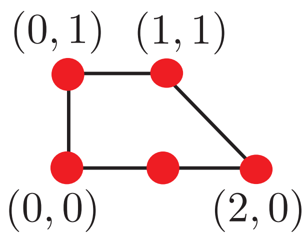
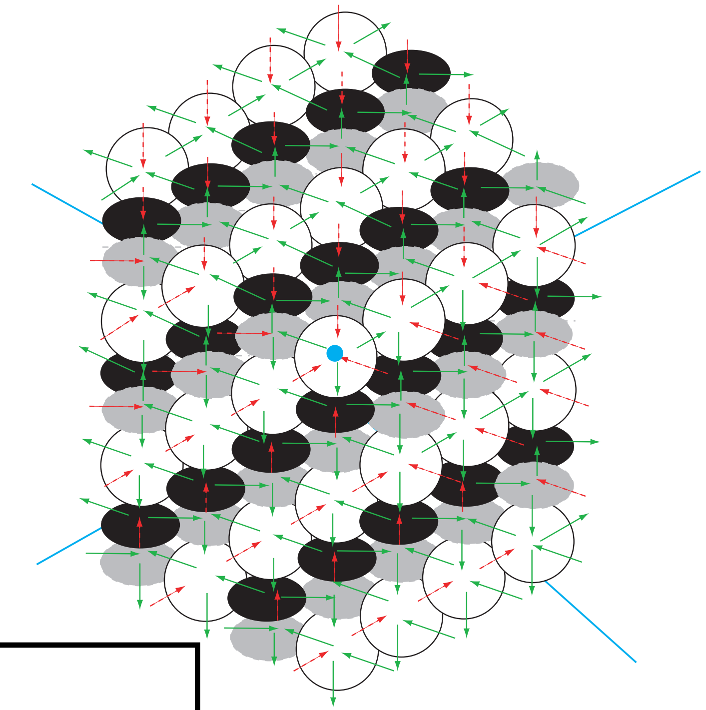


$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\,,\\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\,,\\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\,,\\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\,,\\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\,,\\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\,,
\end{aligned}
$$



Quiver Yangians and Crystal Melting

Masahito Yamazaki

IPMU INSTITUTE FOR THE PHYSICS AND MATHEMATICS OF THE UNIVERSE

Berkeley String-Math seminar

October 26, 2020

Based on

Wei Li + MY

(2003.08909 [hep-th])

Dmitry Galakhov + MY

(2008.07006 [hep-th])

Many related papers, in particular

M. Rapcak, Y. Soibelman, Y. Yang, G. Zhao

(1810.10402, 2007.13365 [math.QA])

Also earlier works, e.g.

Hirosi Ooguri + MY (0811.2810 [hep-th])

MY (Ph.D. thesis, 1002.1709 [hep-th])

MY (Master thesis, 0803.4474 [hep-th])

Overview

Geometry

string theory
supersymmetric gauge theory

BPS states

BPS degeneracy
Enumerative Invariants

Geometric Representation theory

BPS state algebra

Many papers, e.g. [Nakajima,⋯, Kontsevich-Soibelman, Alday-Gaiotto-Tachikawa, Schiffman-Vasserot, Maulik-Okounkov,⋯]

$CY_3 : X$

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} \mathfrak{z}^\gamma \qquad \gamma \in H^{\text{even}}(X)$$

toric CY_3 : X

type IIA string theory

$$\begin{array}{c} \mathbb{R}^{3,1} \times X \\ \cup \\ \mathbb{R} \times \{\text{hol. cycle}\} \end{array}$$

BPS particles wrapping hol. cycle

$$Z^X_{BPS} = \sum_\gamma \underbrace{\Omega^X_\gamma(\cdots)}_{\text{BPS degeneracy}} q^\gamma \qquad \gamma \in H^{even}(X)$$

$= Z_{crystal}$ ← fixed point

BPS quiver Yangian

toric CY_3 : X

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^{X}_{BPS} = \sum_{\gamma} \underbrace{\Omega^{X}_{\gamma}(\cdots)}_{\text{BPS degeneracy}} \mathfrak{z}^{\gamma} \qquad \gamma \in H^{even}(X)$$

$= Z_{crystal}$ fixed point

SUSY QM of BPS particles

BPS quiver Yangian

Plan

— Crystal Melting [Ooguri-Y]

— Quiver Yangian: Algebra [Li-Y]

— Quiver Yangian: Representation

— Derivation from Quantum Mechanics

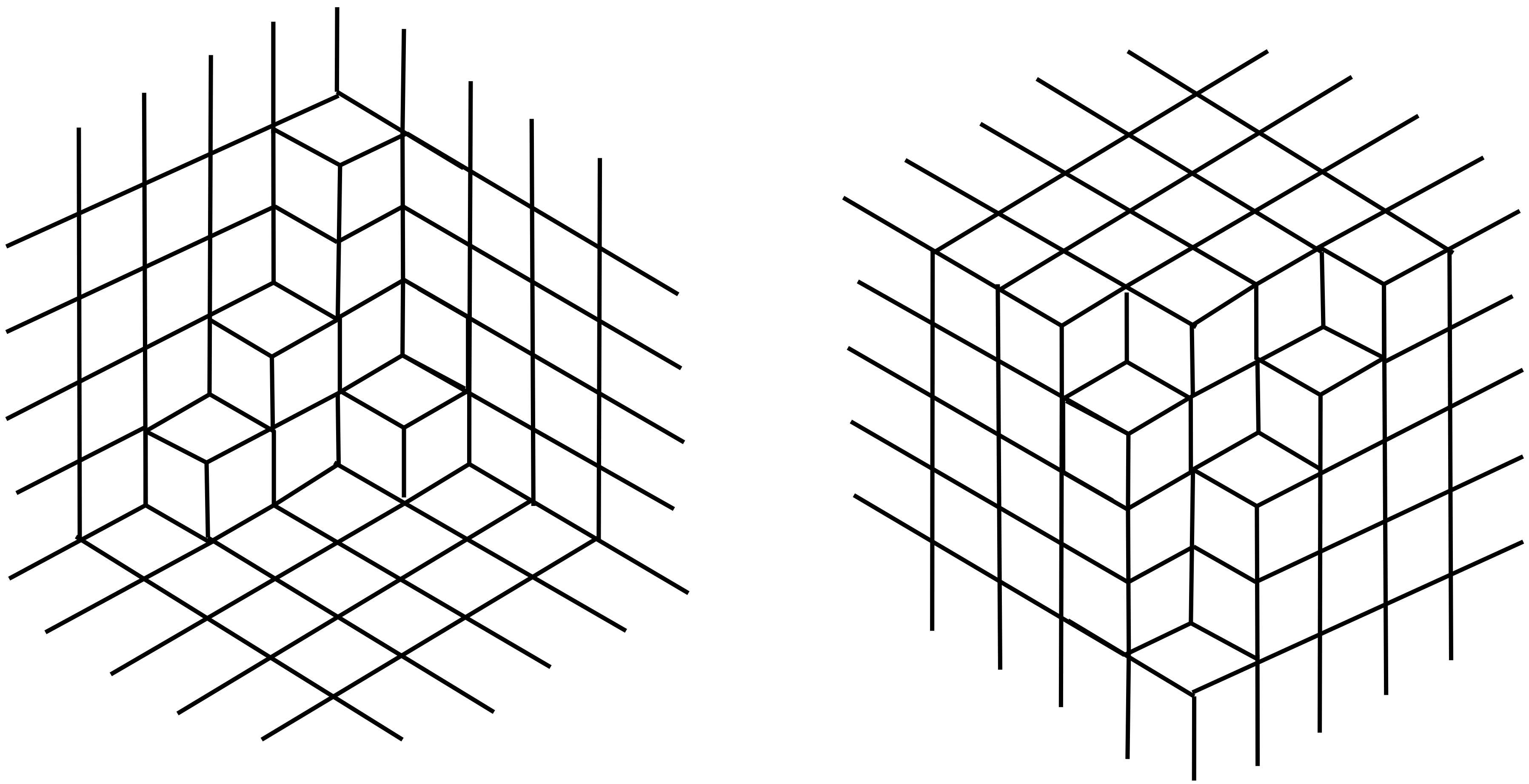
— Summary

[Galakhov-Y]

Crystal Melting

[Szendroi; Mozgovoy, Reineke; Nagao, Nakajima; Ooguri, MY; Jafferis, Chuang, Moore; Sulkowski; Aganagic, Vafa; …]

$\mathbb{C}^3$: crystal melting [Okounkov-Reshetikhin-Vafa; Iqbal, Nekrasov,...]



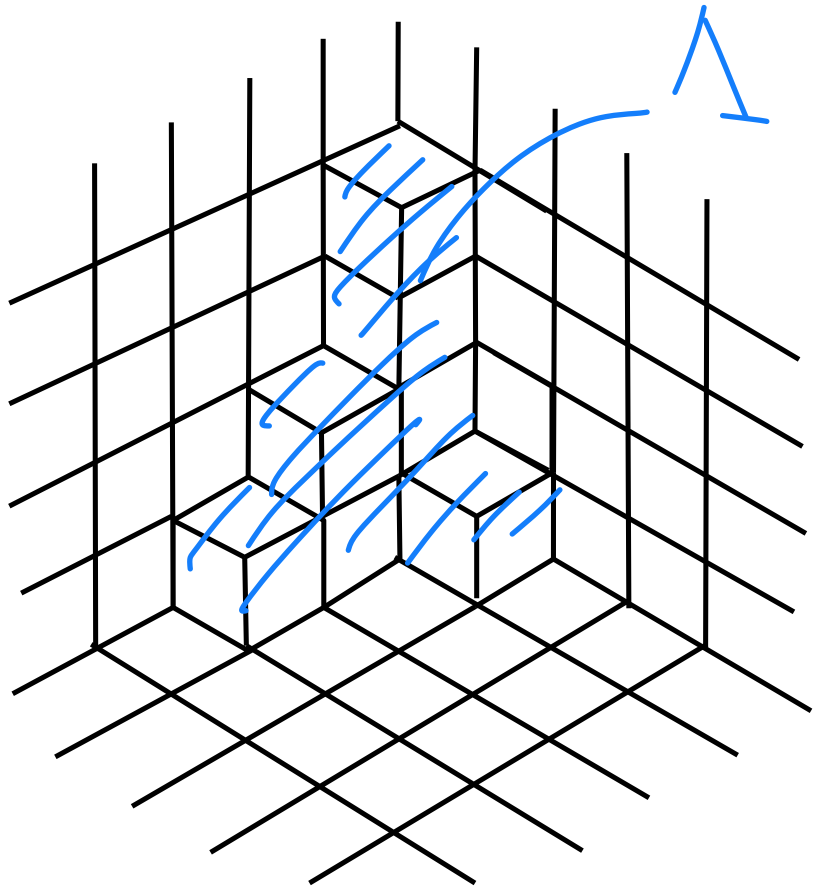
plane partition

$$M(q) \equiv \sum_{\Lambda \in \text{ plane partition}} q^{|\Lambda|} = \prod_{k=1}^{\infty} \frac{1}{(1-q^k)^k}$$

$$= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \ldots \ ,$$

$$= Z^{\mathbb{C}^3}_{\text{top A-model}}$$

crystal melting



ideal sheaf

$$I_\Lambda \subset \mathbb{C}[x, y, z]$$

$$\shortparallel$$

$$\mathrm{Span}\{x^i y^j z^k \mid (i,j,k) \notin \Lambda\}$$

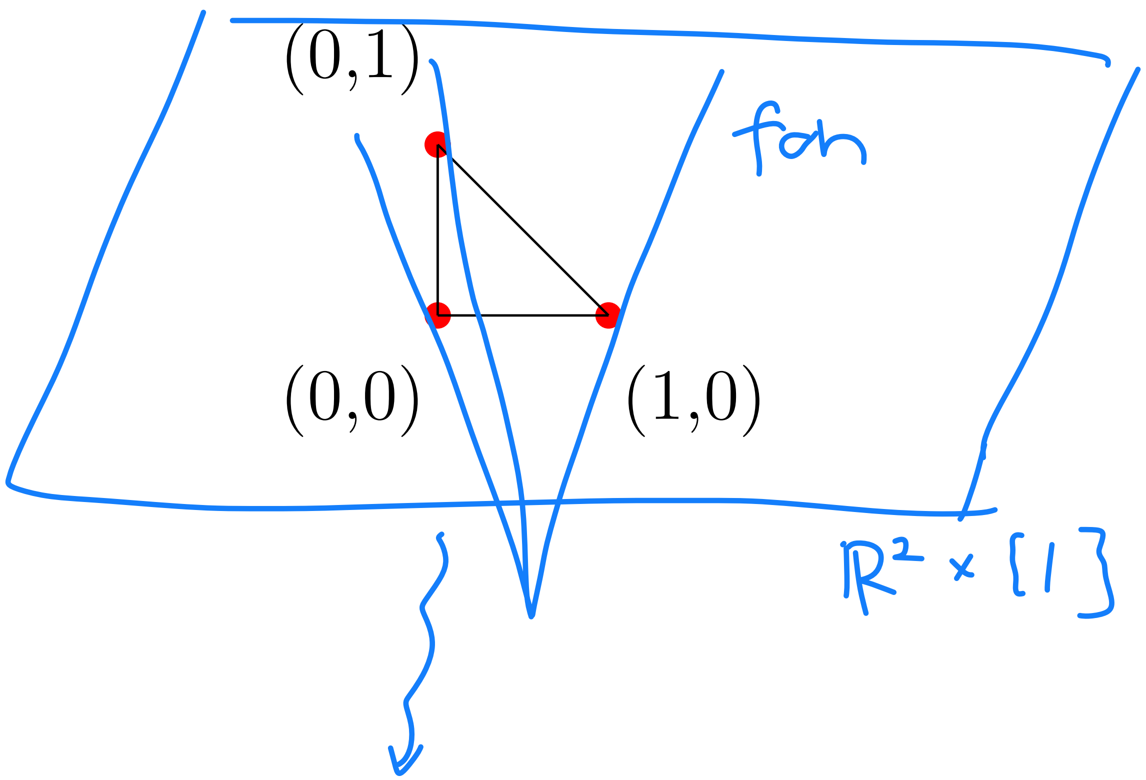
$$x \cdot I_\Lambda,\ y \cdot I_\Lambda,\ z \cdot I_\Lambda \subset I_\Lambda$$

