Geometric characterisation of topological string partition functions

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Topological string partition functions

Consider A/B model topological string on Calabi-Yau manifold X/Y.

World-sheet definition of Z_{top} yields formal series

$$\log Z_{\text{top}} \sim \sum_{g=0}^{\infty} \lambda^{2g-2} \mathcal{F}_g \tag{1}$$

F_g have mathematical definition through Gromov-Witten invariants.

Non-perturbative definitions?

Do there exist functions Z_{top} having (1) as asymptotic expansion?

(Functions on which space? Functions, sections of a line bundle, or what?)

Z_{top} could be locally defined functions on $\mathcal{M}_{\text{Käh}}(X)$ or $\mathcal{M}_{\text{cplx}}(Y)$.

$$Z_{\text{top}} = Z_{\text{top}}(t), \quad t = (t_1, \ldots, t_d) : \text{ coordinates on } \mathcal{M}_{\text{Käh}}(X).$$

Dream: There exists a natural geometric structure on $\mathcal{M}_{\text{cplx}}(Y)$ allowing us to represent Z_{top} as "local sections".

Our playground: Local Calabi-Yau manifolds Y_Σ of class Σ:

$uv - f_\Sigma(x,y) = 0$ s.t. $\Sigma = \big\{(x,y) \in T^*C; f_\Sigma(x,y) = 0\big\} \subset T^*C$ smooth,

$f_\Sigma(x,y) = y^2 - q(x)$, $q(x)(dx)^2$: quadratic differential on cplx. surface C.

Moduli space $\mathcal{B} \equiv \mathcal{M}_{\mathsf{cplx}}(Y)$: Space of pairs (C,q), C: Riemann surface, q: quadratic differential.

Special geometry: Coordinates

$$a^r = \int_{\alpha^r} \sqrt{q}, \qquad \check{a}^r = \int_{\check{\alpha}_r} \sqrt{q} = \frac{\partial}{\partial a^r}\mathcal{F}(a),$$

where $\{(\alpha^r, \check{\alpha}_r); r = 1,\dots,d\}$ is a canonical basis for $H_1(\Sigma,\mathbb{Z})$.

Integrable structure: (Donagi-Witten, Freed) $\exists$ canonical torus fibration

$$\pi : \mathcal{M} \to \mathcal{B}, \qquad \Theta_b := \pi^{-1}(b) = \mathbb{C}^d/(\mathbb{Z}^d + \tau(b)\cdot\mathbb{Z}^d),$$

$\tau(b)_{rs} = \frac{\partial}{\partial a_\imath^r}\frac{\partial}{\partial a_\imath^s}\mathcal{F}(a_\imath)$, coordinates $\theta_\imath^r$, $r = 1,\dots,d$, on torus fibers.

First part

Alternative representations of $\mathcal{M}$:

(a) $\mathcal{M}$ moduli space of pairs $(\Sigma, \mathcal{D})$, $\mathcal{D}$: divisor on Σ

– Abel map: Divisors $\mathcal{D}$ to points in Θ_b

(b) $\mathcal{M} \simeq \mathcal{M}_{\text{Hit}}(Y)$, moduli space of Higgs pairs $(\mathcal{E}, \varphi)$

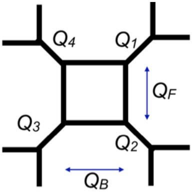
– Hitchin: Map Higgs pairs $(\mathcal{E}, \varphi)$ to pairs $(\Sigma, \mathcal{D})$, Σ defined from $q = \frac{1}{2}\mathrm{tr}(\varphi^2)$ as above, $\mathcal{D}$ (roughly): divisor characterising the bundle of eigen-lines of φ.

(c) $\mathcal{M} \simeq$ intermediate Jacobian fibration (Diaconescu-Donagi-Pantev)

A possible starting point

Some of Y_Σ: limits of toric CY $\Rightarrow$ compute Z_{top} with topological vertex[1].

Basic example:



$$Z_{\text{top}} = z^{\sigma^2-\theta_1^2-\theta_2^2} Z_{\text{out}}\, Z_{\text{in}}\, Z_{\text{inst}}$$

$$Z_{\text{out}} = \frac{\mathcal{M}(Q_F)\mathcal{M}(Q_3Q_4Q_F)}{\prod_{i=3}^{4}\mathcal{M}(Q_i)\mathcal{M}(Q_iQ_F)},$$

$$Z_{\text{in}} = \frac{\mathcal{M}(Q_F)\mathcal{M}(Q_1Q_2Q_F)}{\prod_{i=1}^{2}\mathcal{M}(Q_i)\mathcal{M}(Q_iQ_F)}.$$

- $\mathcal{M}(Q)$ is defined as $\mathcal{M}(Q) = \prod_{i,j=0}^{\infty}(1 - Qq^{i+j+1})^{-1}$ for $|q| < 1$.
- Z^{inst} is $d = 5$, $\mathcal{N} = 2$, $SU(2)$ instanton partition function[2].

$$Q_i = e^{-t_i},\ t_i = \mathcal{O}(R) \textit{ for } i = 1,2,3,4,F \ \Rightarrow \begin{cases} \text{Limit from } 5d \text{ to } 4d, \\ \text{mirror: local CY of class } \Sigma. \end{cases}$$

AGT-correspondence: $Z^{\text{inst}} \sim$ conformal block of Virasoro VOA at $c = 1$.

