Refined holomorphic anomaly equation for local $\mathbb{P}^2$

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References

- Talk based on arXiv:2001.05347 "Holomorphic anomaly equation for $(\mathbb{P}^2, E)$ and the Nekrasov-Shatashvili limit of local $\mathbb{P}^2$", joint with Honglu Fan, Shuai Guo, and Longting Wu.
- Relies on some of my previous works:
  - arXiv:1706.07762 "Tropical refined curve counting from higher genera and lambda classes"
  - arXiv:1806.11495 "The quantum tropical vertex"
  - arXiv:1909.02985 "Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$"
  - arXiv:1909.02992 "A proof of N.Takahashi's conjecture for $(\mathbb{P}^2, E)$ and a refined sheaves/Gromov-Witten correspondence"
- Closely related:
  - arXiv:1808.07336 "Quantum mirrors of log Calabi-Yau surfaces and higher genus curve counting"
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Overview

- Refined topological string on non-compact Calabi-Yau 3-folds is expected to satisfy the refined holomorphic anomaly equation.
- The case of the ordinary/unrefined topological string starts to be well-understood mathematically.
- Our result: first mathematical result going non-trivially in the "refined direction".
- Focus on local $\mathbb{P}^2$: total space of the canonical line bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$ of $\mathbb{P}^2$, and the Nekrasov-Shatashvili limit of the refined topological string.
- Purely mathematical result: construction of quasimodular forms from Betti numbers of moduli spaces of one-dimensional coherent sheaves on $\mathbb{P}^2$.

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Plan

- Review: topological string and holomorphic anomaly equation.
- Review: refined topological string and refined holomorphic anomaly equation.
- Statement of the main result: mathematical derivation of the refined holomorphic anomaly equation for local $\mathbb{P}^2$ in the Nekrasov-Shatashvili limit.
- Main idea: reformulation of the NS limit of the refined (closed) topological string as an open topological string on a dual geometry.
- Proof of this reformulation: scattering diagram form of the Kontsevich-Soibelman wall-crossing formula and tropical geometry. Need to consider the full D4-D2-D0 BPS spectrum.
- Derivation of the holomorphic anomaly equation on the open string side by some degeneration from the (unrefined) closed topological string on local $\mathbb{P}^2$. Corollary: a new formula relating the unrefined topological string and the NS limit of the refined topological string (proved for local $\mathbb{P}^2$, conjectured to hold at least for all local del Pezzo surfaces).

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Physics overview

- Fix X a Calabi-Yau 3-fold.
- From coupling of appropriate twists of the two-dimensional supersymmetric sigma model of target X with two-dimensional topological gravity, define the A/B models topological string on X. [Witten, 1988-1990]
- Genus g amplitudes are obtained by integration over the moduli space of genus g Riemann surfaces.
- Moduli space M of parameters, complex manifold. For the B-model, M is the moduli space of complex structures on X. For the A-model, M is the "stringy complexified Kähler moduli space of X".
- Genus g amplitudes of the topological string are functions $F_g(t, \bar{t})$ on M, where t and $\bar{t}$ are holomorphic and antiholomorphic coordinates on M.

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Holomorphic anomaly equation

- Naive arguments suggest that $F_g(t, \bar{t})$ is an holomorphic function on M (independent of $\bar{t}$).
- It is not the case: subtelty comes from the fact that the moduli space of genus g Riemann surfaces is not compact, need to add (stable) nodal curves to get a compact moduli space. The holomorphic anomaly equation expresses the fact that the defect of holomorpy comes entirely from there. Schematically (without details of special geometry)

$$\bar{\partial} F_g = \frac{1}{2} \sum_{i=1}^{g-1} (DF_i)(DF_{g-i}) + \frac{1}{2} D^2 F_{g-1}$$

where D is a first order differential operator and D^2 is a second order differential operator. [Bershadsky-Cecotti-Ooguri-Vafa, 1993]

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Holomorphic anomaly equation

- The holomorphic anomaly equation is a very powerful tool to solve the topological string: it recursively computes F_g up to ambiguities which are holomorphic functions on $\mathcal{M}$. These holomorphic ambiguities can sometimes be fixed simply from their boundary behavior.
- The A-model of the quintic 3-fold can be solved for $g \geq 51$ using the holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap, Castelnuovo bound). [Huang-Klemm-Quackenbush, 2006]
- The A-model of some non-compact Calabi-Yau 3-folds, such as local $\mathbb{P}^2$, is uniquely determined by the holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap). [Aganagic- Bouchard-Klemm, 2006][Haghighat-Klemm-Rauch, 2008]

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Focus on the A-model.

