponding to the map $\pi\colon T^*T_{\mathbb{C}}/W \to \mathfrak{t}_{\mathbb{C}}/W$. This gives a quantum integrable system.
- We have monopole operators $\mathfrak{m}_\lambda$ for dominant coweights $\lambda\colon U(1) \to G$, coming from the corresponding functions on $T_{\mathbb{C}}$.

We can combine these to get a basis of "dressed monopole operators." It's very hard to multiply in this basis, unfortunately.

Idea from math:

There should be "fractional monopole operators" attached to any path in the torus T which relate different line defects, and straight closed paths give usual monopole operators.

Composition of operators corresponds to concatenation of paths, and then application of local relations:



Here we allow coupons with polynomial functions on $\mathfrak{t}_{\mathbb{C}}$.

Theorem

This gives an algebraic presentation of a category $\mathscr{B}$ (a subcategory of line defects), in which the BFN Coulomb branch is an endomorphism algebra.

This has some advantages:

- You can define Coulomb branches for people who don't know about the affine Grassmannian.
- If you hope to find Coulomb branches in some other context, you have a much easier set of relations to check.

In fact, this category depends on a choice of mass parameters $\vec{m}$, valued in the Lie algebra of the torus of the flavor group $\mathrm{Norm}_{U(N)}(G)/G$.

Theorem

If $\mathfrak{M}_C$ can be smoothed by deforming mass parameters, then the category $\mathscr{B}_{\vec{m}}$ at $h = 0$ gives a noncommutative crepant resolution of $\mathfrak{M}_C$ for generic $\vec{m}$. In fact, $D^b(\mathscr{B}_{\vec{m}}\text{-mod}) \cong D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}))$ for a commutative resolution $\tilde{\mathfrak{M}} \to \mathfrak{M}$.

These resolutions played a key role in recent work with Gammage and McBreen on homological mirror symmetry for hypertoric varieties (i.e. the case $G = U(1)^n$). We hope this is a more general phenomenon.

Considering the larger group Q generated by G and the flavor torus allows us to connect these different mass parameters.

Inside the category $\mathscr{B}^Q$, we can construct $\mathscr{B}_{\vec{m}}$ - $\mathscr{B}_{\vec{n}}$ bimodules ${}_{\vec{m}}\mathscr{T}_{\vec{n}}$. We can compute the action on these bimodules as before using local moves on paths.

We call the functor of derived tensor product with ${}_{\vec{m}}\mathscr{T}_{\vec{n}}$ a **wall-crossing functor**.

Theorem

Wall-crossing functors induce an action of $\pi_1(\mathring{T}_{F,\mathbb{C}}/W)$ *on* $D^b(\mathscr{B}_{\vec{m}}\text{-mod})$. *In fact, they give a* **Schober***: a "perverse sheaf of derived categories" on the space* $T_{F,\mathbb{C}}/W$ *with:*

- *the fiber over* $\exp(\vec{m})$ *given by* $D^b(\mathscr{B}_{\vec{m}}\text{-mod})$.
- *the fiber over a generic point given by* $D^b(\mathsf{Coh}(\tilde{\mathfrak{M}}))$.

All of this has been about the $h = 0$ case. What about $h = 1$?

I'm a representation theorist, so naturally, I want to understand representions.

- For physicists: this will hint at boundary defects with verious different properties.
- For mathematicians: we will learn something new about very familiar algebras like $U(\mathfrak{gl}_n)$.

The category of modules of most interest to us are those with classical limit on the fiber $\pi^{-1}(0)$ of the integrable system. These are called **Gelfand-Tsetlin modules.**

Theorem (W.)

The principal block of the category of Gelfand-Tsetlin A_C-modules is Koszul dual to a category of "pre-$\mathscr{O}$ D-modules" on the quotient N/G.

In particular, the category of G-T modules has a graded lift where the dimensions of S-weight spaces can be computed by a Kazhdan-Lusztig type algorithm.

You can do explicit calculations in this category, along the lines of the monopole operators discussed. KLR algebras appear!

The G-T integrable system for $U(\mathfrak{gl}_n)$ is a special case:

Theorem (Kazhdan-Lusztig-Beilinson-Bernstein-Brylinksi-Kashiwara-Soergel-$\cdots$)

If we consider the principal block $\mathscr{O}_0$ of $U(\mathfrak{gl}_n)$, then $K^0(\mathscr{O}_0) \cong (\mathbb{C}^n)^{\otimes n}_{(1,\dots,1)}$, with simple modules matching the dual canonical basis.

Theorem (KTWWY)

The principal block $\mathcal{C}_0$ of simple Gelfand-Tsetlin modules over $U(\mathfrak{gl}_n)$ has $K^0(\mathcal{C}_0) = (U(\mathfrak{n}_-) \otimes (\mathbb{C}^n)^{\otimes n})_{(1,\dots,1)}$ with classes of simples matching dual canonical basis.

A hint of duality:

D-modules on N/G are modules over a quantization of $T^*(N/G) = \mu^{-1}(0)/G$, the stacky Higgs branch.

This duality looks more symmetric if we pass to category $\mathscr{O}$, the category where a fixed grading element (depending on FI parameters $\vec{t}$) in A_C acts with eigenvalues that are bounded above.

Theorem

Under geometric assumptions that hold in all cases we understand, the Koszul dual of category $\mathscr{O}$ for A_C with $\vec{m}, \vec{t}$ is category $\mathscr{O}$ for A_H with grading element induced by $\vec{m}$ and quantization parameter depending on $\vec{t}$.

This is related to the previous statement by quantum Hamiltonian reduction.

In the $h = 1$ case, we also have Schobers:

- one induced by changing grading elements (Coulomb $\leftrightarrow$ FI, Higgs $\leftrightarrow$ mass): shuffling functors
- one induced by changing quantization parameters (Coulomb $\leftrightarrow$ mass, Higgs $\leftrightarrow$ FI): twisting functors

Theorem

Under Koszul duality, these two Schobers switch places.

Can we find the non-commutative resolution discussed before on the Higgs side?

Work of Costello, Creuzig and Gaiotto suggests the correct thing to consider is a log VOA coming from a holomorphic boundary condition.

The line defects discussed earlier should give modules over this VOA where the fractional monopole operators act "Koszul dually" i.e. with the grading on $\mathbb{C}[\mathfrak{M}]$ becoming homological.

Thanks for listening.