melting rule

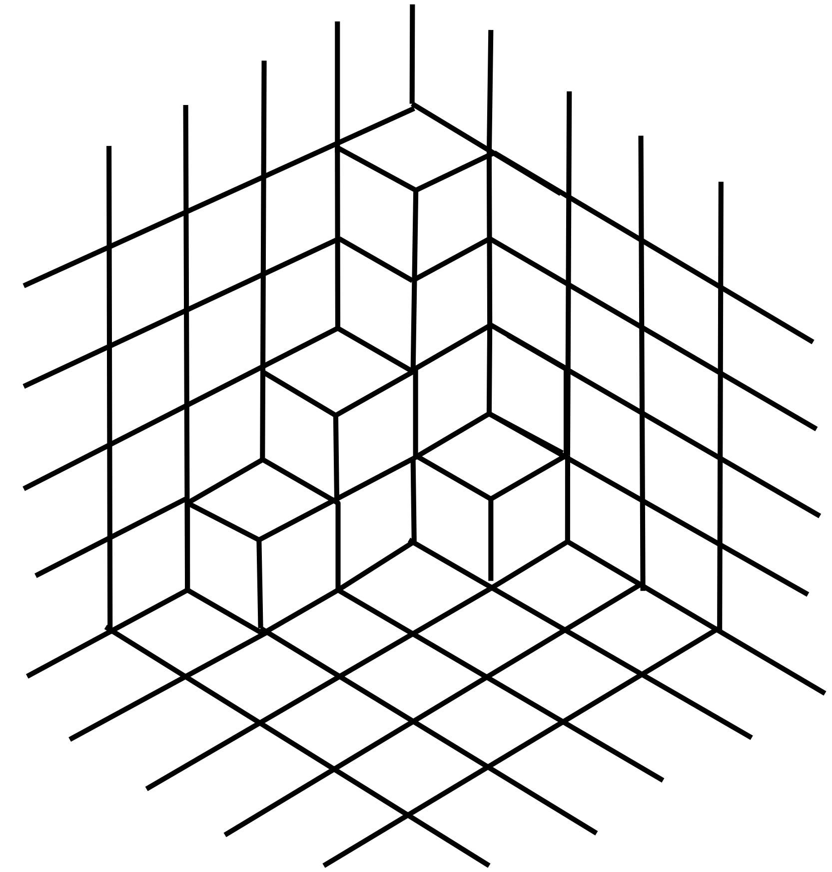
$$(i+1, j, k) \text{ or } (i, j+1, k) \text{ or } (i, j, k+1) \in \Lambda$$

$$\rightsquigarrow (i, j, k) \in \Lambda$$

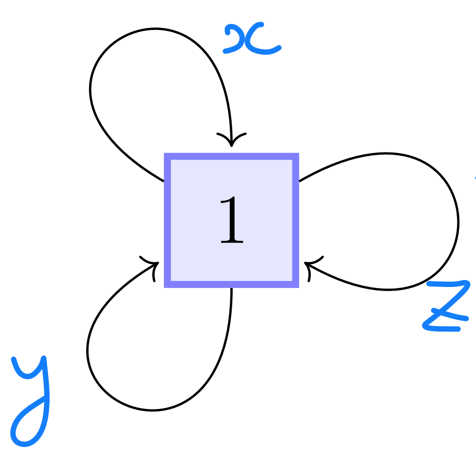
toric diagram $\subset \mathbb{Z}^2$



crystal melting



quiver



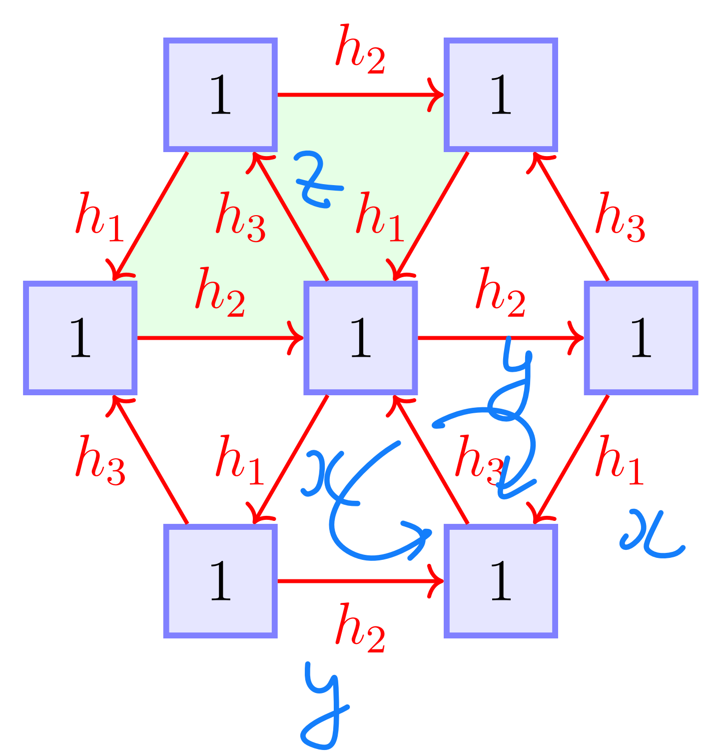
$$W = \mathrm{Tr}(xyz - xzy)$$

path algebra

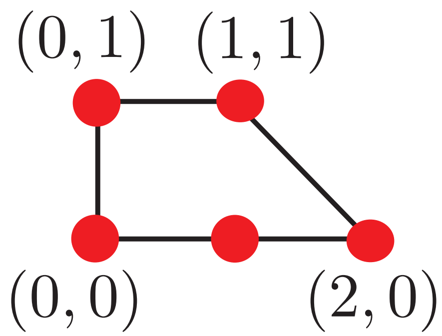
$$\mathbb{C}\langle x, y, z\rangle / (\partial W)$$

$$\|$$

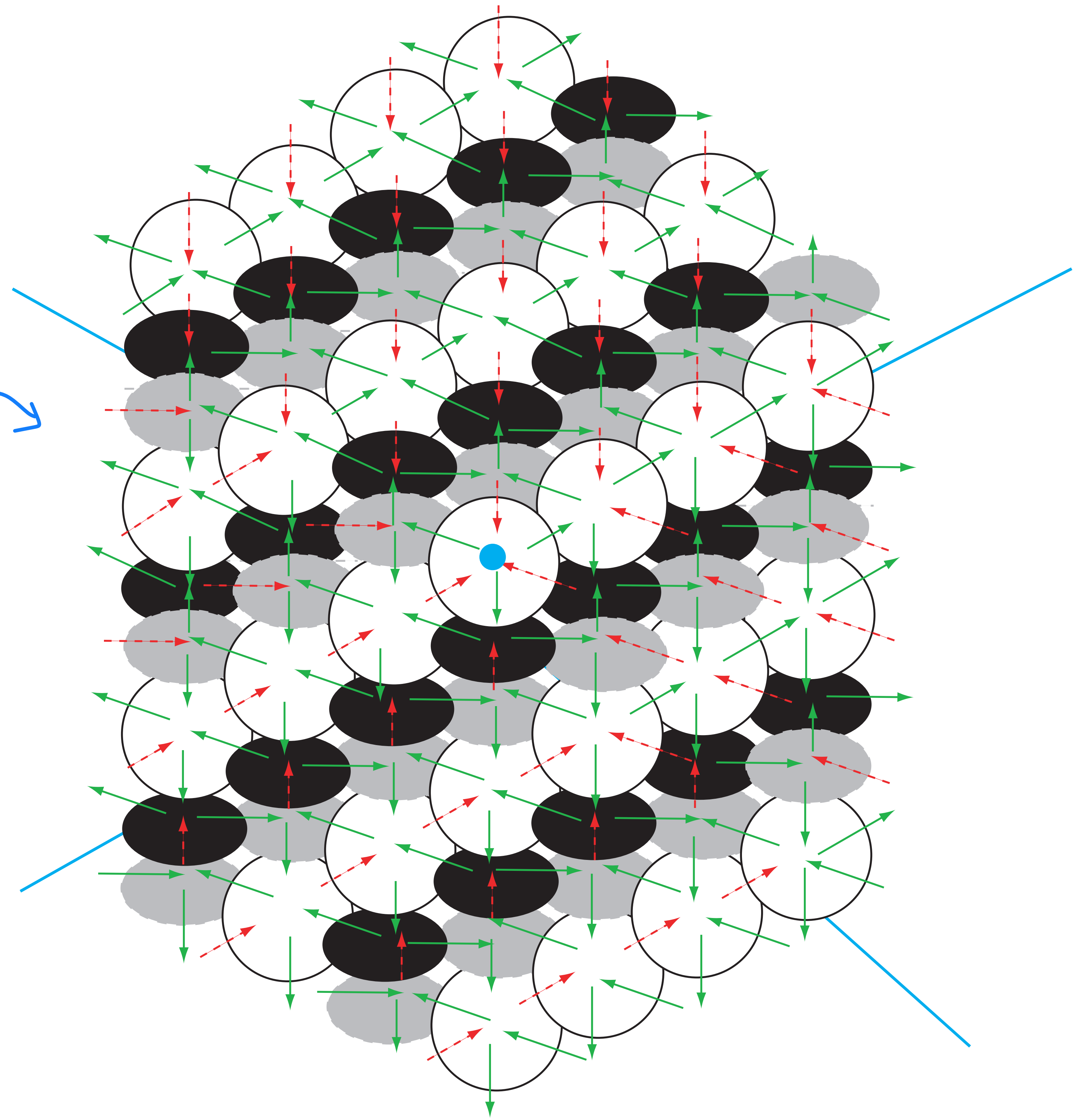
$$\mathbb{C}[x, y, z]$$



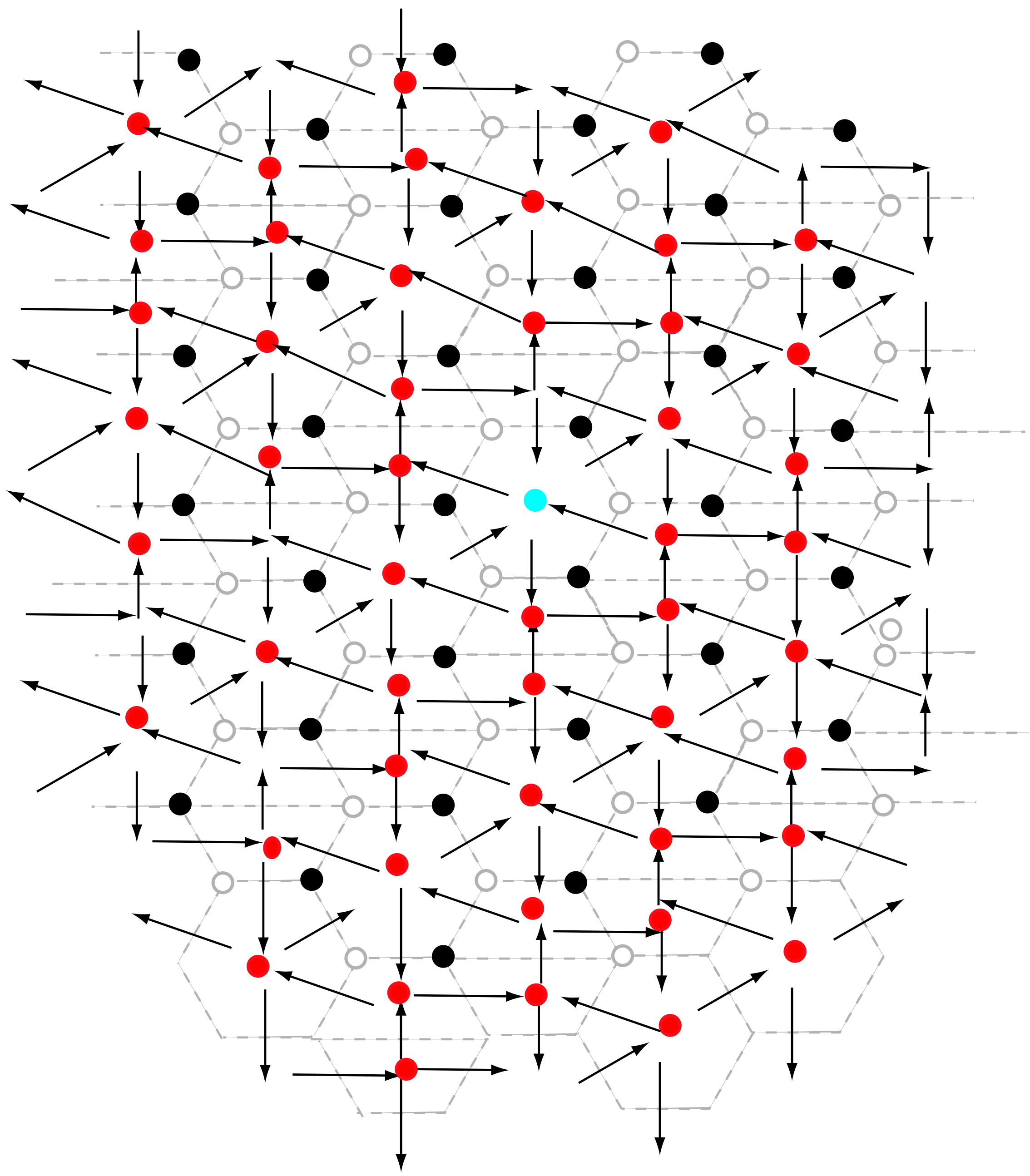
The story generalizes to an arbitrary toric CY3 toric diagram (SPP $xy = zw^2$)