- There does not exist yet a general definition mathematical definition of the "stringy complexified Kähler moduli space". When we know that X is mirror to Y, it is the moduli space of complex structures on Y.
- In particular, there does not exist yet a general mathematical definition of the genus g amplitudes $F_g(t, \bar{t})$.
- What exists mathematically is the power expansion around the large volume point $t = +\infty$ of the holomorphic limit $F_g(t) := \lim_{\bar{t} \to 0} F_g(t, \bar{t})$ where t is a linear coordinate on $H^2(X, \mathbb{C})$: up to known perturbative terms,

$$F_g(t) = \sum_{\beta \in H_2(X, \mathbb{Z})} N_{g,\beta} e^{-\int_\beta t}$$

where $N_{g,\beta}$ is the "number of holomorphic curves in X of genus g and class β", more precisely Gromov-Witten invariants.

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- The physics prediction of the existence of $F_g(t, \bar{t})$ implies that the mathematically defined series $F_g(t)$ should have non-zero radius of convergence and analytic continuation to the universal cover of $\mathcal{M}$, with quasi-modular-like transformations with respect to the action of the fundamental group.
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Topological string: mathematical results

Theorem (Conjecture of (Aganagic-Bouchard-Klemm, 2006), proofs: (Fang-Liu-Zong, 2016), (Lho-Pandharipande, 2017), (Coates-Iritani, 2018))

For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, the generating series $F_g(t)$ of Gromov-Witten invariants are (after the mirror map change of variables) quasi-modular forms for the group $\Gamma_1(3)$ and satisfy the holomorphic anomaly equation.

The topological vertex ([Aganagic-Klemm-Marino-Vafa, 2003], [Li-Liu-Liu-Zhou, 2004]) gives a way to compute A-model/Gromov-Witten invariants of toric Calabi-Yau 3-folds. However, it is not known how to deduce quasimodularity and holomorphic anomaly equation from the topological vertex.

Topological string from M-theory

The genus g A-model amplitudes of a Calabi-Yau 3-fold compute F-terms in the low-energy effective action of Type IIA string theory on $\mathbb{R}^{1,3} \times X$. Using that the strong coupling limit of Type IIA string theory is M-theory, one obtains a M-theory description of the topological string [Gopakumar-Vafa, 1998]

$$\sum_{g \geq 0} F_g(t) g_s^{2g-2} = - \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k \geq 1} \frac{1}{k} \frac{Tr_{\mathcal{H}_\beta}[(-1)^{2(J_L+J_R)} q^{2kJ_L}]}{(q^{\frac{k}{2}} - q^{-\frac{k}{2}})^2} e^{-k \int_\beta t}$$

where $\mathcal{H}_\beta$ is the spaces of BPS states of M2 branes of charge β, representation of the 5d little group $SO(4) \sim SU(2)_L \times SU(2)_R$, and $q = e^{ig_s}$.

Refined topological string

Let X be a Calabi-Yau 3-fold with a $U(1)_T$ symmetry scaling non-trivially the holomorphic volume form, e.g. $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ with $U(1)_T$ rotating the fibers of the natural projection $\mathcal{O}_{\mathbb{P}^2}(-3) \to \mathbb{P}^2$. Define the refined topological string amplitudes $F_{g_1,g_2}(t)$ by

$$\sum_{g_1,g_2 \geq 0} F_{g_1,g_2}(t)(\epsilon_1+\epsilon_2)^{2g_1}(-\epsilon_1\epsilon_2)^{g_2-1}$$

$$:= \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k \geq 1} \frac{1}{k} \frac{Tr_{\mathcal{H}_\beta}((-1)^{2(J_L+J_R)} q_L^{2kJ_L} q_R^{2k(J_R+I_T)})}{(q_1^{\frac{k}{2}} - q_1^{-\frac{k}{2}})(q_2^{\frac{k}{2}} - q_2^{-\frac{k}{2}})} e^{-k\int_\beta t}$$

where

$$q_1 = e^{\epsilon_1}, q_2 = e^{\epsilon_2}, q_L = e^{\epsilon_L}, q_R = e^{\epsilon_R},$$

$$\epsilon_R = \frac{\epsilon_1+\epsilon_2}{2}, \epsilon_L = \frac{\epsilon_1-\epsilon_2}{2}.$$

[Nekrasov, 2002][Hollowood-Iqbal-Vafa, 2003]