[1] Aganagic, Klemm, Marino, Vafa

[2] Moore-Nekrasov-Shatashvili; Losev-Nekrasov-Shatashvili; Nekrasov

String dualities predict[3] that $Z_{\rm top}(t;\hbar) \overset{\rm [MNOP]}{\sim} Z_{\rm D0\text{-}D2\text{-}D6}(t;\hbar)$ is related to

$$Z_{\rm dual}(\xi,t;\hbar) := Z_{\rm D0\text{-}D2\text{-}D4\text{-}D6}(\xi,t;\hbar) = \sum_{p\in H^2(Y,\mathbb{Z})} e^{p\xi} Z_{\rm top}(t+\hbar p;\hbar) :$$

free fermion partition function on non-commutative*) deformation of Σ.

*) Equation $y^2 = q(x)$ defining Σ admits canonical quantisation $y \to \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$,

$$\rightsquigarrow \textbf{ quantum curve } \quad \hbar^2\frac{\partial^2}{\partial x^2} - q(x) \quad \leftrightsquigarrow \quad \text{oper } \nabla_\hbar = \hbar\frac{\partial}{\partial x} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}.$$

And indeed[4],

$$\boxed{Z_{\rm dual}(\xi,t;\hbar) = \mathcal{T}(t,\xi) \equiv Z_{\rm ff}(\xi,t;\hbar),}$$

where $\mathcal{T}(t,\xi)$: Tau-function for isomonodromic deformations of "deformed quantum curves", $q(x) \to q_\hbar(x) = q(x) + \mathcal{O}(\hbar)$, canonical ξ-dependent deformation of $q(x)$ (more later).

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[3] Dijkgraaf-Hollands-Sulkowski-Vafa [DHSV] using Maulik-Nekrasov-Okounkov-Pandharipande [MNOP]

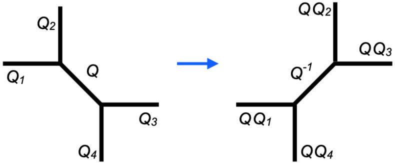
[4] Coman-Pomoni-J.T., based on Gamayun-Iorgov-Lisovyy, Iorgov-Lisovyy-J.T.

$\mathcal{T}(t,\xi) \equiv Z_{\mathrm{ff}}(\xi, t; \hbar)$ admits Fredholm determinant representation[5]

$$\Rightarrow \quad Z_{\text{dual}} \text{ and } Z_{\text{top}} \text{ are locally } \textbf{holomorphic} \text{ functions of } t.$$

But: Partition functions Z_{top} are only **piecewise holomorphic** over $\mathcal{B}$!!!

Example: Flop [Konishi, Minabe]



Analytic continuation of Z^{top} from chamber $|Q| < 1$ to $|Q| > 1$ is related to actual value as

$$Z_{\text{top}} \to Z_{\text{top}} \frac{M(Q)}{M(Q^{-1})}.$$

More complicated **wall-crossing** relations expected to describe jumps across other walls in moduli space $\mathcal{B}$.

Main question: How do we continue Z_{top} over all of moduli space?

Important hint (Coman-Pomoni-J.T.): Relation to abelianisation (Hollands-Neitzke).

[5] Gavrylenko-Marshakov, Cafasso-Gavrylenko-Lisovyy

Our proposal in a nutshell: (compare with Alexandrov, Persson, Pioline – later!)

Main geometric players:

- Moduli space $\mathcal{B} \equiv \mathcal{M}_{\text{cplx}}(Y)$ of complex structures,
- torus fibration $\mathcal{M}$ over $\mathcal{B}$ canonically associated to the special geometry on $\mathcal{B}$ ($\sim$ intermediate Jacobian fibration).