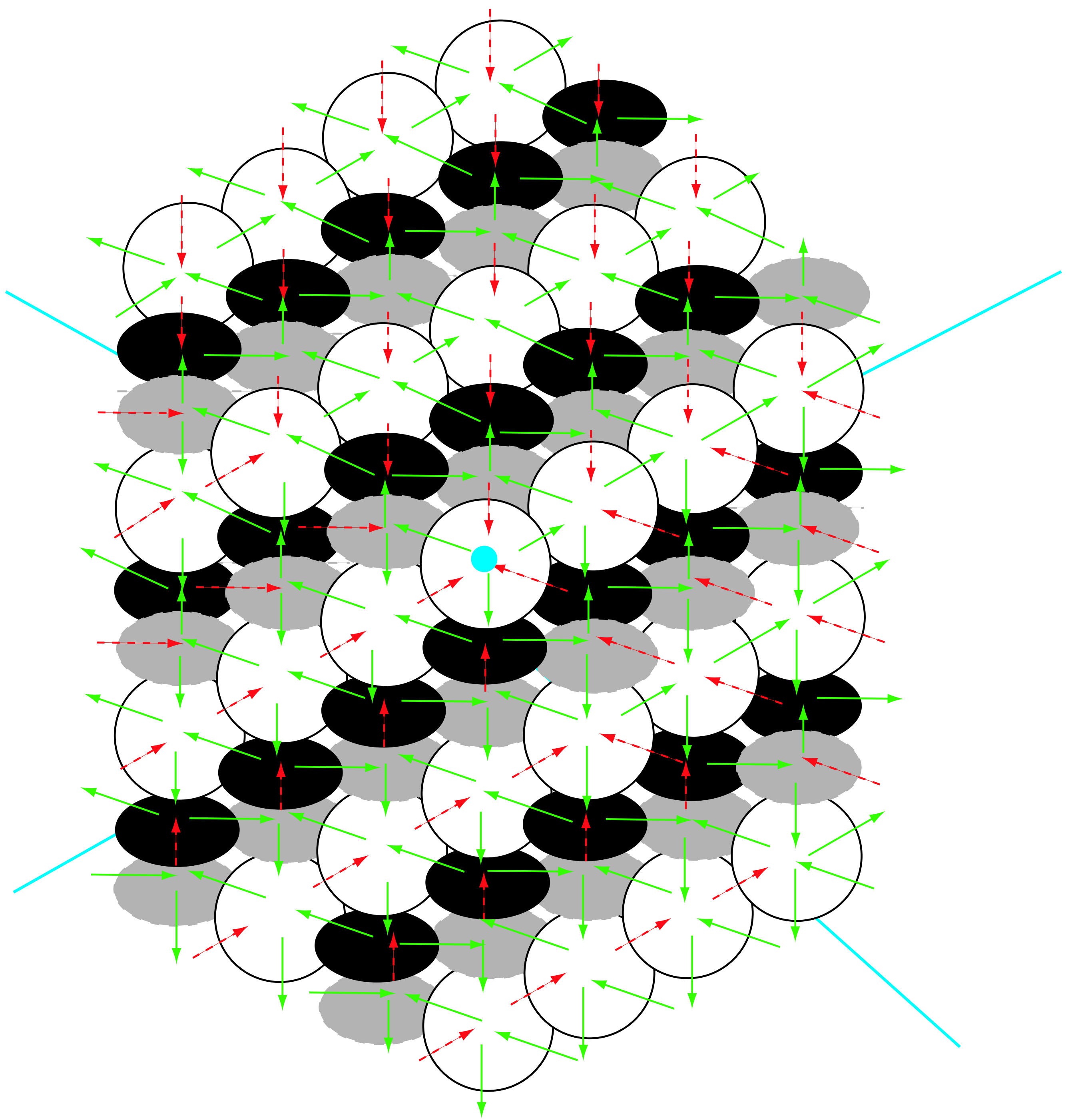
[Ooguri-MY '08]



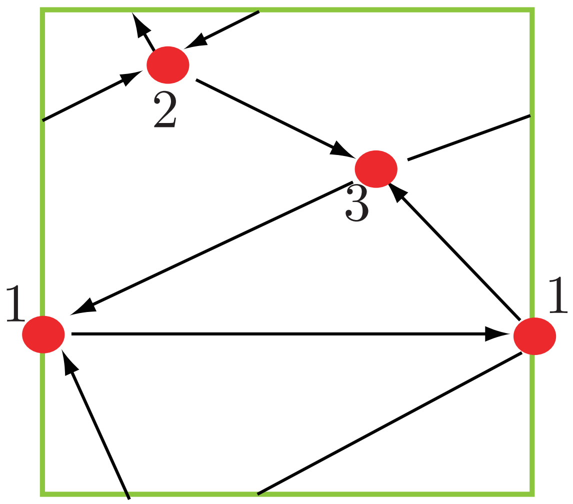
2D projection



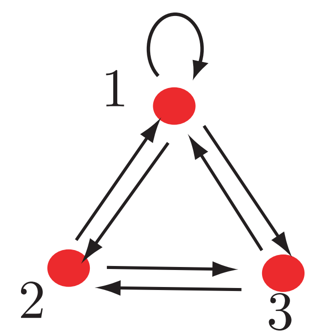
3D crystal



2D projection of the crystal is a tesselation of the periodic quiver on T^2
studied by [Hanany et al.]



$\rightsquigarrow$



$Q = (Q_0, Q_1, Q_2)$

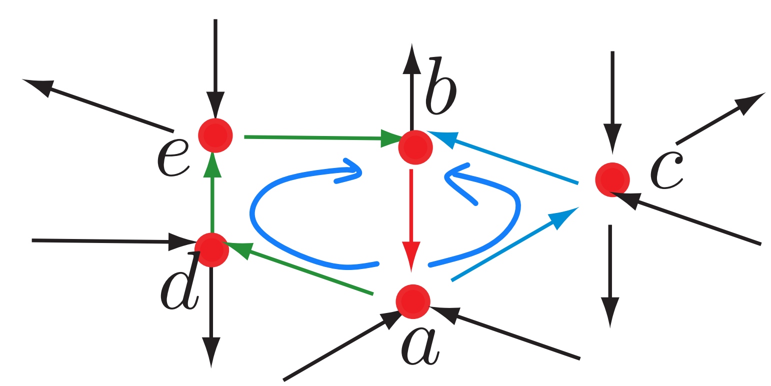
quiver vertex — quiver arrow — superpotential W

Superpotential / F-term relations

$$W = \mathrm{Tr}(\Phi_{ba}\Phi_{ac}\Phi_{cb} - \Phi_{ba}\Phi_{ad}\Phi_{de}\Phi_{eb})$$

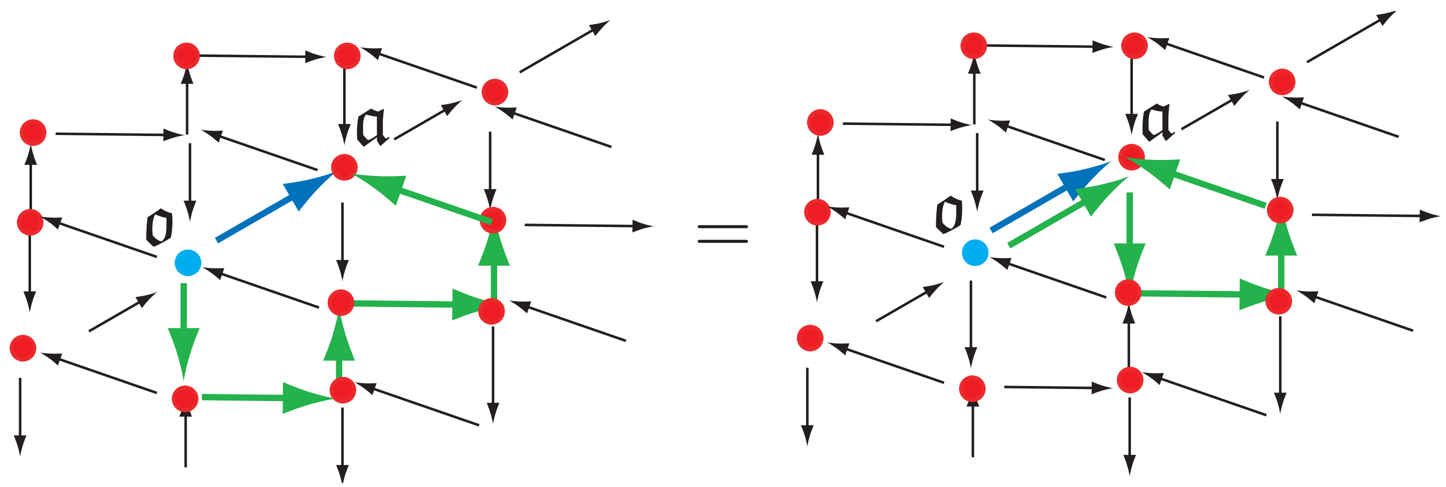
$$\partial W/\partial \Phi_{ba} = \Phi_{ac}\Phi_{cb} - \Phi_{ad}\Phi_{de}\Phi_{eb} = 0$$

$\rightsquigarrow$



$\mathbb{C}Q/(\partial W)$: path algebra (non-commutative in general)

We can lift the 2D projection of the crystal into 3D by keeping track of "depth"



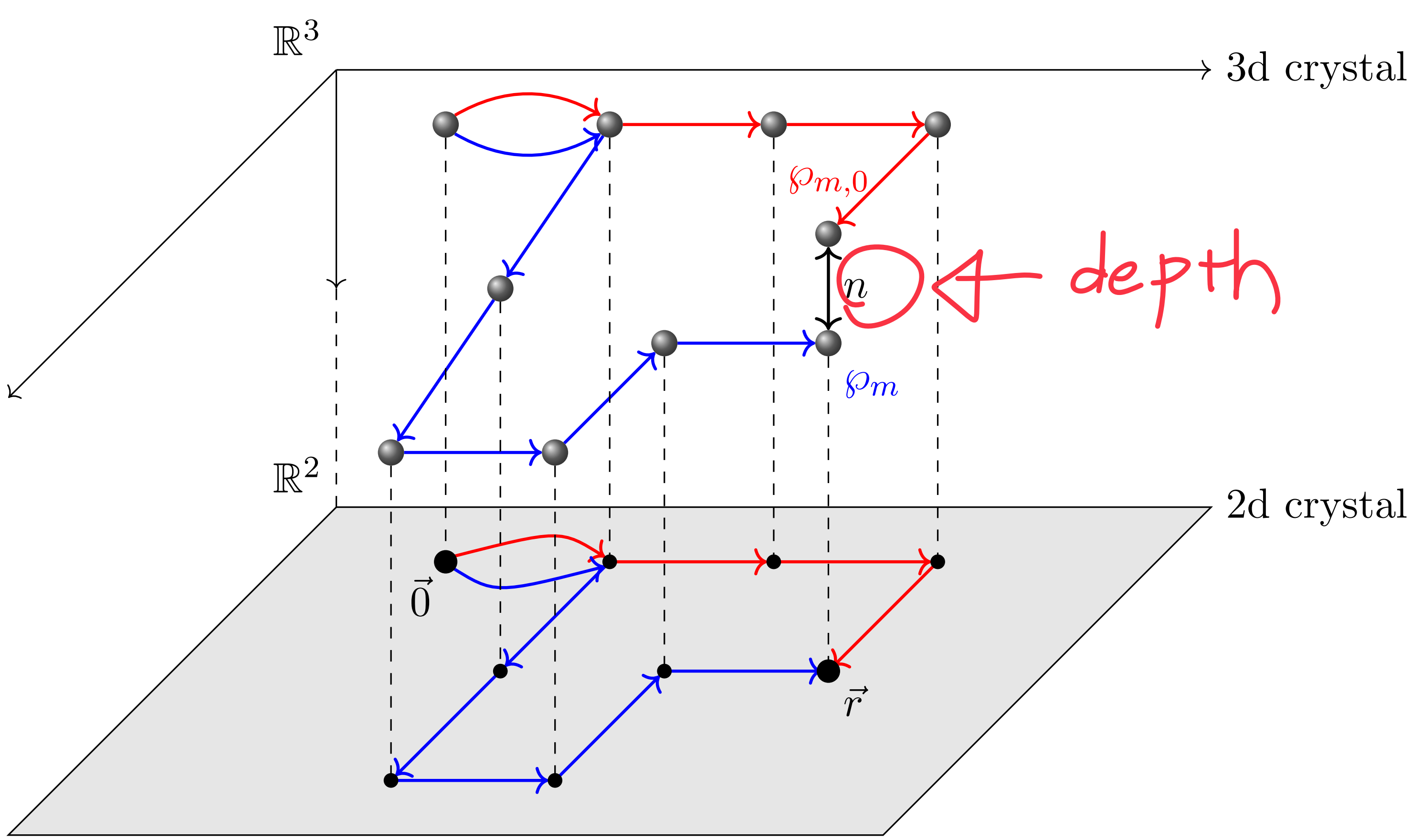
any path $\mathfrak{o} \to \mathfrak{a}$

of the form

depth

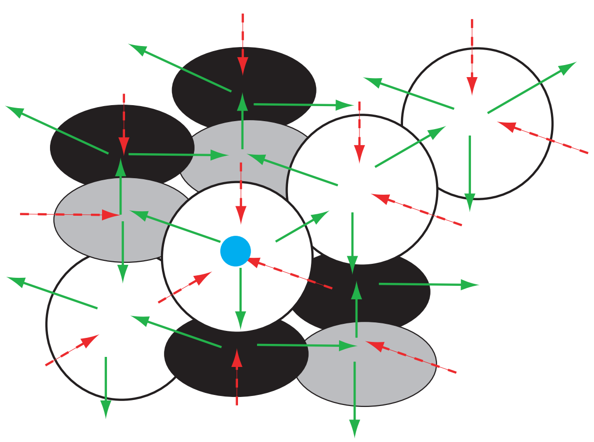
(minimal path)·(loop)