Refined topological string

- Unrefined limit: $\epsilon_1 = -\epsilon_2 = ig_s$, so $\epsilon_L = ig_s, \epsilon_R = 0, q_L = e^{ig_s}, q_R = 1$, and so $F_{0,g}(t) = F_g(t)$.
- From the M-theory definition, one expects $F_{g_1,g_2}(t)$, a priori a formal power series in $Q = e^{-t}$, to be convergent and to come from a globally defined function $F_{g_1,g_2}(t, \bar{t})$ on the space M of parameters.
- No obvious worldsheet definition of $F_{g_1,g_2}(t, \bar{t})$, so no BCOV derivation for an holomorphic anomaly equation.
- Nevertheless, a refined holomorphic anomaly equation was guessed [Krefl-Walcher, 2010][Huang-Klemm, 2010]

$$\bar{\partial} F_{g_1,g_2} = \frac{1}{2} \sum_{\substack{0 \leq j_1 \leq g_1 \\ 0 \leq j_2 \leq g_2 \\ (j_1,j_2) \neq (0,0) \\ (j_1,j_2) \neq (g_1,g_2)}}^{g-1} (DF_{j_1,j_2}) \left(DF_{(g_1-j_1,g_2-j_2)} \right) + \frac{1}{2} D^2 F_{g_1,g_2-1} \, .$$

Refined topological string

- Unrefined limit: $\epsilon_1 = -\epsilon_2 = ig_s$, so $\epsilon_L = ig_s, \epsilon_R = 0, q_L = e^{ig_s}, q_R = 1$, and so $F_{0,g}(t) = F_g(t)$.
- From the M-theory definition, one expects $F_{g_1,g_2}(t)$, a priori a formal power series in $Q = e^{-t}$, to be convergent and to come from a globally defined function $F_{g_1,g_2}(t,\bar{t})$ on the space M of parameters.
- No obvious worldsheet definition of $F_{g_1,g_2}(t,\bar{t})$, so no BCOV derivation for an holomorphic anomaly equation.
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Refined topological string

- The refined topological string of some non-compact Calabi-Yau 3-folds, such as local $\mathbb{P}^2$, is uniquely determined by the refined holomorphic anomaly equation and known/expected boundary conditions (large volume leading behavior, orbifold regularity, conifold gap). [Huang-Klemm, 2010]
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The spaces of $M2$-branes $\mathcal{H}_\beta$ can be defined as cohomology of moduli spaces of $D2 - D0$ branes, i.e. moduli spaces of coherent sheaves on X supported on curves, and the $SU(2)_L \times SU(2)_R$ action can be understood geometrically ([Gopakumar-Vafa, 1998], subtleties coming from the fact that these moduli spaces are singular [Hosono-Saito-Takahashi, 2001],[Kiem-Li, 2016],[Maulik-Toda, 2016]).

Refined topological string: mathematics

$$\sum_{g_1,g_2\geq 0} F_{g_1,g_2}(t)(\epsilon_1+\epsilon_2)^{2g_1}(-\epsilon_1\epsilon_2)^{g_2-1}$$

$$:= \sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{k\geq 1}\frac{1}{k}\frac{Tr_{\mathcal{H}_\beta}((-1)^{2(J_L+J_R)}q_L^{2kJ_L}q_R^{2k(J_R+I_T)})}{(q_1^{\frac{k}{2}}-q_1^{-\frac{k}{2}})(q_2^{\frac{k}{2}}-q_2^{-\frac{k}{2}})}e^{-k\int_\beta t}$$

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Main difficulty: the refined holomorphic anomaly equation is a recursion on (g_1,g_2), but $F_{g_1,g_2}(t)$ is defined through the non-trivial change of variables $q_1=e^{\epsilon_1}$ and $q_2=e^{\epsilon_2}$. Without some geometric/worldsheet-like interpretation of the parameters (g_1,g_2), it seems very difficult to prove anything. It is not known how to deduce quasimodularity and holomophic anomaly from the refined topological vertex for example.

The Nekrasov-Shatashvili limit

- NS limit: $\epsilon_1 = \hbar$, $\epsilon_2 = 0$, and so $\epsilon_L = \epsilon_R = \frac{\hbar}{2}$.
- Denote $F_g^{NS}(t) := F_{g,0}(t)$ and $y = e^{\hbar}$

$$\sum_{g \geq 0} F_g^{NS}(t) \hbar^{2g-1} = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_\beta(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-k \int_\beta t}$$

where

$$\Omega_\beta(y) := \mathrm{Tr}_{\mathcal{H}_\beta}((-1)^{2(J_L+J_R)} y^{J_L+J_R+I_T}) \, .$$

Refined 4d BPS index for D2-D0 branes ($J_L + J_R$ is the 4d diagonal $SU(2)$ in the 5d $SU(2)_L \times SU(2)_R$), (roughly) the Hirzebruch χ_y-genus of the moduli space of D2-D0 branes.