There then exist

(A) a canonical one-parameter ($\hbar$) family of deformations of the complex structures on $\mathcal{M}$, defined by an atlas of Darboux coordinates $x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ on $\mathcal{Z} := \mathcal{M} \times \mathbb{C}^*$,

(B) a canonical pair $(\mathcal{L}_\Theta, \nabla_\Theta)$ consisting of

- $\mathcal{L}_\Theta$: line bundle on $\mathcal{Z}$, transition functions: Difference generating functions of changes of coordinates $x_\imath$,
- ∇_Θ: connection on $\mathcal{L}_\Theta$, flat sections: Tau-functions $\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$,

defining the topological string partition functions via

$$\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \sum_{\mathrm{n} \in \mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n, \check{\mathrm{x}}^\imath)} Z^\imath_{\text{top}}(\mathrm{x}_\imath - \mathrm{n}).$$

(A) The BPS Riemann-Hilbert problem (Gaiotto-Moore-Neitzke; Bridgeland)

Define $\hbar$-deformed complex structures by atlas of coordinates on $\mathcal{Z} \simeq \mathcal{M} \times \mathbb{C}^\times$ with charts $\{\mathcal{U}_\imath; \imath \in \mathbb{I}\}$, Darboux coordinates

$$x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = x_\imath(\hbar), \qquad \Omega = \sum_{r=1}^{d} dx_\imath^r \wedge d\check{x}_r^\imath, \quad \text{such that}$$

- changes of coordinates across $\{\hbar \in \mathbb{C}^\times; a_\gamma/\hbar \in \mathrm{i}\mathbb{R}_-\}$ represented as

$$X_{\gamma'}^\jmath = X_{\gamma'}^\imath (1 - X_\gamma)^{\langle\gamma',\gamma\rangle\Omega(\gamma)}, \qquad X_\gamma^\jmath = e^{2\pi\mathrm{i}\langle\gamma,x_\imath\rangle} = e^{2\pi\mathrm{i}(p_r^\imath x_\imath^r - q_\imath^r \check{x}_r^\imath)},$$
$$\text{if } \gamma = (q_\imath^1, \ldots, q_\imath^d; p_1^\imath, \ldots, p_d^\imath),$$

  determined by data $\Omega(\gamma)$ satisfying Kontsevich-Soibelman-WCF.
- asymptotic behaviour

$$\mathrm{x}_\imath^r \sim \frac{1}{\hbar} a_\imath^r + \vartheta_\imath^r + \mathcal{O}(\hbar), \qquad \check{\mathrm{x}}_\imath^r \sim \frac{1}{\hbar} \check{a}_r^\imath + \check{\vartheta}_r^\imath + \mathcal{O}(\hbar),$$

  with $(a_\imath^r, \check{a}_r^\imath)$ coordinates on $\mathcal{B}$, $\theta_r^\imath := \vartheta_r^\imath - \tau \cdot \check{\vartheta}_\imath^r$ coordinates on Θ_b.

Solving the BPS-RH problem

$1^{\rm st}$ Solution: NLIE (Gaiotto-Moore-Neitzke (GMN); Gaiotto)

$$X_\gamma(\hbar) = X_\gamma^{\rm sf}(\hbar)\exp\left[-\frac{1}{4\pi \mathrm{i}}\sum_{\gamma'}\langle\gamma,\gamma'\rangle\Omega(\gamma')\int_{l_{\gamma'}}\frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1-X_{\gamma'}(\hbar'))\right]$$

with $\log X_\gamma^{\rm sf}(\hbar) = \frac{1}{\hbar}a_\gamma + \vartheta_\gamma$. (Gaiotto: Conformal limit of GMN-NLIE)

$2^{\rm nd}$ Solution: Quantum curves

Quantum curves: Opers, certain pairs $(\mathcal{E},\nabla_\hbar) = (\text{bundle},\text{connection}) \leftrightsquigarrow$

$$\text{differential operators} \quad \hbar^2\partial_x^2 - q_\hbar(x).$$

Coordinates $X_\gamma^\imath(\hbar)$, $\check{X}_\imath^\gamma(\hbar)$ for space of monodromy data defined by Borel summation of exact WKB solution $\rightsquigarrow$ charts $\mathcal{U}_\imath$ labelled by spectral networks (Gaiotto-Moore-Neitzke; Hollands-Neitzke).