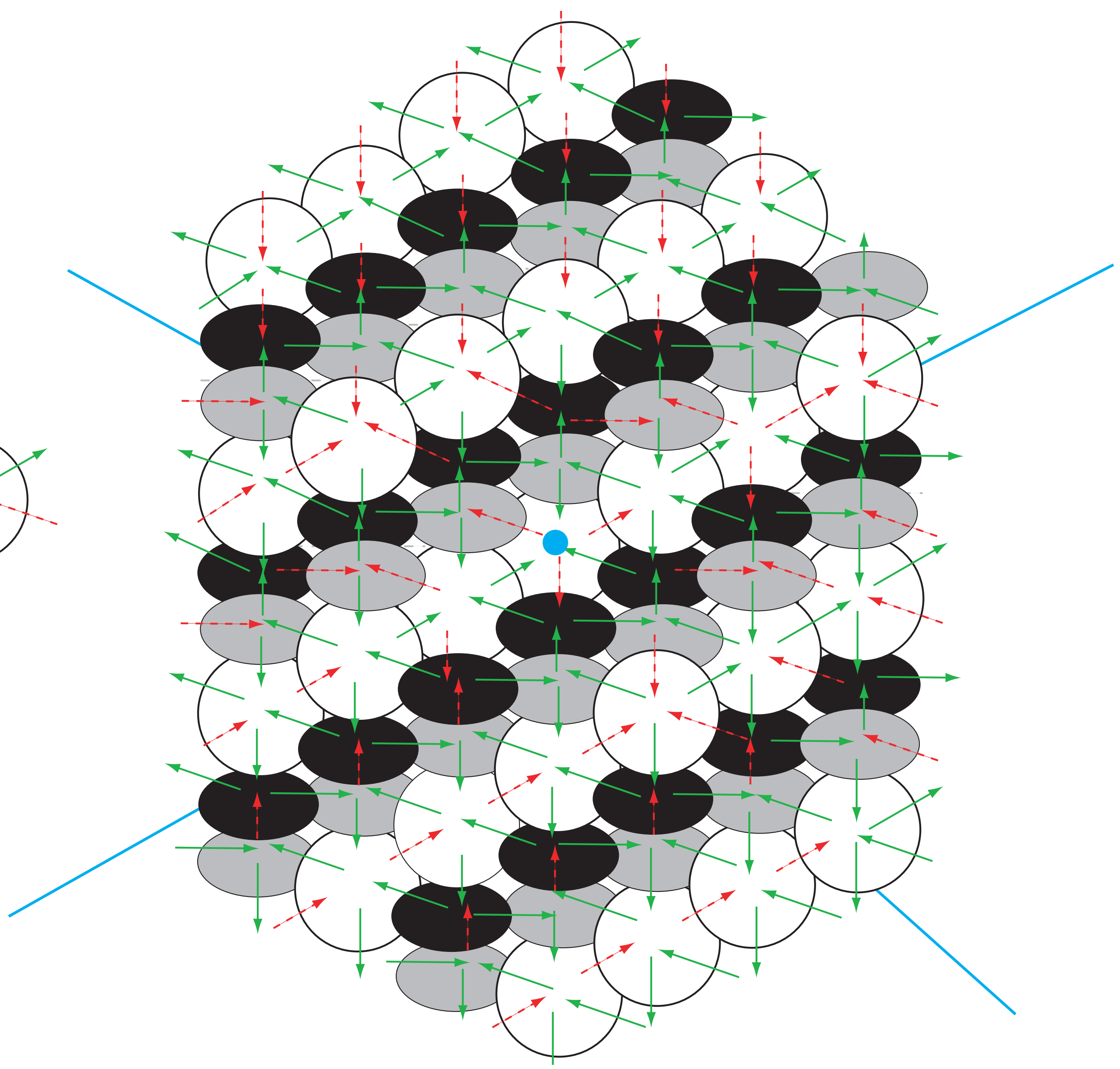
mod F-term rel.



melting rule:

$\square \in \mathrm{K}$ whenever there exists an edge $I \in Q_1$ such that $I \cdot \square \in \mathrm{K}$



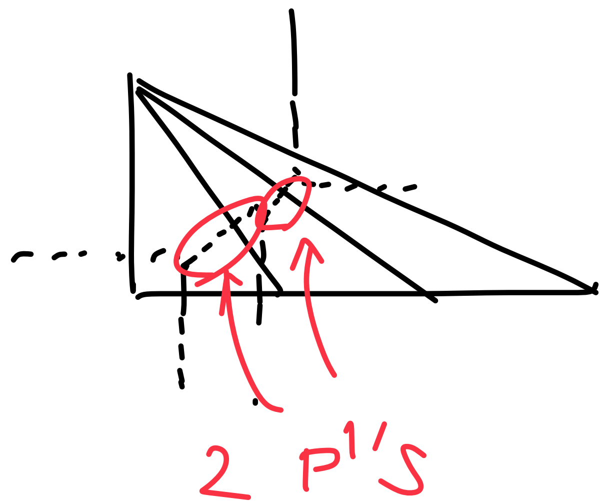


$$Z(q_1, \ldots, q_{|Q_0|}) = \sum_{\mathrm{K}} \prod_{a \in Q_0} q_a^{|\mathrm{K}(a)|}$$

formal variable for each quiver vertex

Infinite-product forms discussed
in [Szendroi, Young, Nagao, Aganagic-Ooguri-Vafa-MY]

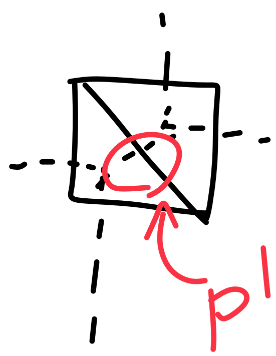
$(\mathbb{C}^2/\mathbb{Z}_3)\times\mathbb{C}$



$$Z \sim \prod_n \frac{1}{1-q^n Q_1}\,\frac{1}{1-q^n Q_2}\,\frac{1}{1-q^n Q_1 Q_2}$$

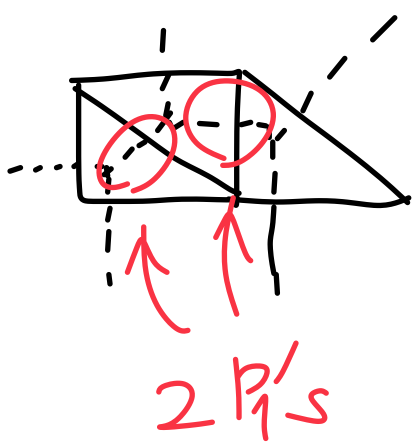
$$\begin{pmatrix} n\delta+\alpha_1, & n\delta+\alpha_2, & n\delta+\alpha_1+\alpha_2 \\ \text{even} & \text{even} & \text{even} \end{pmatrix}$$

conifold



$$Z \sim \prod_n (1-q^n Q) \qquad \begin{pmatrix} n\delta+\alpha \\ \text{odd} \end{pmatrix}$$

SPP



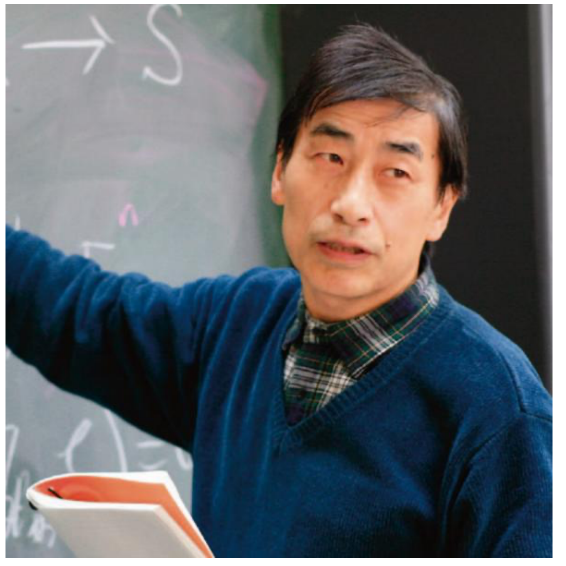
$$Z \sim \prod_n (1-q^n Q_1)(1-q^n Q_1 Q_2)\,\frac{1}{1-q^n Q_2}$$

$$\begin{pmatrix} n\delta+\alpha_1, & n\delta+\alpha_2, & n\delta+\alpha_1+\alpha_2 \\ \text{odd} & \text{even} & \text{odd} \end{pmatrix}$$

[Nagao-MY] discussed chamber structures in terms of affine Weyl groups]

Lie superalgebra ?

Circa 2009-2010



Elliptic !!

Quantum toroidal !!

Later important developments on quantum toroidal algebras (Ding-Iohara-Miki) and affine Yangians by

[B. Feigin, E. Feigin, Jimbo, Miwa, Mukhin; Tsymbaulik; Prochazka, $\cdots$]

which in particular constructed representations on plane partitions.

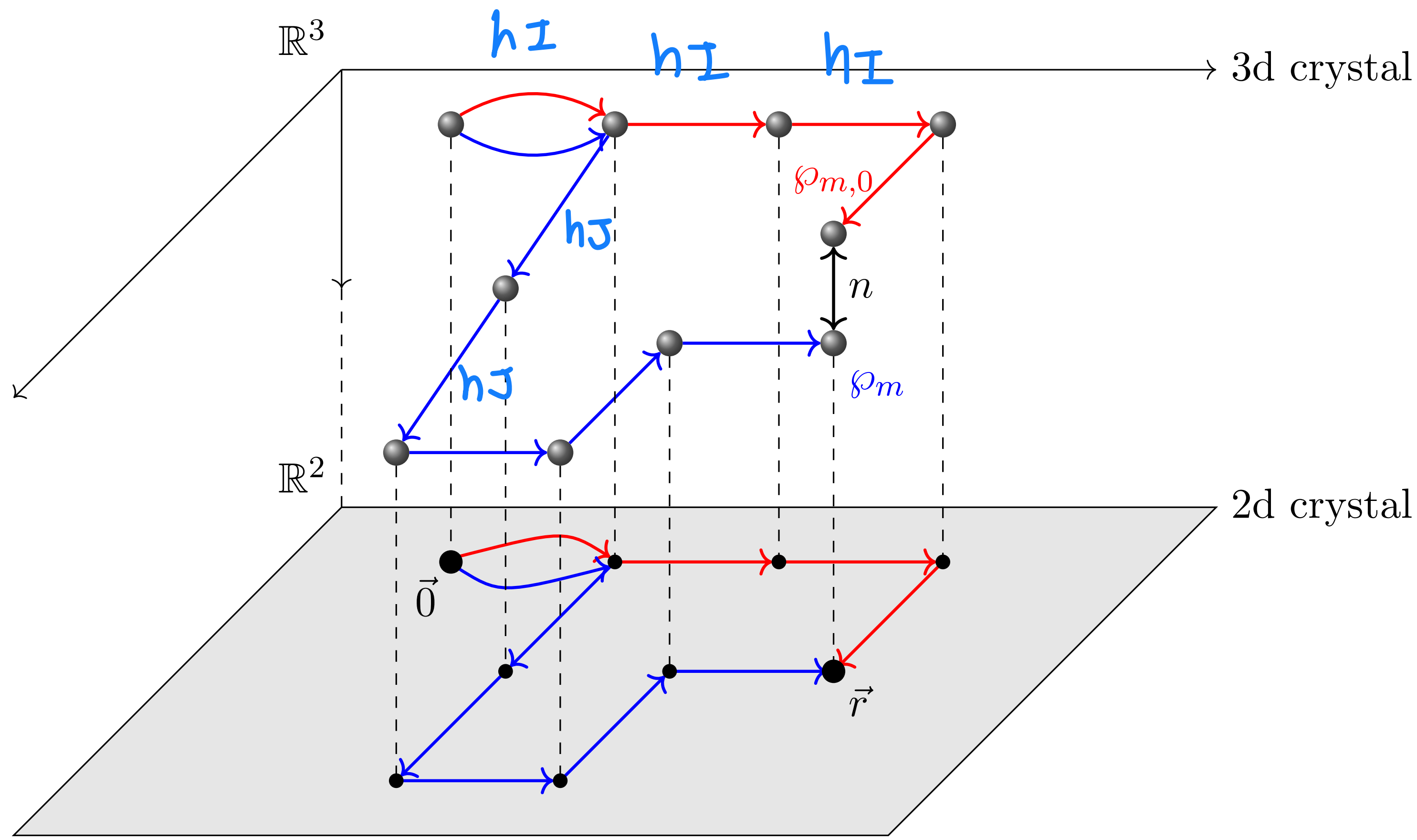
Affine Yangians also appear in higher spin algebras [Gaberdiel, Gopakumar; Li, Peng,$\cdots$]

Quiver Yangian

: Algebra

[Li-MY '20]

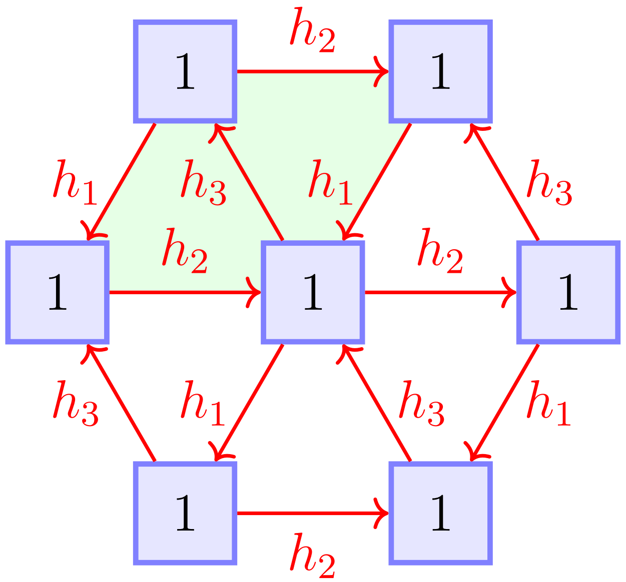
A. equivariant parameters



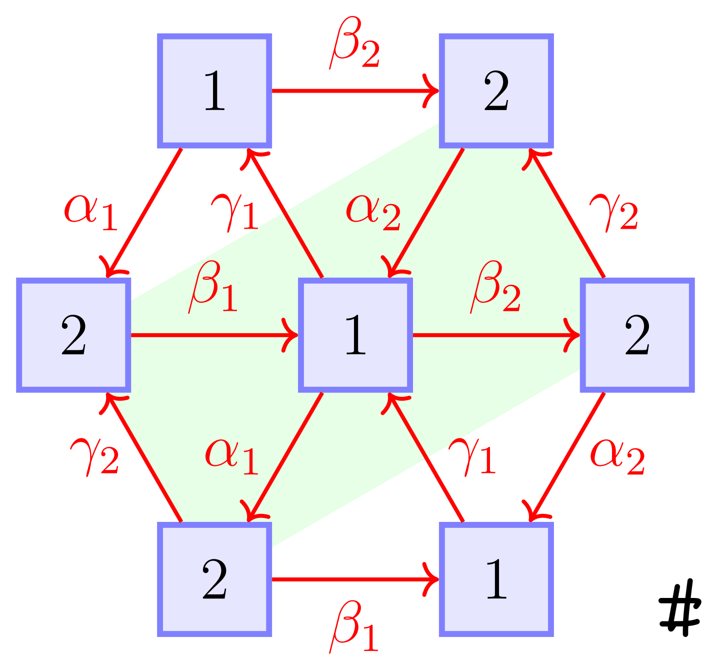
$$h(\boxed{a}) \equiv \sum_{I \in \text{path}[\mathfrak{o} \to \boxed{a}]} h_I \ .$$

loop constraint: $\sum_{I \in L} h_I = 0$,

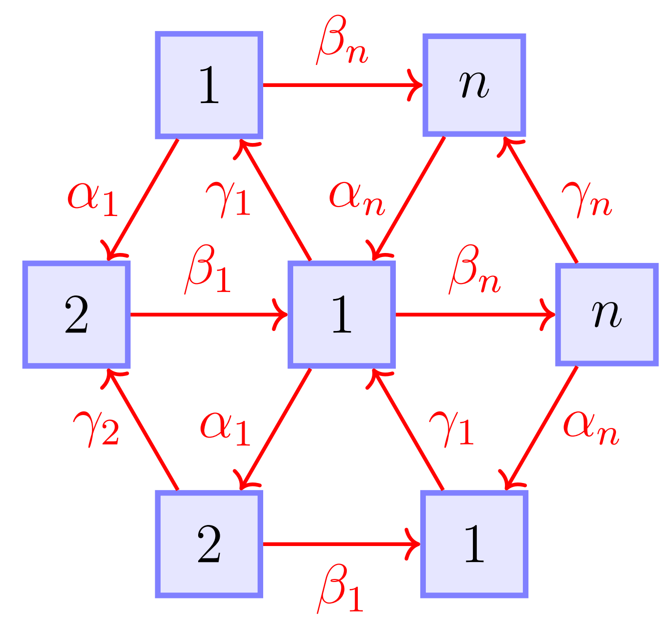
A. equivariant parameters



$\mathbb{C}^3$



$\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$



$\mathbb{C}^2/\mathbb{Z}_n \times \mathbb{C}$

external lattice pt

internal lattice pt

loop constraint: $\displaystyle\sum_{I\in L} h_I = 0\ ,$ $\longleftarrow$ $E + 2I - 1$ parameters