- Refined holomorphic equation in the NS limit (note: no loop term)

$$\overline{\partial} F_g^{NS} = \frac{1}{2} \sum_{i=1}^{g-1} \left(D F_i^{NS} \right) \left(D F_{g-i}^{NS} \right) \, .$$

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Main result

Theorem (Conjecture of (Huang-Klemm, 2010), proof: (B-Fan-Guo-Wu, 2020))

For $X = \mathcal{O}_{\mathbb{P}^2}(-3)$, the series $F_g^{NS}(t)$, defined in terms of Betti numbers of one-dimensional coherent sheaves on $\mathbb{P}^2$), are (after the mirror map change of variables) quasi-modular forms for the group $\Gamma_1(3)$ and satisfy the NS limit of the refined holomorphic anomaly equation.

Precise definitions

$X = \mathcal{O}_{\mathbb{P}^2}(-3)$, $H_2(X,\mathbb{Z}) = H_2(\mathbb{P}^2,\mathbb{Z}) = \mathbb{Z}$

$$\sum_{g \geq 0} F_g^{NS}(t)\hbar^{2g-1} = \sum_{d \geq 1}\sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

- Precise mathematical definition of the Laurent polynomials $\Omega_d(y)$?
- Moduli space $M_{d,\chi}$ of D2-D0 branes: (coarse) moduli space of semistable coherent sheaves F on $\mathbb{P}^2$ supported on curves of degree d and with $\chi(F) = \chi$. Very natural in algebraic geometry (Abel-Jacobi in family)[Simpson, 1990], [Le Potier, 1993].
- Define $\Omega_{d,\chi}(y)$ as the Poincaré polynomial of Betti numbers of $M_{d,\chi}$ for the intersection cohomology (physics: think about L^2-cohomology of the stable (smooth) locus. $M_{d,\chi}$ is smooth if $\gcd(d,\chi) = 1$, singular in general.

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Precise definitions

Conjecture

$\Omega_{d,\chi}$ only depends on d and not on χ.

We have $M_{d,\chi} \simeq M_{d,\chi+d}$ ($F \mapsto F \otimes \mathcal{O}_{\mathbb{P}^2}(1)$) and $M_{d,\chi} \simeq M_{d,-\chi}$ ($F \mapsto F^\vee$) but $M_{d,\chi} \neq M_{d,\chi'}$ if $d \geq 3$ and $\chi \neq \pm\chi' \mod d$.

Theorem (B, 2019)

- *$\Omega_{d,\chi}$ only depends on d and on $\gcd(d,\chi)$.*
- *The full conjecture is true for $d \leq 4$.*

Proof using Gromov-Witten theory of a basic result on topology of moduli space of sheaves.
Without knowing the general conjecture, we define $\Omega_d(y)$ by average:

$$\Omega_d(y) := \frac{1}{d} \sum_{\chi \mod d} \Omega_{d,\chi}(y)\,.$$

Proof strategy

$y = e^{\hbar}$

$$\sum_{g \geq 0} F_g^{NS}(t)\hbar^{2g-1} = \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

Theorem (B-Fan-Guo-Wu, 2020)

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We would like a genus g worldsheet definition of $F_g^{NS}(t)$. This cannot be the unrefined topological string on $\mathcal{O}_{\mathbb{P}^2}(-3)$: $F_g^{NS}(t) \neq F_g(t)$ for $g \geq 1$.

Hint

$y = e^{\hbar}$

$$\sum_{g \geq 0} F_g^{NS}(t) \hbar^{2g-1} = \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k^2} \frac{\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

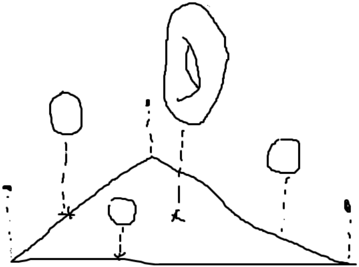
Take the derivative d/dt

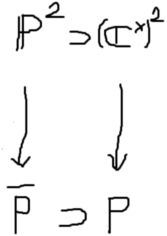
$$\sum_{g \geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = - \sum_{d \geq 1} \sum_{k \geq 1} \frac{1}{k} \frac{d\Omega_d(y^k)}{y^{\frac{k}{2}} - y^{-\frac{k}{2}}} e^{-kdt}$$

The derivative of the NS limit of the refined topological string looks exactly like an open topological string! [Ooguri-Vafa, 1999]
Main claim: for $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ (and more generally local del Pezzo sufaces), we can find such open topological string.