Focus on 2^{nd} solution: **Quantum curves**

Equation $y^2 = q(x)$ defining Σ admits canonical quantisation $y \to \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x}$,

$$\rightsquigarrow \quad \text{oper} \quad \hbar^2 \frac{\partial^2}{\partial x^2} - q(x) \quad \leftrightsquigarrow \quad \nabla_\hbar = \hbar \frac{\partial}{\partial x} - \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix}.$$

Observation: There is an essentially canonical generalisation $\hbar$-deforming pairs $(\Sigma, \mathcal{D})$, representable by opers with apparent singularities. $C = C_{0,4}$:

$$q_\hbar(x) = q(x) - \hbar\left(\frac{u(u-1)}{x(x-1)(x-u)} + \frac{2u-1}{x(x-1)}\frac{u-z}{x-z}\right)v + \frac{3}{4}\frac{\hbar^2}{(x-u)^2},$$

$$q(x) = \frac{a_1^2}{x^2} + \frac{a_2^2}{(x-z)^2} + \frac{a_3^2}{(x-1)^2} - \frac{a_1^2+a_2^2+a_3^2-a_4^2}{x(x-1)} + \frac{z(z-1)}{x(x-1)(x-z)}H.$$

with $v^2 = q(u)$. Pair $(u, v) \leftrightsquigarrow$ point on $\Sigma \leftrightsquigarrow$ divisor $\mathcal{D}$.

Conjecture

Solution of BPS-RH-problem given by composition of holonomy map with coordinates on character variety[a] having Borel summable $\hbar$-expansion.

[a] coordinate ring generated by trace functions $\mathrm{tr}(\mathrm{Hol}(\nabla_\hbar))$

Expansion in $\hbar$ - exact WKB: Solutions to $\left(\hbar^2\frac{\partial^2}{\partial x^2} - q_\hbar(x)\right)\chi(x) = 0$,

$$\chi_\pm^{(b)}(x) = \frac{1}{\sqrt{S_{\rm odd}(x)}}\exp\left[\pm\int^x dx'\; S_{\rm odd}(x')\right],$$

with $S_{\rm odd} = \frac{1}{2}(S^{(+)} - S^{(-)})$, $S^{(\pm)}(x)$ being formal series solutions to

$$q_\hbar = \lambda^2(S^2 + S'), \qquad S(x) = \sum_{k=-1}^{\infty}\hbar^k S_k(x), \qquad S_{-1}^{(\pm)} = \pm\sqrt{q_0}. \qquad (2)$$

It is believed[6] that series (2) is **Borel-summable** away from **Stokes-lines**,

$$\mathrm{Im}(w(x)) = \mathrm{const.}, \qquad w(x) = e^{-\mathrm{i}\arg(\lambda)}\int^x dx'\,\sqrt{q(x')}$$

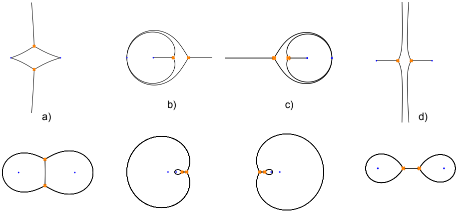
Voros symbols $V_\beta := \int_\beta dx\; S_{\rm odd}(x)$ can be Borel-summable, then representing ingredients of the solution to the BPS-RH-problem.

[6]Probably proven by Koike-Schäfke (unpublished), and by Nikolaev (to appear).

Borel summability depends on the topology of Stokes graph formed by Stokes lines (determined by $q \sim$ point on $\mathcal{B}$). Two "extreme" cases:

- FG Stokes graph $\leftrightsquigarrow$ triangulation of C
- FN Stokes graph $\leftrightsquigarrow$ pants decomposition

In between there exist several hybrid types of graphs.



Case FG: D. Allegretti has proven conjecture of T. Bridgeland: $V_\beta \rightsquigarrow$ Fock-Goncharov (FG) $(x_\imath, \check{x}^\imath)$ coordinates solving BPS-RH problem.

Case FN: Coordinates $(x_\imath, \check{x}^\imath)$ of Fenchel-Nielsen (FN) type

Extension to case FN needed for topological string applications:

Case FN: **Real**[7] "skeleton" in $\mathcal{B}$, described by FN-type Stokes graphs.

– Transitions from FG-type to FN-type: "Juggle" (Gaiotto-Moore-Neitzke).

[7] Real values of $\hbar$ and special coordinates $a^r_\imath$

Second half of our proposal:

There exists a canonical pair $(\mathcal{L}_\Theta, \nabla_\Theta)$ consisting of

$\mathcal{L}_\Theta$: line bundle on $\mathcal{Z}$, transition functions: Difference generating functions of changes of coordinates $x_\imath$

∇_Θ: connection on $\mathcal{L}_\Theta$, flat sections: Tau-functions $\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$, determining Z_{top} with the help of

$$\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \sum_{\mathrm{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\,(n,\check{\mathrm{x}}^\imath)} Z^\imath_{\text{top}}(\mathrm{x}_\imath - \mathrm{n}).$$

This means that there are wall-crossing relations

$$\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = F_{\imath\jmath}(\mathrm{x}_\imath, \mathrm{x}_\jmath)\mathcal{T}_\jmath(\mathrm{x}_\jmath, \check{\mathrm{x}}^\jmath),$$

on overlaps $\mathcal{U}_\imath \cap \mathcal{U}_\jmath$ of charts, with transition functions $F_{\imath\jmath}(\mathrm{x}_\imath, \mathrm{x}_\jmath)$: difference generating functions, defined by the changes of coordinates $x_\imath = x_\imath(x_\jmath)$.