+

vertex constraint: $\displaystyle\sum_{I\in a} \mathrm{sign}_a(I)\, h_I = 0$ $\longleftarrow$ 2 parameters

(2-D coordinate of 2-D projection)

B. Chevally-type generators

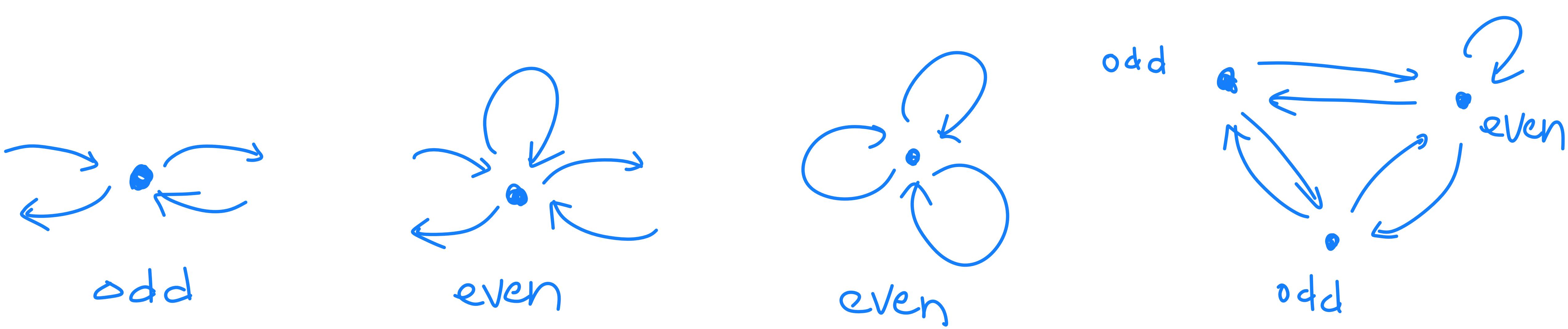
(z : spectral parameter)

$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e_n^{(a)}}{z^{n+1}}\,, \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi_n^{(a)}}{z^{n+1}}\,, \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f_n^{(a)}}{z^{n+1}}\,,$$

$$e^{(a)}(u) : \ \text{creation}\,, \qquad \psi^{(a)}(u) : \ \text{charge}\,, \qquad f^{(a)}(u) : \ \text{annihilation}$$

Z2-grading (super algebra)

$$|a| = \begin{cases} 0 & (\exists I \in Q_1 \quad \text{such that} \quad s(I) = t(I) = a)\,, \\ 1 & (\text{otherwise})\,, \end{cases}$$



C. “OPE relations”

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ , \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\ , \\
e^{(a)}(z)\,e^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,e^{(a)}(z)\ , \\
\psi^{(a)}(z)\,f^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,\psi^{(a)}(z)\ , \\
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ , \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,
\end{aligned}
$$

“$\simeq$” means equality up to $z^n w^{m\geq 0}$ terms

“$\sim$” means equality up to $z^{n\geq 0} w^m$ and $z^n w^{m\geq 0}$ terms

$$
\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}
$$

$$
\begin{aligned}
\psi^{(a)}(z)\,\psi^{(b)}(w) &= \psi^{(b)}(w)\,\psi^{(a)}(z)\ , \\
\psi^{(a)}(z)\,e^{(b)}(w) &\simeq \varphi^{b\Rightarrow a}(\Delta)\,e^{(b)}(w)\,\psi^{(a)}(z)\ , \\
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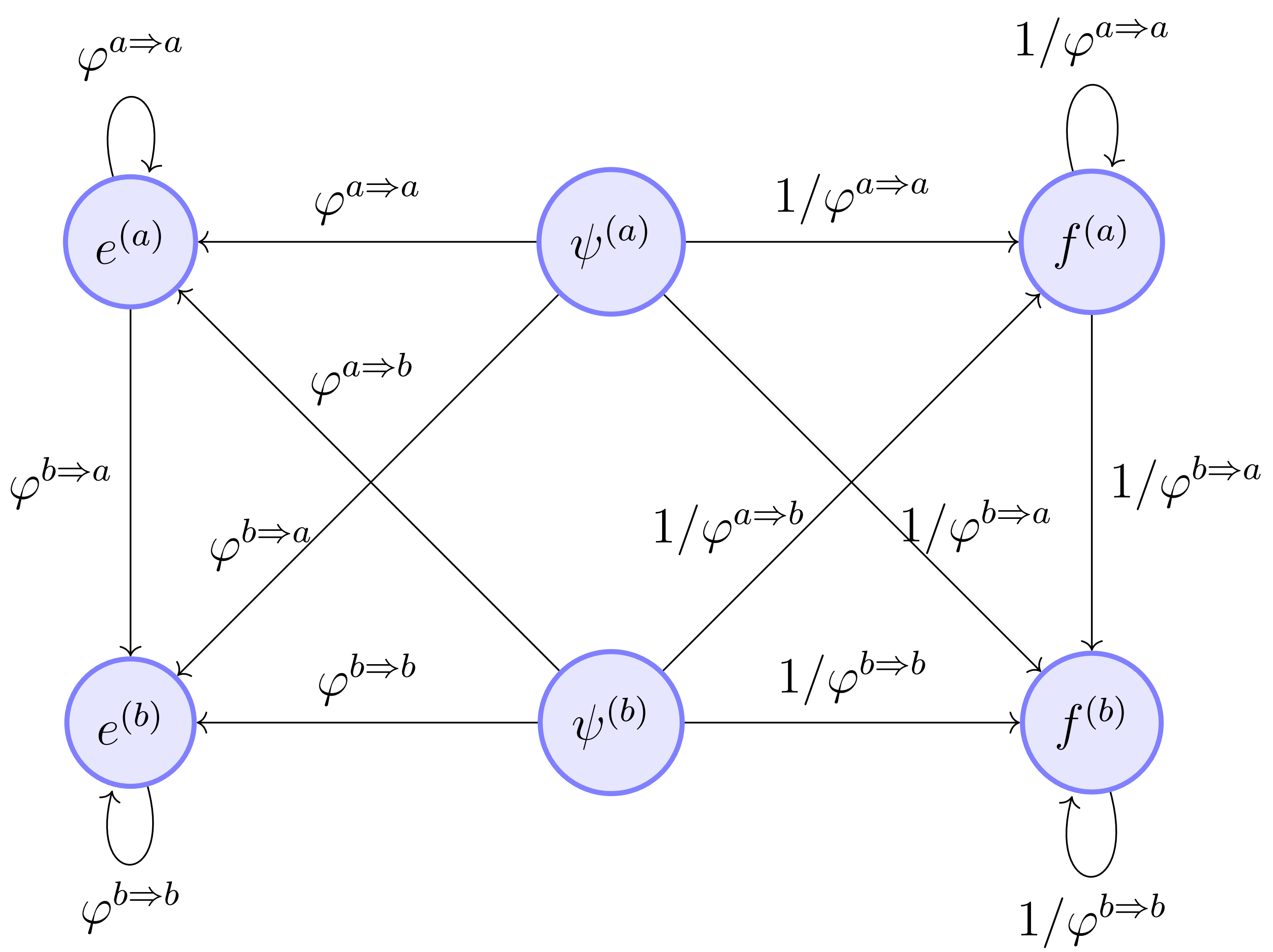
f^{(a)}(z)\,f^{(b)}(w) &\sim (-1)^{|a||b|}\varphi^{b\Rightarrow a}(\Delta)^{-1}\,f^{(b)}(w)\,f^{(a)}(z)\ , \\
\left[e^{(a)}(z), f^{(b)}(w)\right\} &\sim -\delta^{a,b}\,\frac{\psi^{(a)}(z)-\psi^{(b)}(w)}{z-w}\ ,
\end{aligned}
$$



when expanded in terms of modes,

$$\left[\psi_n^{(a)}\,,\,\psi_m^{(b)}\right]=0\,,$$

$$\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[\psi_n^{(a)}\,e_m^{(b)}]_k=\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[e_m^{(b)}\,\psi_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[e_n^{(a)}\,e_m^{(b)}]_k=(-1)^{|a||b|}\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[e_m^{(b)}\,e_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[\psi_n^{(a)}\,f_m^{(b)}]_k=\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[f_m^{(b)}\,\psi_n^{(a)}]^k\,,$$

$$\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,[f_n^{(a)}\,f_m^{(b)}]_k=(-1)^{|a||b|}\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,[f_m^{(b)}\,f_n^{(a)}]^k,$$

$$\left[e_n^{(a)}\,,\,f_m^{(b)}\right\}=\delta^{a,b}\,\psi_{n+m}^{(a)}\,,$$

$$\prod_{I\in\{a\to b\}}(z-w+h_I)=\sum_{k=0}^{|a\to b|}\sigma^{a\to b}_{|a\to b|-k}\,(z-w)^k\,,$$

$$\prod_{I\in\{b\to a\}}(z-w-h_I)=\sum_{k=0}^{|b\to a|}(-1)^{|b\to a|-k}\,\sigma^{b\to a}_{|b\to a|-k}\,(z-w)^k\,,$$

$$[A_n\,B_m]_k\equiv\sum_{j=0}^{k}(-1)^j\,\binom{k}{j}\,A_{n+k-j}\,B_{m+j}\,,$$

$$[B_m\,A_n]^k\equiv\sum_{j=0}^{k}(-1)^j\,\binom{k}{j}\,B_{m+j}\,A_{n+k-j}\,.$$

Example

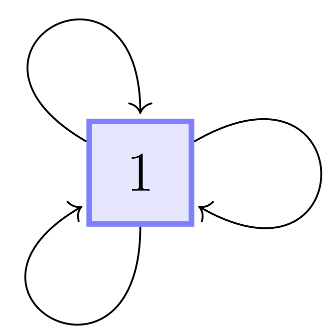
OPE relation

$$\psi(z)\,\psi(w) \sim \psi(w)\,\psi(z)\,,$$
$$\psi(z)\,e(w) \sim \varphi_3(\Delta)\,e(w)\,\psi(z)\,,$$
$$\psi(z)\,f(w) \sim \varphi_3^{-1}(\Delta)\,f(w)\,\psi(z)\,,$$
$$e(z)\,e(w) \sim \varphi_3(\Delta)\,e(w)\,e(z)\,,$$
$$f(z)\,f(w) \sim \varphi_3^{-1}(\Delta)\,f(w)\,f(z)\,,$$
$$[e(z)\,,f(w)] \sim -\frac{1}{\sigma_3}\,\frac{\psi(z)-\psi(w)}{z-w}\,,$$

$$\varphi_3(z) \equiv \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}\,.$$

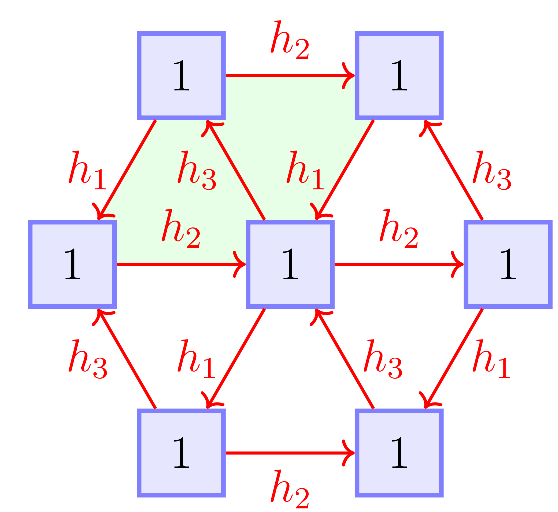
$$h_1+h_2+h_3=0\,,$$

$$\sigma_3 \equiv h_1\,h_2\,h_3\,.$$



Serre relation

$$\mathrm{Sym}_{z_1,z_2,z_3}(z_2-z_3)[\,e(z_1)\,,[\,e(z_2)\,,e(z_3)]] = 0\,;$$
$$\mathrm{Sym}_{z_1,z_2,z_3}(z_2-z_3)[f(z_1)\,,[f(z_2)\,,f(z_3)]] = 0\,.$$



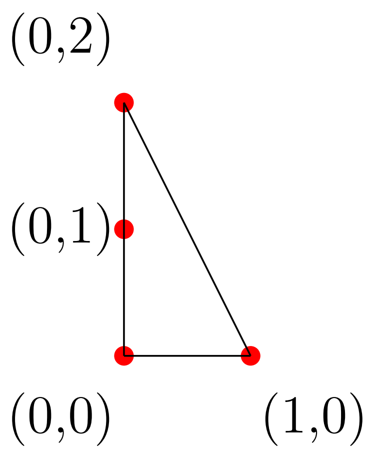
This gives $Y(\hat{gl}_1)$: affine Yangian