Towards the open string geometry

- $\mathbb{P}^2$: complex projective plane, complex dimension 2, real dimension 4. Standard symplectic structure, T^2-Hamiltonian action leading to a toric description.
- Moment map: : $\mathbb{P}^2 \to \bar{P}$. Over the interior P of $\bar{P}$, $(\mathbb{C}^*)^2 \to P$, Lagrangian T^2-fibration. $(\mathbb{C}^*)^2$: complement of a triangle of lines in $\mathbb{P}^2$.





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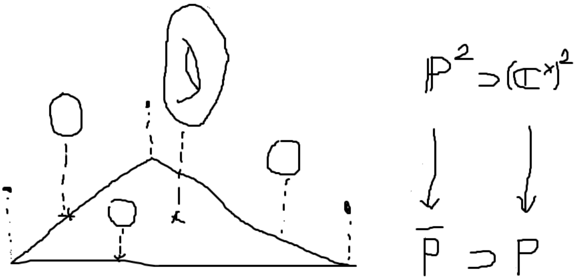

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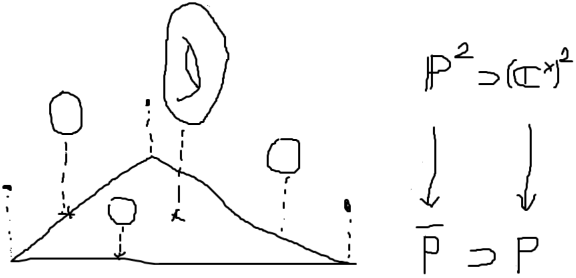
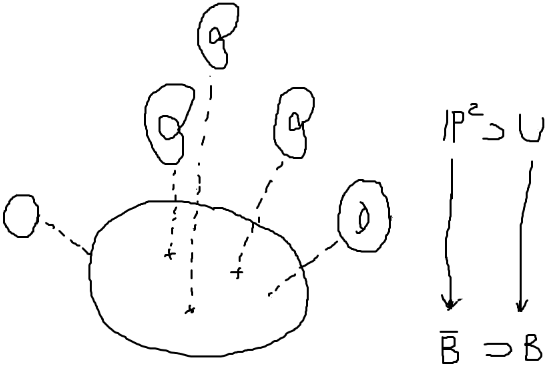


Towards the opens string geometry: $(\mathbb{P}^2, E)$

- Replace the triangle of lines by a smooth cubic E (genus one curve) and $(\mathbb{C}^*)^2$ by $U := \mathbb{P}^2 - E$. U is a non-compact Calabi-Yau surface.
- Over the interior B of $\bar{B}$, $U \to B$, Lagrangian T^2-fibration with 3 nodal singular fibers. Topological check: $\chi_{top}(U) = \chi_{top}(\mathbb{P}^2) - \chi_{top}(E) = 0$. In fact, there exists such special Lagrangian fibration. [Collins,Jacob,Lin, 2019]



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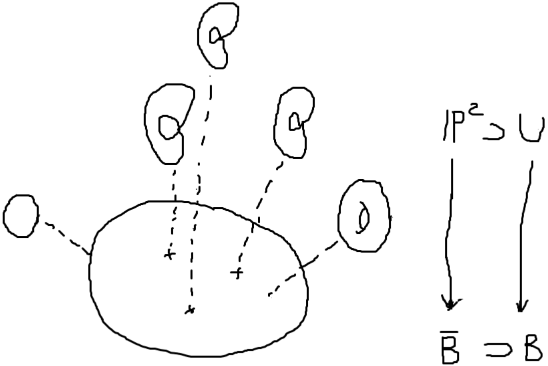

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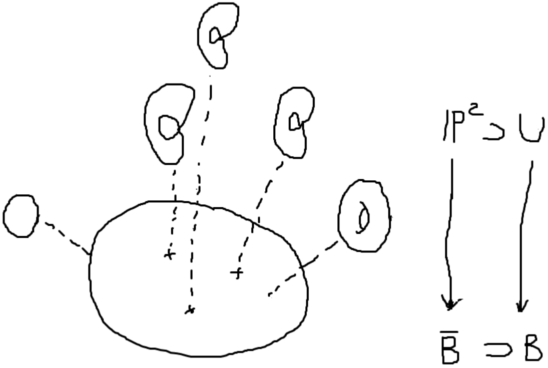
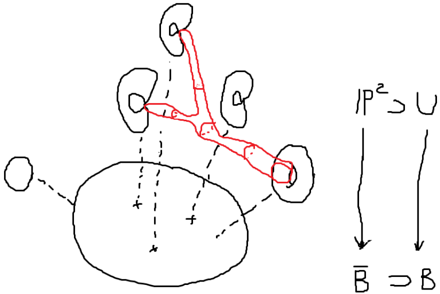