Difference generating functions:

$$\mathcal{T}(\mathsf{x},\check{\mathsf{x}}) = \sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}(\mathsf{n},\check{\mathsf{x}})} Z(\mathsf{x}-\mathsf{n}) \;\Leftrightarrow\; \begin{cases} \mathcal{T}(\mathsf{x},\check{\mathsf{x}}+\delta_r) = \mathcal{T}(\mathsf{x},\check{\mathsf{x}}) \\ \mathcal{T}(\mathsf{x}+\delta_r,\check{\mathsf{x}}) = e^{2\pi\mathrm{i}\,\check{\mathsf{x}}_r}\mathcal{T}(\mathsf{x},\check{\mathsf{x}}) \end{cases} \qquad (3)$$

Coordinates considered here are such that $\mathsf{x}_\imath = \mathsf{x}_\imath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath)$ can be solved for $\check{\mathsf{x}}^\jmath$ in $\mathcal{U}_\imath \cap \mathcal{U}_\jmath$, defining $\check{\mathsf{x}}^\jmath(\mathsf{x}_\imath,\mathsf{x}_\jmath)$. Having defined tau-functions $\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath)$ and $\mathcal{T}_\jmath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath)$ on charts $\mathcal{U}_\imath$ and $\mathcal{U}_\jmath$, respectively, there is a relation of the form

$$\mathcal{T}_\imath(\mathsf{x}_\imath,\check{\mathsf{x}}^\imath) = F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)\mathcal{T}_\jmath(\mathsf{x}_\jmath,\check{\mathsf{x}}^\jmath),$$

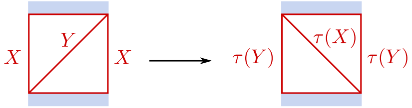
on the overlaps $\mathcal{U}_{\imath\jmath} = \mathcal{U}_\imath \cap \mathcal{U}_\jmath$. To ensure that both $\mathcal{T}_\imath$ and $\mathcal{T}_\jmath$ satisfy the relations (3), $F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)$ must satisfy

$$F_{\imath\jmath}(\mathsf{x}_\imath+\delta_r,\mathsf{x}_\jmath) = e^{+2\pi\mathrm{i}\,\check{\mathsf{x}}^\imath_r} F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath), \qquad (4a)$$

$$F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath+\delta_r) = e^{-2\pi\mathrm{i}\,\check{\mathsf{x}}^\jmath_r} F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath). \qquad (4b)$$

We will call functions $F_{\imath\jmath}(\mathsf{x}_\imath,\mathsf{x}_\jmath)$ satisfying the relations (4) associated to a change of coordinates $x_\imath = x_\imath(x_\jmath)$ difference generating functions.

Basic example:



$$X' = \tau(X) = Y^{-1}, \qquad (5)$$

$$Y' = \tau(Y) = X(1 + Y^{-1})^{-2}.$$

Introduce logarithmic variables x, y, x', y',

$$X = e^{2\pi \mathrm{i}\, x}, \qquad Y = -e^{2\pi \mathrm{i}\, y}, \qquad X' = -e^{2\pi \mathrm{i}\, x'}, \qquad Y' = e^{2\pi \mathrm{i}\, y'}.$$

The equations (5) can be solved for Y and Y',

$$Y(x, x') = -e^{-2\pi \mathrm{i}\, x'}, \qquad Y'(x, y) = e^{2\pi \mathrm{i}\, x}(1 - e^{2\pi \mathrm{i}\, x'})^{-2}.$$

The difference generating function $\mathcal{J}(x, x')$ associated to (5) satisfies

$$\frac{\mathcal{J}(x+1, x')}{\mathcal{J}(x, y)} = -(Y(x, x'))^{-1}, \qquad \frac{\mathcal{J}(x, x'+1)}{\mathcal{J}(x, y)} = Y'(x, x').$$

A function satisfying these properties is

$$\mathcal{J}(x, x') = e^{2\pi \mathrm{i} x x'}(E(x'))^2, \qquad E(z) = (2\pi)^{-z} e^{-\frac{\pi \mathrm{i}}{2} z^2} \frac{G(1+z)}{G(1-z)},$$

where $G(z)$ is the Barnes G-function satisfying $G(z+1) = \Gamma(z)G(z)$.

Tau-functions as solutions to the secondary RH problem

In arXiv:2004.04585 and work in progress we explain how to define solutions $\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ to the secondary RH problem by combining

free fermion CFT with exact WKB.

