4D/2D duality and representation theory

Informal Berkeley String Math meetings

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October 4, 2020

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$C_2(V)$ is the subspace of V spanned by the elements of the above form with $n_1 + \cdots + n_r \geq 1$.

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The Higgs branch Higgs($\mathcal{T}$) is a hyperkähler cone, while the associated variety X_V of a VOA V is only a Poisson variety in general.

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$$X_{L_k(\mathfrak{g})} \subset \mathfrak{g}^*, \quad G\text{-invariant and conic.}$$

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Xie-Yan-Yau'16, Song-Xie-Yan'17

$L_k(\mathfrak{g})$ appears as $\mathbb{V}(\mathcal{T})$ for some Argyres-Douglas theory $\mathcal{T}$ if k is boundary admissible.

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Together with Beem-Rastelli conjecture, the above theorem implies the modularity of the Schur index of a 4D $\mathcal{N} = 2$ SCFT.

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According to Moore-Tachikawa, it is sufficient to describe $\mathrm{Higgs}(S_G(\Sigma))$ for genus zero Σ.

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$\Rightarrow$ MT_r has a finitely many symplectic leaves ([Weekes]).

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Moreover, by Theorem (c), we conclude that Beem-Rastelli conjecture is true for the class $\mathcal{S}$ theory.

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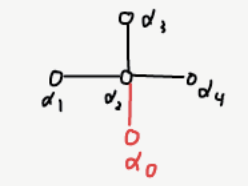

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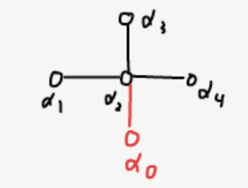

$\mathbf{V}_4 = L_{-2}(D_4)$, the simple affine vertex algebra associated with D_4 at level -2 (conjectured by [BL^{2}PRvR]).

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Also, the MLDE method gives

$$\mathrm{tr}_{L_{-2}(D_4)}(q^{L_0 - c/24}) = \frac{E_4'(\tau)}{240\eta(\tau)^{10}}$$

([A.-Kawasetsu]).

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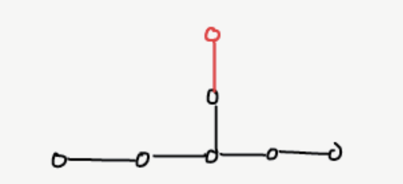

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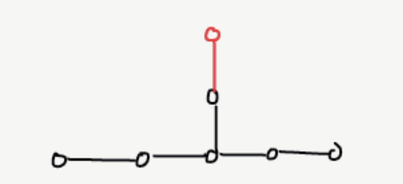
$\mathrm{MT}_3 = \overline{\mathbb{O}_{min}}$ in E_6.



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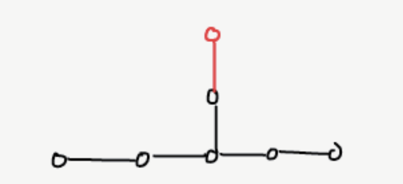

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In general, neither MT_r nor $\mathbf{V}_r$ has a simple description.

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where $\mathcal{D}^{ch}_{G,-h^\vee}$ itself corresponds to the δ-function D-module δ_e at the identity.

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By restricting this equivalence, we get

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and one can check this isomorphism holds for any $\widehat{\mathfrak{g}}$-module M at the critical level on which the $\mathfrak{g}[t]$-action integrates to the action of $G[[t]]$ ([Arkhipov-Gaitsgory]).

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$\mathbf{V}_1 = H^0_{DS}(\mathcal{D}^{ch}_{G,-h^\vee})$ should satisfy

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Equivalently, we want to recover everything from $\mathbf{V}_1$.

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No, because the action of the two Feigin-Frenlel center on $\mathbf{V}_2 = \mathcal{D}^{ch}_{G,-h^\vee}$ is the same.

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We can kill the difference of the two action of the center on $\mathbf{V}_1 \otimes \mathbf{V}_1$, or more generally, on $\mathbf{V}_1^{\otimes r}$, using a certain BRST cohomology.

Construction of $\mathbf{V}_r$

$\mathfrak{z}(\widehat{\mathfrak{g}})$: Feigin-Frenkel center of $\widehat{\mathfrak{g}}$ at the critical level generated by $p_1(z), \ldots, p_{\mathrm{rk}(\mathfrak{g})}(z)$.

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$$\mathbf{V}_r := H^0_{BRST}(\mathbf{V}_1^{\otimes r} \otimes (\otimes_{i=1}^{\mathrm{rk}(\mathfrak{g})}(b_i, c_i))^{\otimes r-1}, Q_{(0)})$$

where

$$Q(z) = \sum_{i=1}^{r-1} Q_{i,i+1}(z),$$

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One can check that the above defined $\mathbf{V}_r$ satisfies the required properties. $\square$.

Thank you!