Open topological string: Holomorphic curves in $(\mathbb{P}^2, E)$

- Study holomorphic curves in U with one boundary on a Lagrangian T^2-fiber.
- In fact U is hyperkähler. Study the open topological string on the Calabi-Yau 3-fold Z obtained as the $\mathbb{C}^*$-twistor family of U, with brane $T_b^2 \times \mathbb{R}$: $N_{g,v}^{open}(b)$ "counts" of genus g Riemanns surfaces in Z with one boundary of class $v \in H_1(T_b^2 \times \mathbb{R}, \mathbb{Z}) \simeq \mathbb{Z}^2$.



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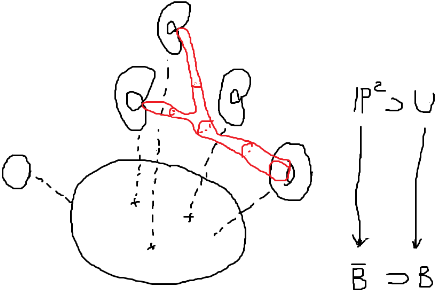

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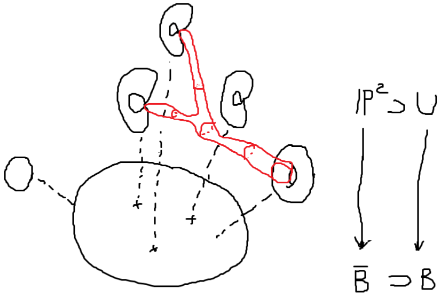
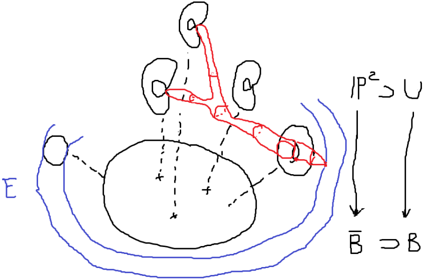


- The open invariants $N_{g,v}^{open}(b)$ jump as a function of b (wall-crossing) and are difficult to define rigorously (e.g. we are not in a toric situation).
- We provide a mathematically precise definition when b is close to E and $v \in H_1(T_b^2, \mathbb{Z})$ is a multiple of the cycle $(1,0)$ collapsing on E. For open curves wrapping a cycle collapsing on E, close them by gluing a disc: get a closed holomorphic curve in $\mathbb{P}^2$ meeting E in one point. Use relative Gromov-Witten theory to define $N_{g,(d,0)}^{open} \in \mathbb{Q}$.



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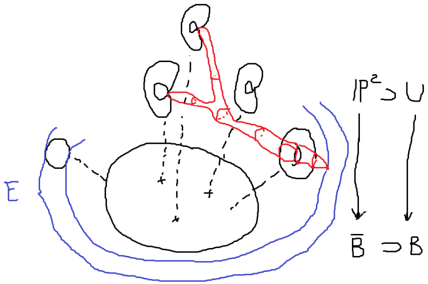

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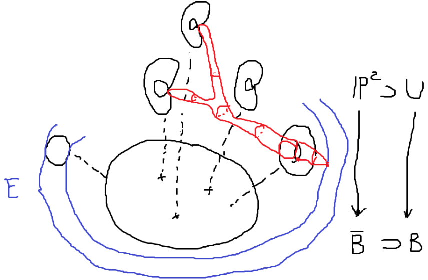
- We provide a mathematically precise definition when b is close to E and $v \in H_1(T_b^2, \mathbb{Z})$ is a multiple of the cycle $(1,0)$ collapsing on E. For open curves wrapping a cycle collapsing on E, close them by gluing a disc: get a closed holomorphic curve in $\mathbb{P}^2$ meeting E in one point. Use relative Gromov-Witten theory to define $N_{g,(d,0)}^{open} \in \mathbb{Q}$.



Open string interpretation of F^{NS}

Theorem (B, 2019)

$$\sum_{g\geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = -\frac{1}{3} \sum_{g\geq 0} \sum_{d\geq 1} (-1)^{d-1} N_{g,(d,0)}^{open} \hbar^{2g-1} e^{-dt}$$

In other words, the refined BPS counts $\Omega_d(y)$ of (closed) D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ are equal (up to simple factor) to Ooguri-Vafa open BPS counts on $Z = U \times \mathbb{C}^* = (\mathbb{P}^2 - E) \times \mathbb{C}^*$.