Key features:

- Proposal covers real slice in $\mathcal{B}$ represented by Jenkins-Strebel differentials using FN type coordinates,
- agrees with topological vertex calculations on the real slice, whenever available,
- and defines canonical extensions into strong coupling regions[8] (for $C = C_{0,2}$ using important work of Its-Lisovyy-Tykhyy).

Exact WKB for quantum curves fixes normalisation ambiguities $\Rightarrow$ the $\hbar$-deformation is "as canonical as possible".

[8] In the sense of Seiberg-Witten theory

Second part

Summary of first part: For local CY

- $\mathcal{M}$: Canonical torus fibration over space $\mathcal{B}$ of complex structures of Σ $\sim$ intermediate Jacobian fibration.
- Quantize classical curve $\Sigma \subset T^*C$ defined by (C, q), $y^2 = q(x)$.
- $\mathcal{M}$ deformed into twistor space $\mathcal{Z} \simeq \mathcal{M} \times \mathbb{C}^*$, for fixed value of $\hbar$ ($\sim$ coordinate on $\mathbb{C}^*$) – monodromy data for quantum curves.
- Exact WKB defines canonical atlas
  - patches $\sim$ types of Stokes graphs/spectral networks defined by (C, q),
  - Holomorphic Darboux coordinates $(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ on $\mathcal{Z}$ (FG, FN or hybrid type)

Tau-functions/free fermion partition functions $\mathcal{T}_\imath$:

Canonical sections of a canonical line bundle $\mathcal{L}_\Theta$ over $\mathcal{Z}$.

- FN-type networks $\rightsquigarrow$ Factorisation of $C \simeq C_2 \sqcup_A C_1$

$$\rightsquigarrow \text{factorisation of } \mathcal{T}_\imath : \qquad \mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \langle \mathfrak{f}_{C_2}, \mathfrak{f}_{C_1} \rangle$$

  $\rightsquigarrow$ $\mathcal{T}_\imath$: free fermion partition function (Fredholm determinant)
- Transition functions: Difference generating functions determined by changes of coordinates $(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) \sim (\mathrm{x}_\jmath, \check{\mathrm{x}}^\jmath)$ $\rightsquigarrow$ extension to all of $\mathcal{B}$.

Tau-functions $\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath)$ related to topological string partition functions $Z^\imath_{\rm top}(\mathsf{x}_\imath)$ via

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = \sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n,\check{\mathsf{x}}^\imath)} Z^\imath_{\rm top}(\mathsf{x}_\imath - \mathsf{n}).$$

For $C = C_{0,4}$: perfect **match** with topological vertex results.

For $C = C_{g,n}$: new result/prediction!

Important features

- Line bundle $\mathcal{L}_\Theta$ defined by **difference generating functions** describing **cluster type** transition functions,
- determined by **spectrum of BPS-states** (DT-invariants)

$\Rightarrow$ Relation to **geometry of space of stability conditions** (Program of T. Bridgeland)

The picture found in the class Σ examples suggests:

The higher genus corrections in the topological string theory on X are encoded in a canonical $\hbar$-deformation of the moduli space $\mathcal{M}_{\text{cplx}}(Y)$ of complex structures on the mirror Y of X.

There are hints that this picture may generalise beyond the class Σ examples:

(A) Relations to geometry of hypermultiplet moduli spaces

(B) Relations to quantisation of moduli spaces of complex structures

– see below

Further relations worth discussing/investigating

(1) Interplay between 2d-4d wall-crossing and free fermion picture

(2) Uplift to 5d, relations to spectral determinants (Marino et.al.)?

(A) Relation to geometry to hypermultiplet moduli spaces

A similar characterisation of $Z_{\rm top}$ follows from the proposal of Alexandrov, Persson, and Pioline (APP) for NS5-brane corrections to the geometry of hypermultiplet moduli spaces:

- SUSY $\rightsquigarrow$ describe quantum corrections using twistor space geometry,

$$\text{locally} \quad \mathcal{Z} \simeq \mathcal{M} \times \mathbb{P}^1,$$

  having atlas of Darboux coordinates $x_\imath = (\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ on $\mathcal{Z}$.

- Combining mirror symmetry, S-duality, and twistor space geometry $\Rightarrow$ quantum correction from one NS5-brane encoded in locally defined holomorphic functions $H_{\rm NS5}(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath)$ having representation of the form

$$H_{\rm NS5}(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) = \sum_{\mathrm{n}\in\mathbb{Z}^d} e^{2\pi \mathrm{i}\,(n,\check{\mathrm{x}}^\imath)} K^\imath_{\rm NS5}(\mathrm{x}_\imath - \mathrm{n}).$$

- Using the DT-GW-relation (MNOP): $K^\imath_{\rm NS5}(\mathrm{x}_\imath) \sim Z^\imath_{\rm top}(\mathrm{x}_\imath)$.