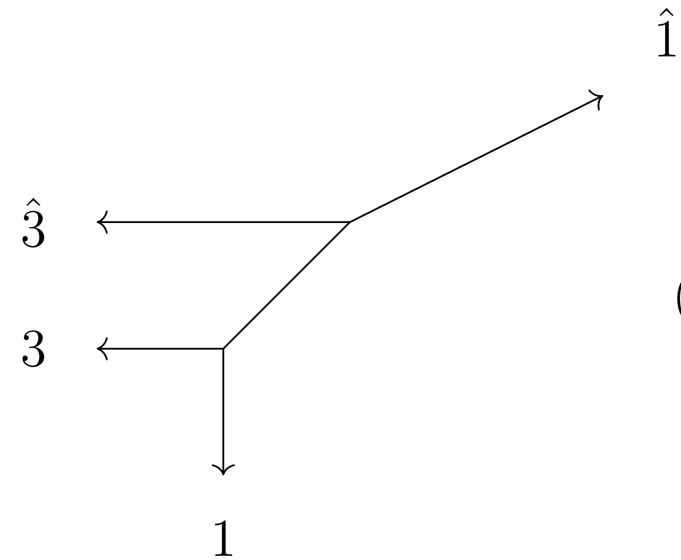
is

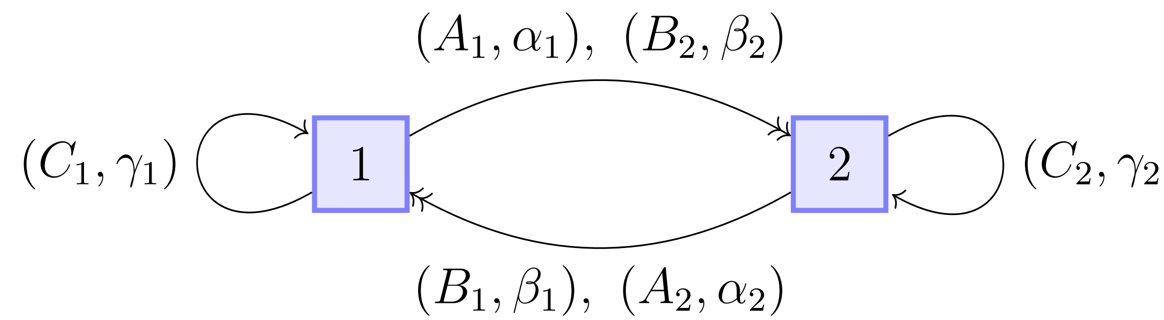
$\mathcal{U}(\mathcal{W}_{1+\infty})$

[Schiffmann-Vasserot; Tsymbaulik; Prochazka; Gaberdiel-Gopakumar-Li-Peng,...]

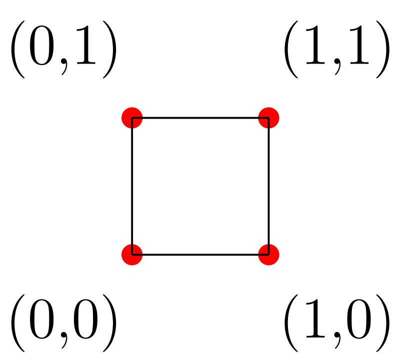
* $(\mathbb{C}^2/\mathbb{Z}_2)\times\mathbb{C} \rightsquigarrow Y(\widehat{gl_2})$

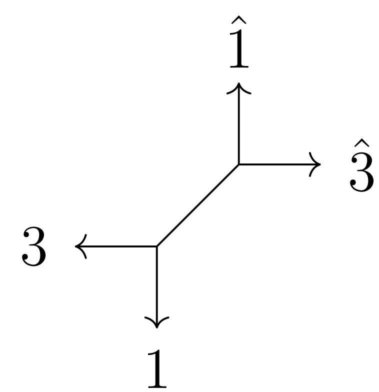


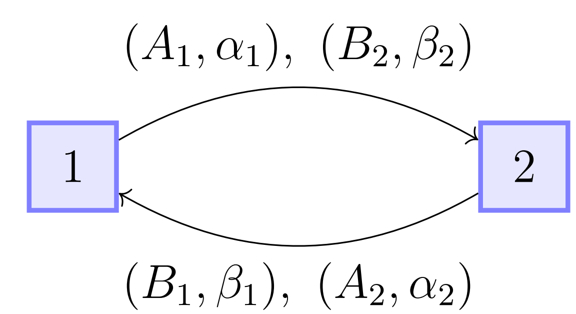




* conifold $\rightsquigarrow Y(\widehat{gl_{1|1}})$







* more generally,

$xy = z^n w^m \rightsquigarrow Y(\widehat{gl_{m|n}})$ [Rapcak; Bezerra-Mukhin]

Some Properties of Quiver Yangians [Li-MY]

a. triangular decomposition

$$\mathsf{Y}_{(Q,W)} = \mathsf{Y}^{+}_{(Q,W)} \oplus \mathsf{B}_{(Q,W)} \oplus \mathsf{Y}^{-}_{(Q,W)} , \qquad e^{(a)}(z) \leftrightarrow f^{(a)}(z) , \quad \psi^{(a)}(z) \leftrightarrow \psi^{(a)}(z)^{-1} ,$$

$\{e_a\}$ $\{\psi_a\}$ $\{f_a\}$

order 2 involution

b. grading

$$\deg_a(e^{(b)}_n) = \delta_{a,b} , \quad \deg_a(\psi^{(b)}_n) = 0 , \quad \deg_a(f^{(b)}_n) = -\delta_{a,b} .$$

$$\deg_{\text{level}}(e^{(b)}_n) = \deg_{\text{level}}(f^{(b)}_n) = n + \tfrac{1}{2} , \quad \deg_{\text{level}}(\psi^{(b)}_n) = n + 1 ,$$

grading when $\deg(hz) = 1$

c. spectral shift

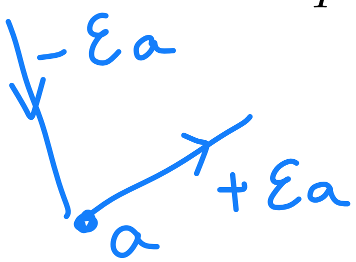
$$e^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{e^{(a)}_n}{z^{n+1}} , \qquad \psi^{(a)}(z) \equiv \sum_{n=-\infty}^{+\infty} \frac{\psi^{(a)}_n}{z^{n+1}} , \qquad f^{(a)}(z) \equiv \sum_{n=0}^{+\infty} \frac{f^{(a)}_n}{z^{n+1}} ,$$

$z \to z - \varepsilon$ causes

$$e'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k e_{l-k} , \quad f'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k f_{l-k} , \quad \psi'_l = \sum_{k=0}^{l} \binom{l}{k} \varepsilon^k \psi_{l-k} \quad (l = 0, 1, \dots) ,$$

$$\psi'_{-l-1} = \sum_{k=l}^{\infty} \binom{k}{l} (-\varepsilon)^{k-l} \psi_{-k-1} \quad (l = 0, 1, \dots ,) .$$

d. gauge shift



$$h_I \to h'_I = h_I + \varepsilon_a \, \mathrm{sign}_a(I) \; , \qquad \mathrm{sign}_a(I) \equiv \begin{cases} +1 & (s(I)=a \; , \quad t(I) \neq a) \; , \\ -1 & (s(I) \neq a \; , \quad t(I) = a) \; , \\ 0 & (\text{otherwise}) \; , \end{cases}$$

consistent with **loop constraint:** $\displaystyle\sum_{I \in L} h_I = 0 \; ,$

$\to$ $E + 2I - 1$ parameters

$$\varphi^{a \Rightarrow b}(u) \to \varphi^{a \Rightarrow b\prime}(u) = \frac{\prod_{I \in \{b \to a\}} (u + h_I + \varepsilon_a \, \mathrm{sign}_a(I))}{\prod_{I \in \{a \to b\}} (u - h_I - \varepsilon_a \, \mathrm{sign}_a(I))} \; .$$

which reshuffles generators

e^a_m mixes w/ e^a_n $(n<m)$

To eliminate this ambiguity,

vertex constraint: $\displaystyle\sum_{I \in a} \mathrm{sign}_a(I) \, h_I = 0$

$\to$ 2 parameters

Quiver Yangian :

Representation

Representation by crystal melting [Li-MY '20], inspired by [FFJMM] and [Prochazka]

$$\psi^{(a)}(z)|\mathrm{K}\rangle = \Psi^{(a)}_{\mathrm{K}}(z)|\mathrm{K}\rangle \ ,$$

$$e^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Add}(\mathrm{K})} \frac{E^{(a)}(\mathrm{K}\to \mathrm{K}+\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}+\boxed{a}\rangle \ ,$$

$$f^{(a)}(z)|\mathrm{K}\rangle = \sum_{\boxed{a}\in \mathrm{Rem}(\mathrm{K})} \frac{F^{(a)}(\mathrm{K}\to \mathrm{K}-\boxed{a})}{z-h(\boxed{a})}|\mathrm{K}-\boxed{a}\rangle \ ,$$

poles in z

add/remove an atom

$$h(\boxed{a}) \equiv \sum_{I\in \mathrm{path}[\mathfrak{o}\to \boxed{a}]} h_I \ .$$

$$\Psi^{(a)}_{\mathrm{K}}(u) = \psi^{(a)}_0(z) \prod_{b\in Q_0} \prod_{\boxed{b}\in \mathrm{K}} \varphi^{b\Rightarrow a}(u-h(\boxed{b})) \ ,$$

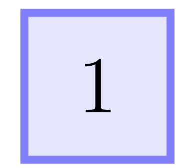
$$\varphi^{a\Rightarrow b}(u) \equiv \frac{\prod_{I\in\{b\to a\}}(u+h_I)}{\prod_{I\in\{a\to b\}}(u-h_I)}$$

In fact, we can "bootstrap" the algebra from this Ansatz

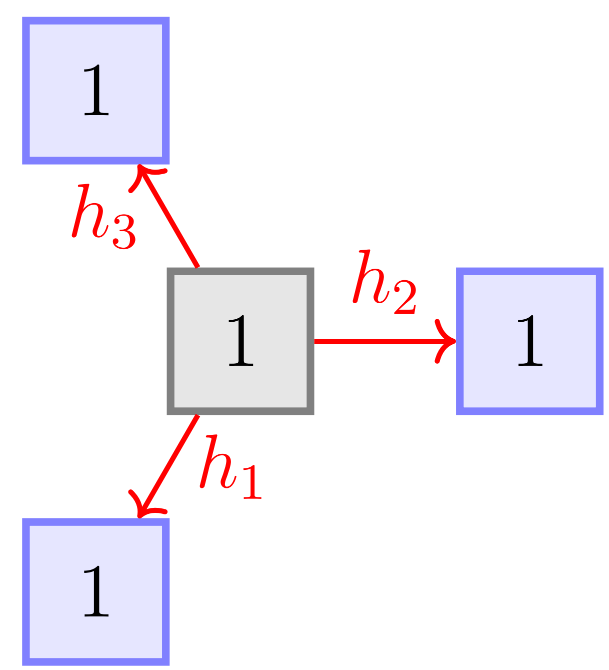
Crucial ingredient: poles keep track of the crystal structure