Open string interpretation of the BPS spectrum of $D4 - D2 - D0$ branes of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$

- $\Omega_d(y)$ is the refined BPS count of D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ in the large volume limit. More generally, consider $\Omega_{d,r}(y)$ the refined BPS count of D4-D2-D0 branes in the large volume limit, where r is the $D4$ charge, i.e. the rank of the coherent sheaf on $\mathbb{P}^2$.
- Assume that we have a definition of $N^{open}_{g,(d,r)}$ for open curves wrapping general cycles $(d,r) \in H_1(T^2_b, \mathbb{Z})$, not only the ones of class $(d,0)$ which collapse to E.

Almost theorem (for any reasonable mathematical definition of the open invariants)(B.2019)

The refined BPS counts $\Omega_{d,r}(y)$ of (closed) D4-D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ are equal (up to simple factor) to Ooguri-Vafa open BPS counts on $Z = U \times \mathbb{C}^* = (\mathbb{P}^2 - E) \times \mathbb{C}^*$ defined by the open invariants $N^{open}_{g,(d,r)}$.

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- It is NOT an example of MNOP-like GW-DT correspondence, but a new kind of coherent sheaves/open Gromov-Witten correspondence.
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Heuristic chain of geometric connections:

- Start with open holomorphic curves in U.
- U is hyperkähler: do some hyperkähler rotation. In some different complex structure, get V an elliptic fibration $W \colon V \to B$. Open holomorphic curves in U becomes open special Lagrangians.
- Suspension construction: S Calabi-Yau 3-fold $uv = W$, fibration in affine conics over V, degenerate over T^2. Open special Lagrangians in V lift to closed special Lagrangians in S.
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- Following the chain of dualities, the base B of the Lagrangian torus fibration on U becomes moduli space of complex moduli on S and so should become the stringy Kähler moduli space of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$.
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- Tropicalize the holomorphic curves in U to graphs on the base B of the T^2-fibration (g=0: Carl-Pumperla-Siebert, Prince, Gabele, $g > 0$: Bousseau), get a scattering diagram computing the Gromov-Witten invariants $N^{open}_{g,(d,r)}(b)$ ("4d spectral network").
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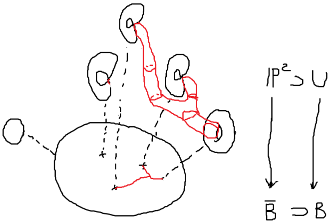

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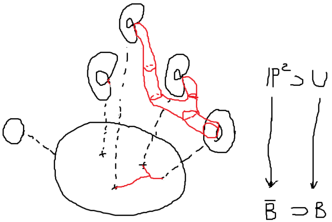

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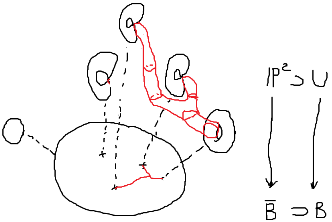
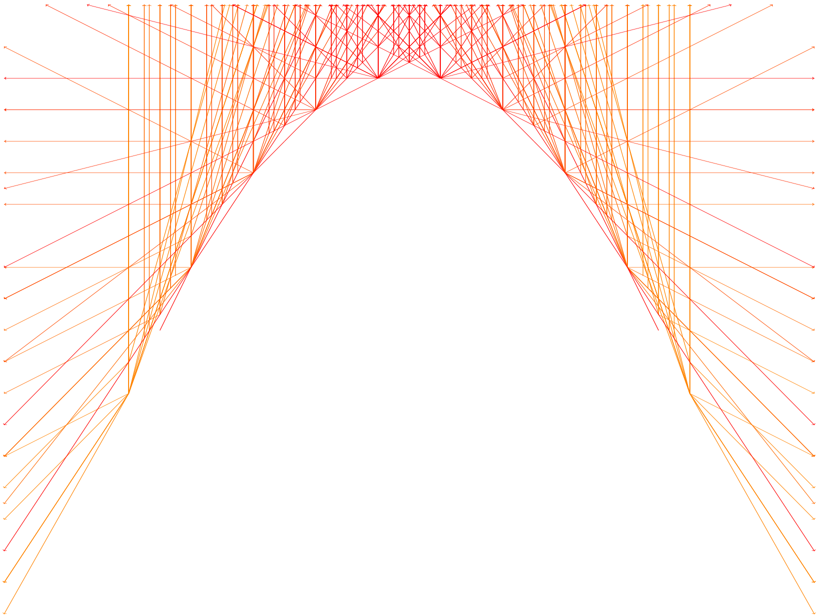