This suggests:
- Our results $\rightsquigarrow$ confirmation of APP-proposal,
- APP-framework predicts generalisations of our results.

(B) Relations classical-quantum:

(B.1) Quantization of moduli of complex structures

There are several conjectures/hints[9] that higher genus corrections in top. string theory can be described in terms of a non-commutative deformation of the geometric structures of $\mathcal{B}$, the moduli space of complex structures on a CY Y, or rather $\mathcal{M}$, the intermediate Jacobian fibration over $\mathcal{B} = \mathcal{M}_{\text{cplx}}(Y)$.

A common feature is an interpretation of $Z^{\imath}_{\text{top}}(\mathsf{x}_{\imath})$ as a **wave-function**.

[9] (Aganagic-Dijkgraaf-Vafa and collaborators; many others)

Magical relation, case $C = C_{0,4}$ for simplicity:

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = \sum_{\mathsf{n}\in\mathbb{Z}} e^{2\pi \mathrm{i}\, n\, \check{\mathsf{x}}^\imath} Z^\imath_{\mathrm{top}}(\mathsf{x}_\imath - \mathsf{n}) \tag{6}$$

expresses duality between I-brane (left) and usual topological string (right).

Invert (6):

$$Z^\imath_{\mathrm{top}}(\mathsf{x}_\imath) = \int_{S^1} d\check{\mathsf{x}}^\imath\ \mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath). \tag{7}$$

$\Rightarrow$ gluing relations

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = F_{\imath\jmath}(\mathsf{x}_\imath, \mathsf{x}_\jmath)\mathcal{T}_\jmath(\mathsf{x}_\jmath, \check{\mathsf{x}}^\jmath)$$

translate[10] into integral transformations

$$Z^\imath_{\mathrm{top}}(\mathsf{x}_\imath) = \int d\mathsf{x}_\jmath\ K(\mathsf{x}_\imath, \mathsf{x}_\jmath) Z^\jmath_{\mathrm{top}}(\mathsf{x}_\jmath).$$

[10] Iorgov-Lisovyy-Tykhyy, Alexandrov-Pioline; J.T., in preparation

This looks like the changes of representation in a **quantum theory** obtained from quantisation of $\mathcal{M}$.

We know this quantum theory: Variant of quantum Teichmüller / complex Chern-Simons theory:

Non-commutative deformation $\mathcal{M}_q$ of $\mathcal{M}(C_{0,4})$, the $SL(2)$-character variety.

Generators $\mathcal{L}_i$, $\begin{cases} \text{corresponding to the trace functions } \mathrm{tr}(\mathrm{Hol}_{\gamma_i}(\nabla_\hbar)) \text{ associated to} \\ \text{the curves around } (z_1, z_2),\ (z_1, z_3),\ (z_2, z_3), \text{ for } i = s, t, u, \text{ respectively.} \end{cases}$

Relations:

$$\begin{cases} q\vartheta_s\vartheta_t - q^{-1}\vartheta_t\vartheta_s = (q^2 - q^{-2})\vartheta_u + (q - q^{-1})R_u, \\ q\vartheta_t\vartheta_u - q^{-1}\vartheta_u\vartheta_t = (q^2 - q^{-2})\vartheta_s + (q - q^{-1})R_s, \\ q\vartheta_u\vartheta_s - q^{-1}\vartheta_s\vartheta_u = (q^2 - q^{-2})\vartheta_t + (q - q^{-1})R_t, \\ \vartheta_s\vartheta_t\vartheta_u + (q + q^{-1})^2 = \\ \qquad = q^2\vartheta_s^2 + q^{-2}\vartheta_t^2 + q^2\vartheta_u^2 + qR_s\vartheta_s + q^{-1}R_t\vartheta_t + qR_u\vartheta_u + R_{stu}. \end{cases}$$

To classical Darboux charts correspond quantum representations, related by the quantum cluster transformations represented by the integral transformations defined above.

$\Rightarrow$ There indeed exists a quantisation of $\mathcal{M}$ such that $Z^{\imath}_{\mathrm{top}}$: wave-functions essentially determined by canonical Darboux coordinates $(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath)$.

Claim: The magic formula

$$\mathcal{T}_\imath(\mathsf{x}_\imath, \check{\mathsf{x}}^\imath) = \sum_{\mathsf{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\,(n,\check{\mathsf{x}}^\imath)} Z^{\imath}_{\mathrm{top}}(\mathsf{x}_\imath - \mathsf{n}) \quad \text{relates} \tag{8}$$

(i) non-commutative deformation of $\mathcal{M}$ from the previous slide to

(ii) the $\hbar$-deformation of $\mathcal{M}$ discussed in this talk.