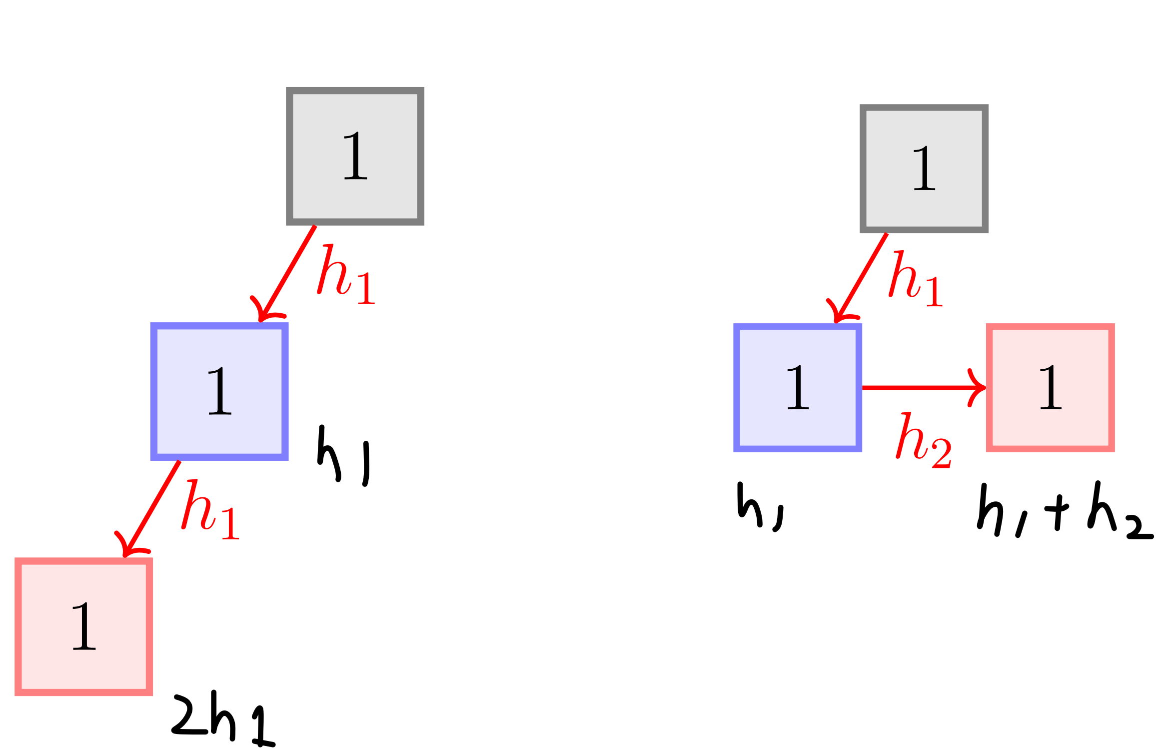
$\sum_a \psi_0^a$: centeral element

$$\Psi_\Lambda(z) = \psi_0(z) = \frac{z+C}{z}$$

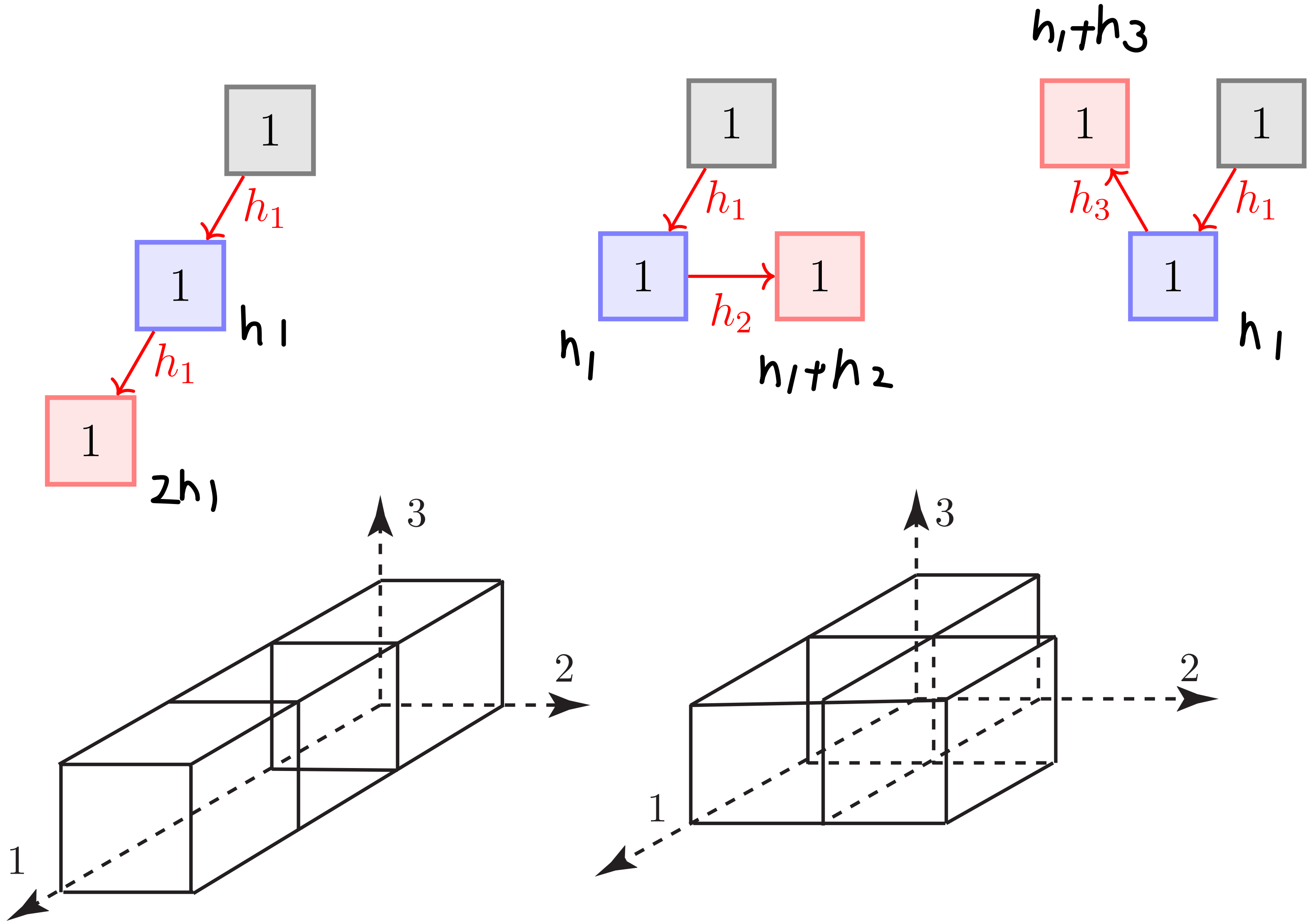


$$\Psi_\Lambda(z) = \psi_0(z)\psi_{\Box_0}(z)$$
$$= \frac{z+C}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}$$





$$\Psi_\Lambda(z) = \psi_0(z)\psi_{\square_0}(z)\psi_{\square_1}(z)$$
$$= \frac{z+C}{z} \cdot \frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)} \cdot \frac{z(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)(z-h_2-h_1)(z-h_3-h_1)}$$



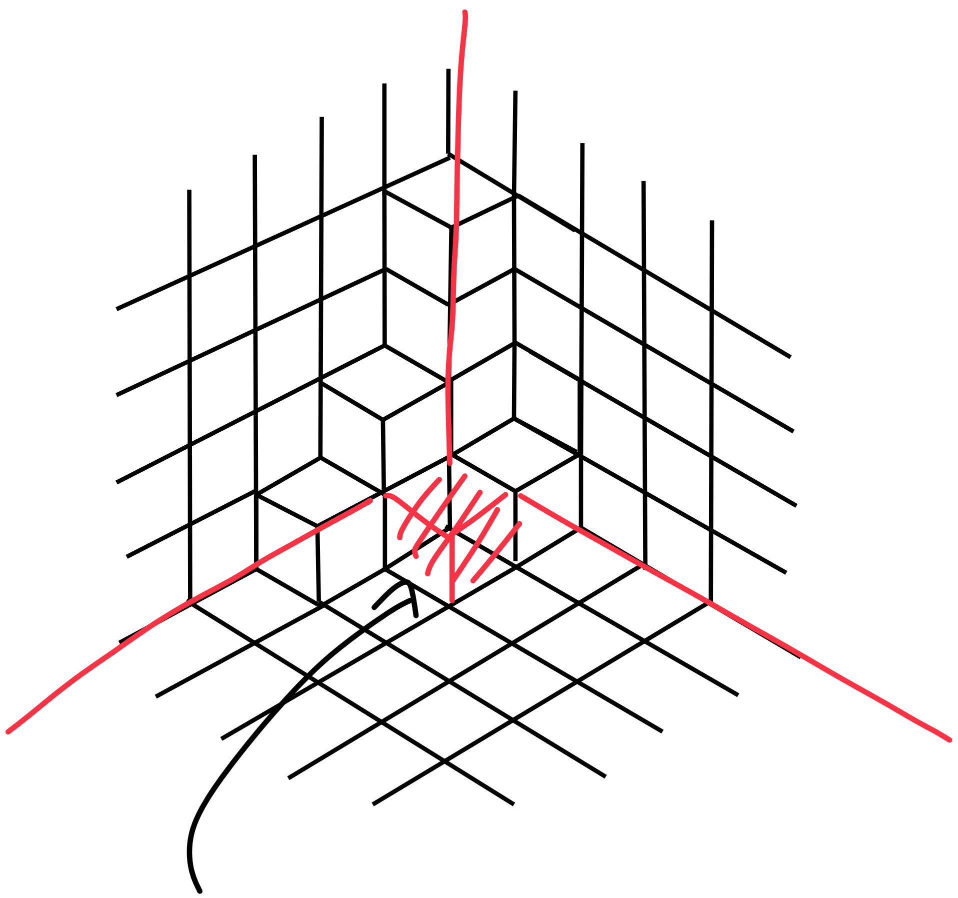
$(h_1+h_2+h_3=0)$

$$
\begin{aligned}
\Psi_\Lambda(z) &= \psi_0(z)\psi_{\square_0}(z)\psi_{\square_1}(z) \\
&= \frac{z+C}{z}\cdot\frac{(z+h_1)(z+h_2)(z+h_3)}{(z-h_1)(z-h_2)(z-h_3)}\cdot\frac{z(z+h_2-h_1)(z+h_3-h_1)}{(z-2h_1)(z-h_2-h_1)(z-h_3-h_1)}
\end{aligned}
$$

In general, loop constraint ensures that poles are in correct positions as dictated by the melting rule of the crystal

Truncations and D4-branes

For non-generic equivariant parameters, we have null states, so that the crystal truncates at the “pit”



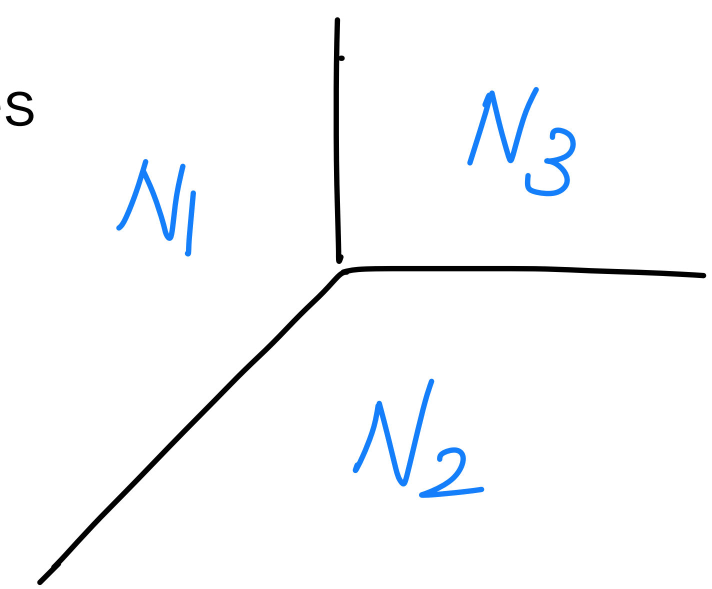
pit: location of null state

$$N_1 h_1 + N_2 h_2 + N_3 h_3 + C = 0$$

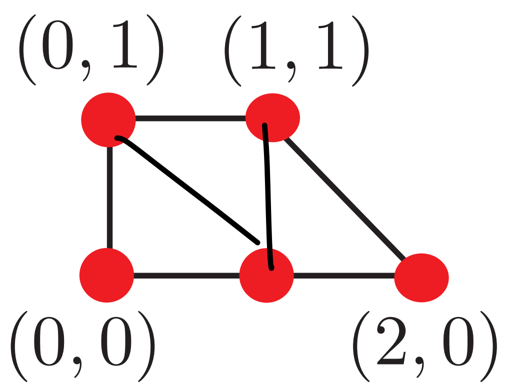
There is a corresponding truncation of the algebra studied by [Gaiotto-Rapcak] (also [Bershtein, Feigin, Merzon])

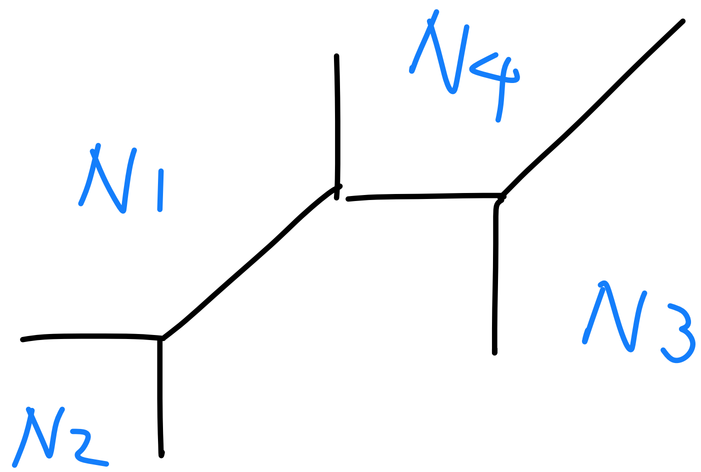
$$Y(\hat{gl}_1) \to Y_{N_1, N_2, N_3}$$

Physically: D4-branes



Generalization?





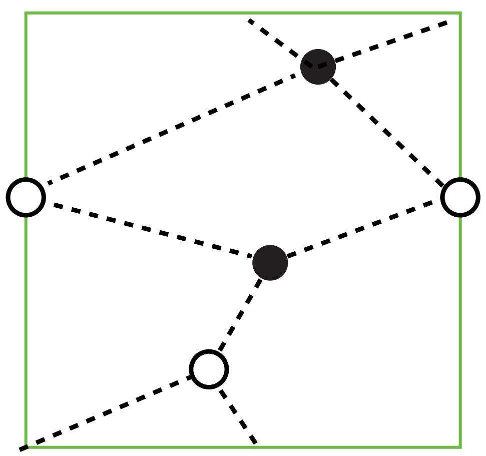
null state happens at

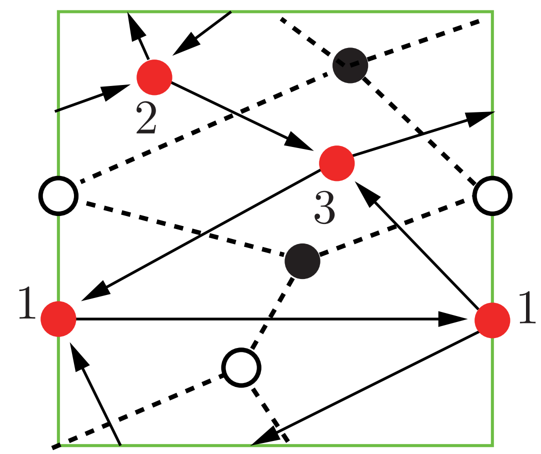
$$\sum_{I} M_I h_I + C = 0$$

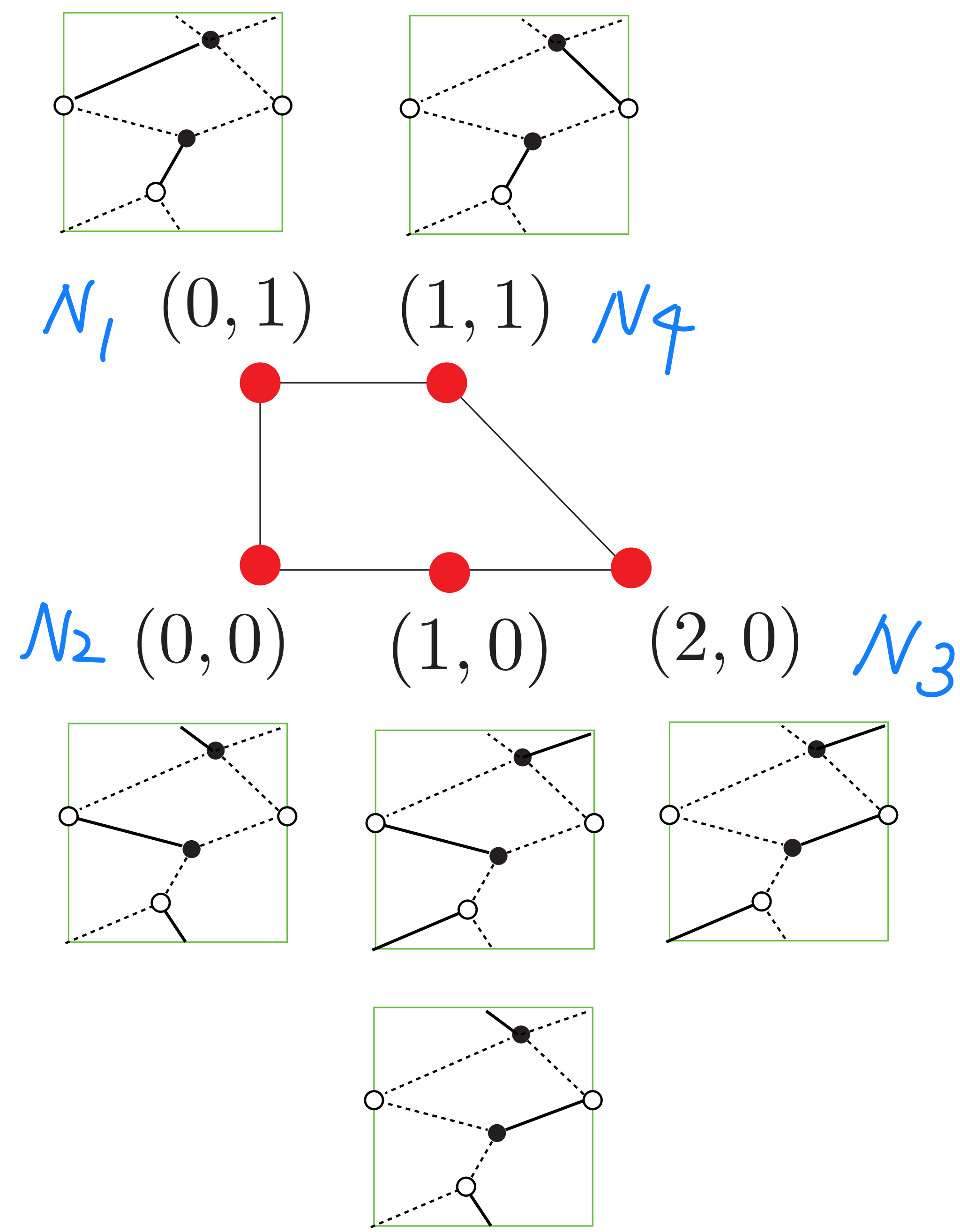
Which combination? $\{M_I\} \longleftrightarrow \{N_\alpha\}$