Scattering diagram



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- Main result (B. 2019): the scattering diagram can be embedded in the space of Bridgeland stability conditions on the derived category $D^b_c Coh(K_{\mathbb{P}^2})$, such that the rays correspond to stability conditions for which there exists stable objects of phase $\pi/2$. Use coordinates on the stringy Kähler moduli space given by the real part of the central charges. Similar to attractor flows in supergravity [Denef,2001][Denef, Moore,2007].
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Remarks

- (Prince) Triangles in the scattering diagrams are indexed by Markov triples (integer solutions of $x^2 + y^2 + z^2 = 3xyz$). Discs potential associated to Vianna's monotone tori can be extracted from the scattering diagram.
- (B.) Viewing the scattering diagrams as living in the space of stability conditions, the sides of the triangles correspond exactly to triples of exceptional objects in the derived category of coherent sheaves on $\mathbb{P}^2$.
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Theorem (B, 2019)

$$\sum_{g\geq 0} \frac{d}{dt} F_g^{NS} \hbar^{2g-1} = -\frac{1}{3} \sum_{g\geq 0} \sum_{d\geq 1} (-1)^{d-1} N_{g,(d,0)}^{open} \hbar^{2g-1} e^{-dt}$$

In other words, the refined BPS counts $\Omega_d(y)$ of (closed) D2-D0 branes on $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ are equal (up to simple factor) to Ooguri-Vafa open BPS counts on $Z = U \times \mathbb{C}^* = (\mathbb{P}^2 - E) \times \mathbb{C}^*$.

To prove the quasimodularity and refined holomorphic equation for F_g^{NS}, it remains to work on the open topological string side. There is some hope because the $\hbar$-expansion is now the geometric genus expansion of the topological string.

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Degeneration argument. Degeneration of $\mathbb{P}^2$ to the normal cone of E. Line bundle defined by the family of divisors E. General fiber: $X = \mathcal{O}_{\mathbb{P}^2}(-3) = \mathcal{O}(-E)$. Special fiber: $\mathbb{P}^2 \times \mathbb{A}^1$, glued along $E \times \mathbb{C}^1$ to a non-trivial line bundle over $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$.
- Localization on the bubble $\mathbb{P}(N_{E|\mathbb{P}^2} \oplus \mathcal{O})$: reduction to equivariant Gromov-Witten theory of $N_{E|\mathbb{P}^2} \oplus N^{\vee}_{E|\mathbb{P}^2} \to E$ with stationary descendent insertions.
- Use Grothendieck-Riemann-Roch (in Coates-Givental form) to reduce to Gromov-Witten theory of E with stationary descendent insertions.

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Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

- Upshot: formula computing open Gromov-Witten invariants $N_{g,(d,0)}^{open}$ of $(\mathbb{P}^2, E)$ in terms of Gromov-Witten invariants of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ and the elliptic curve E.
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$$F_g = (-1)^g F_g^{NS} + \sum_{n\geq 0} \sum_{\substack{g=h+g_1+\cdots+g_n, \\ \mathbf{a}=(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0} \\ (a_j,g_j)\neq(0,0),\, \sum_{j=1}^n a_j=2h-2}} \frac{(-1)^{h-1} F^E_{h,\mathbf{a}}}{|\mathrm{Aut}(\mathbf{a},\mathbf{g})|} \prod_{j=1}^n (-1)^{g_j-1} D^{a_j+2} F^{NS}_{g_j}.$$
- $F^E_{h,\mathbf{a}}$: Gromov-Witten theory of E with stationary descendent insertions, known in closed form by [Okounkov-Pandharipande, 2002]
- New formula relating the unrefined topological string F_g and the NS limit F_g^{NS} of the refined topological string (conjecturally valid at least for all local del Pezzo surfaces).

Modularity from the Gromov-Witten side (with Fan, Guo, Wu, 2020)

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- Use quasimodularity and holomorphic anomaly equation for Gromov-Witten invariants of $X = \mathcal{O}_{\mathbb{P}^2}(-3)$ [Fang-Liu-Zhong, 2016] [Lho-Pandharipande,2017][Coates-Iritani, 2018].
- Slightly miraculous combination of these modularity results gives the desired result (the $SL(2,\mathbb{Z})$ quasimodularity of the elliptic curve needs to become $\Gamma_1(3)$ quasimodularity after mirror map).

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End

Thank you for your attention!