(Main observation from Iorgov-Lisovyy-J.T. : Transform (8) diagonalises the realisations of the quantised algebras of functions on $\mathcal{M}(C)$ at $q = -1$.)

In this sense:

> **The higher genus corrections in the topological string theory on X are encoded in a canonical quantum deformation of the moduli space $\mathcal{M}_{\text{cplx}}(Y)$ of complex structures on the mirror Y of X.**

Furthermore:

> **Topological string partition functions Z_{top}: local sections of an infinite-dimensional vector bundle over $\mathcal{B}$, with transition functions being the quantized changes of coordinates between canonical local charts.**

Probably also related to:

Recent math work[11] $\Rightarrow$ Higher genus corrections (GW invariants) deform mirror of the cubic surface $\sim \mathcal{M}_{\mathrm{Hit}}(C_{0,4})$ into $\mathcal{M}_{q}$.

[11] P. Bousseau, arXiv:2009.02266, based on Gross-Hacking-Keel-Siebert, arXiv:1910.08427

B.2 Relation to geometric quantisation of intermediate Jacobian?

Let us use the isomonodromic tau-functions to define $\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar)$,

$$\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar) := \mathcal{T}\big(\sigma(\mathrm{a},\theta;\hbar)\,,\,\tau(\mathrm{a},\theta;\hbar)\,;\,z\,;\,\hbar\big), \qquad (9)$$

when $d=1$, $\sigma\equiv x_\imath^1$, $\eta\equiv\check{x}_1^\imath$, $\theta=\theta_1^\imath$.

Claim

The limit

$$\log\Theta_\Sigma(\mathrm{a},\theta;z) := \lim_{\hbar\to 0}\Big[\log\Theta_{\Sigma_\hbar}(\mathrm{a},\theta;z;\hbar) - \log\mathcal{Z}_{\mathrm{top}}(\sigma(\mathrm{a},\theta);z;\hbar)\Big] \qquad (10)$$

exists, with function $\Theta_\Sigma(\mathrm{a},\theta;z)$ defined in (10) being the theta function

$$\Theta_\Sigma(\mathrm{a};\theta;z) = \sum_{n\in\mathbb{Z}} e^{2\pi\mathrm{i}n\theta}\, e^{\pi\mathrm{i}n^2\tau_\Sigma(\mathrm{a})}, \qquad (11)$$

with $\tau_\Sigma(\mathrm{a})$ related to $\mathcal{F}(\mathrm{a},z)$ by $\tau_\Sigma = \dfrac{1}{2\pi\mathrm{i}}\dfrac{\partial^2\mathcal{F}}{\partial\mathrm{a}^2}$.

Relation to quantisation of intermediate Jacobian fibers (Witten, several others)?

(1) Interplay between 2d-4d wall-crossing and free fermion picture

Background Y_Σ can be modified to open-closed background by inserting Aganagic-Vafa branes located at points of Σ. Generalisation of the formula

$$\mathcal{T}_\imath(\mathrm{x}_\imath, \check{\mathrm{x}}^\imath) \equiv \langle\, \Omega\,,\, \mathfrak{f}_\Psi \,\rangle = \sum_{\mathrm{n}\in\mathbb{Z}^d} e^{2\pi\mathrm{i}\,(n,\check{\mathrm{x}}^\imath)} Z^\imath_{\mathrm{top}}(\mathrm{x}_\imath - \mathrm{n})$$

due to Iorgov-Lisovyy-J.T. will then relate free fermion expectation values

$$\Psi(x,y) = \langle\!\langle\, \bar{\psi}(x)\psi(y) \rangle\!\rangle = \frac{\langle \Omega, \bar{\psi}(x)\psi(y)\mathfrak{f}_\Psi \rangle}{\langle\, \Omega\,,\, \mathfrak{f}_\Psi \,\rangle},$$

to expectation values of degenerate fields of the Virasoro algebra, representing the fermions of Aganagic-Dijkgraaf-Klemm-Marino-Vafa in our context.

Noting that $\Psi(x,y)$ represents the solution to the classical RH-problem associated to the tau-function $\mathcal{T}_\imath = \langle\, \Omega\,,\, \mathfrak{f}_\Psi \rangle$ one sees that:

relation between classical RH-problem to BPS-RH problem:
Example for 4d-2d wall crossing (GMN).

Exact WKB fixes the normalisations for $\Psi(x,y)$, via 4d-2d wall crossing determining the normalisations of $\mathcal{T}_\imath$.