Answer given by perfect matchings [Li-MY]

Bipartite graph (brane tiling): dual of periodic quiver



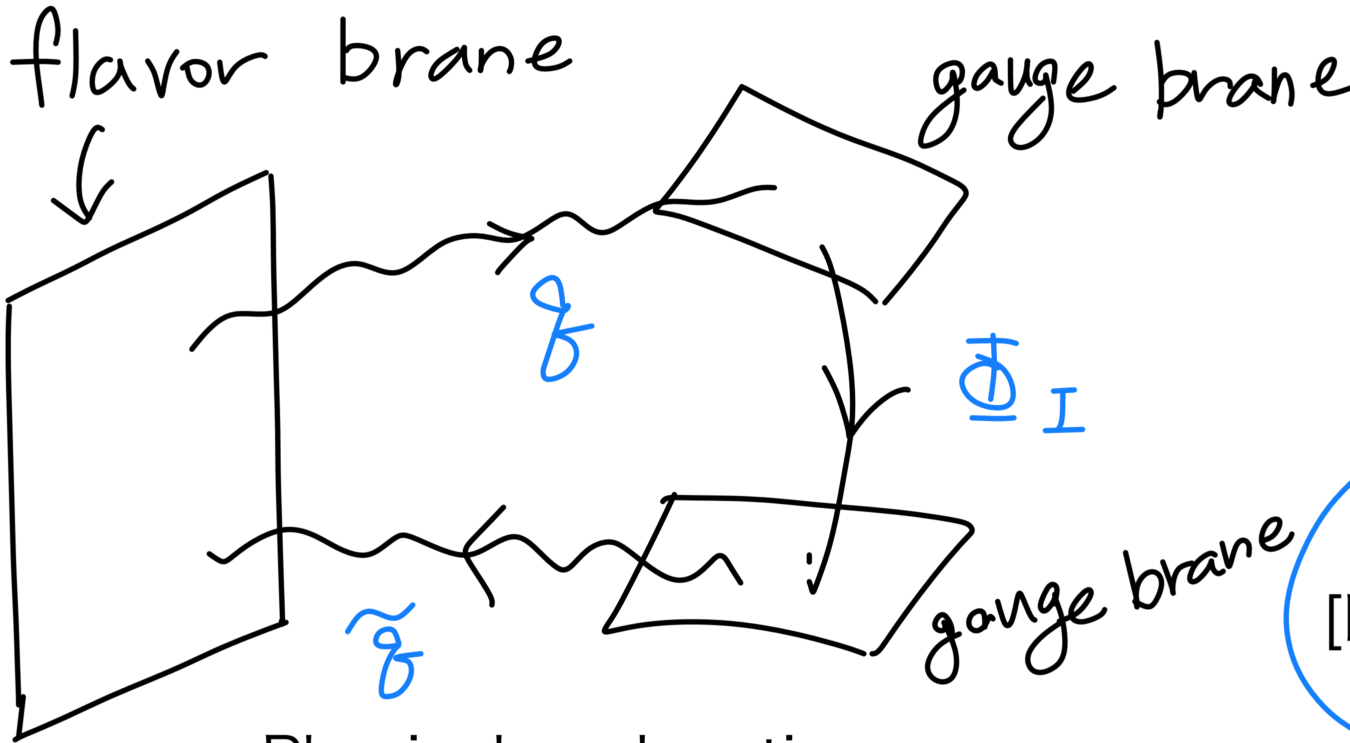




Perfect matching specifies which edges should be "eliminated"

$$\sum_p N_p \left(\sum_{I \in p} h_I \right) + C = 0 \ .$$

perfect matching



GLSM fields

studied in “forward algorithm”
[Franco-Hanany-Kennaway-Vegh-Wech

Physical explanation:
D4-brane = flavor brane, with extra superpotential

$$W = \tilde{q}\,\Phi_I\,q\,. \qquad \Phi_I = \prod_{p \ni I} \tilde{\Phi}_p\,.$$

(※ F-term relation
$\partial W = 0$ solved
in terms of $\tilde{\Phi}_p$)

This describes the divisor, $\Phi_I = 0$
represented by
regions filled by D4-branes
[Imamura-Kimura-Y]

Derivation from

Quantum Mechanics

[Galakhov-MY]

toric CY_3 : X

type IIA string theory

$$\mathbb{R}^{3,1} \times X$$
$$\cup$$
$$\mathbb{R} \times \{\text{hol. cycle}\}$$

BPS particles wrapping hol. cycle

$$Z^{X}_{BPS} = \sum_{\gamma} \underbrace{\Omega^{X}_{\gamma}(\cdots)}_{\text{BPS degeneracy}} q^{\gamma} \qquad \gamma \in H^{even}(X)$$

$$= Z_{crystal}$$ fixed point

SUSY QM of BPS particles

BPS quiver Yangian

Step 1: SQM and its equivariant cohomology

Vect mult at vertex

$(A_\mu, X^3_v, \bar{\Phi}_v)$

$\bar{\Phi}_v = X^1_v + i X^2_v$

We have the vacuum moduli space from supersymmetric quiver quantum mechanics (e.g. [Denef])

$$\mathcal{M}_{\rm SQM}: \quad \begin{array}{l} X^3_v \in \mathfrak{u}(n_v),\ \Phi_v \in \mathfrak{gl}(n_v,\mathbb{C}) \ , \\ q_{(a:\ v\to w)} \in \mathrm{Hom}(\mathbb{C}^{n_v},\mathbb{C}^{n_w}) \ , \end{array}$$

chiral mult. at edge

BPS Hilbert space [Witten]: $\mathcal{H}_{\rm BPS} \cong H^*_{\hat{G}}(\bar{Q}_{\rm i})$.

Supercharge [Galakhov-MY]

$$\bar{Q}_{\rm i} = e^{-\mathfrak{H}}\left(d_{X^3} + \bar{\partial}_{\Phi,q} + \iota_V + dW\wedge\right)e^{\mathfrak{H}} \ ,$$

$$\mathfrak{H} := \sum_{v\in\mathcal{V}} \mathrm{Tr}\, X^3_v \left(\frac{1}{2}\left[\Phi_v,\bar{\Phi}_v\right] - \mu_{\mathbb{R},v}\right) \ ,$$

$$V := \sum_{(a:\ v\to w)\in\mathcal{A}} \left(\Phi_w q_a - q_a\Phi_v\right)\frac{\partial}{\partial q_a} \ ,$$

$$\mu_{\mathbb{R},v} := \theta_v \mathbb{1}_{n_v\times n_v} - \sum_{x\in\mathcal{V}}\sum_{(a:v\to x)\in\mathcal{A}} q_a q_a^\dagger + \sum_{y\in\mathcal{V}}\sum_{(b:y\to v)\in\mathcal{A}} q_b^\dagger q_b \ .$$

stability param.

Step 2: Omega-deformation

We introduce Omega-deformation [Nekrasov,⋯]
to "smooth out" the singular geometry

$$V := \sum_{(a\colon v\to w)\in\mathcal{A}} (\Phi_w q_a - q_a \Phi_v) \frac{\partial}{\partial q_a} ,$$

$$\rightsquigarrow$$

$$V(q_a) = \sum_{(a\colon v\to w)\in\mathcal{A}} (\Phi_w q_a - q_a \Phi_v - \epsilon_a q_a) \frac{\partial}{\partial q_a} .$$

$$\bar{Q}_{\bar{1}}^2 = -4 \sum_{a\in\mathcal{A}} \epsilon_a \, \mathrm{Tr}\left(q_a \frac{\partial}{\partial q_a} \right) W = 0 .$$

The equivariant parameters should be consistent with W (loop constraint),
and hence can be identified with h_I introduced previously

$$\textbf{loop constraint:} \qquad \sum_{I\in L} h_I = 0 ,$$

Step 3: Higgs branch localization

1-parameter deformation of supercharge

Ω-bgd   UV scale   FI/stability

$$\bar{Q}_{\mathrm{i}}^{(\mathbf{s})} = e^{-\mathbf{s}\mathfrak{H}} \left(d_{X^3} + \bar{\partial}_{\Phi,q} + \iota_{\mathbf{s}V} + \mathbf{s}\, dW \wedge \right) e^{\mathbf{s}\mathfrak{H}} \ .$$

$$|\epsilon| \ll \Lambda_{\mathrm{cf}} \ll |\theta|^{\frac{1}{2}} \ .$$

$$X_i = \langle x_i \rangle + \mathbf{s}^{-\frac{1}{2}} x_i \ .$$

$$H = \mathbf{s} H_0 + O\left(\mathbf{s}^{\frac{1}{2}}\right) , \quad \bar{Q}_{\mathrm{i}}^{(s)} = \mathbf{s}^{\frac{1}{2}} \bar{Q}_{\mathrm{i}}^{(0)} + \bar{Q}_{\mathrm{i}}^{(1)} + O(\mathbf{s}^{-\frac{1}{2}}) \ .$$

$$H_0 \sim \sum_i \left(-\partial_{x_i}^2 + \omega_i^2 x_i^2\right) + \sum_i \omega_i \left(\psi_i \psi_i^\dagger - \psi_i^\dagger \psi_i\right) , \quad \bar{Q}_{\mathrm{i}}^{(0)} \sim \sum_i \psi_i \left(\partial_{x_i} + \omega_i x_i\right) \ .$$

Wilsonian decomposition of wave function

$$\Psi = \Psi_{|\omega|<\Lambda_{\mathrm{cf}}} \left(x_{|\omega|<\Lambda_{\mathrm{cf}}}\right) \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \left(x_{|\omega|<\Lambda_{\mathrm{cf}}}, x_{|\omega|>\Lambda_{\mathrm{cf}}}\right) + O(\mathbf{s}^{-1}) \ .$$

$$\left(\bar{Q}_{\mathrm{i}}^{(0)}\right)^\dagger \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} = \bar{Q}_{\mathrm{i}}^{(0)} \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} = 0 \ .$$

$$Q_{\mathrm{eff}}^\dagger \Psi_{|\omega|<\Lambda_{\mathrm{cf}}} = 0, \quad Q_{\mathrm{eff}}^\dagger := \left\langle \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \middle| \bar{Q}_{\mathrm{i}}^{(1)} \middle| \Psi_{|\omega|>\Lambda_{\mathrm{cf}}} \right\rangle \ .$$

Choose a basis such that the gauge action V is diagonal:

$$V = \sum_i w_i \, m_i \frac{\partial}{\partial m_i} \ ,$$

We can then solve for the effective wavefunction as

$$Q_{\text{eff}} \Psi_\Lambda = Q_{\text{eff}}^\dagger \Psi_\Lambda = 0 \ . \qquad Q_{\text{eff}}^\dagger = \sum_i \left(d\bar{m}_i \, \partial_{\bar{m}_i} + w_i m_i \, \iota_{\partial/\partial_{m_i}} \right) \ .$$

$$\Psi_\Lambda = \left(\prod_i \left(w_i - |w_i| \, \bar{\psi}_{1,i} \psi_{2,i} \right) e^{-|w_i||m_i|^2} \right) \prod_i \bar{\psi}_{\dot{2},i} |0\rangle \ .$$

$$= \left(\prod_i w_i \right) \prod_i \bar{\psi}_{\dot{2},i} |0\rangle + \left(Q_{\text{eff}}^\dagger\text{-exact term} \right) \ .$$

to find the Euler class

$$\boxed{\Psi_\Lambda \sim \text{Eul}_\Lambda := \prod_i w_i \ .}$$

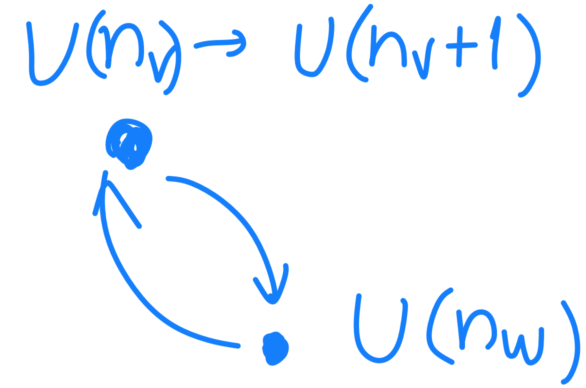
$$\int \Psi_\Lambda = 1 \ , \quad \int \Psi_\Lambda \wedge \Psi_{\Lambda'} = \text{Eul}_\Lambda \, \delta_{\Lambda,\Lambda'} \ .$$

Step 4: Hecke modification

Raising/lowering operators of the algebra obtained by “Hecke modification”

$\hat{e}$ $\hat{f}$ shifting the dimension vectors at the quiver nodes:

$$n'_v = n_v \pm 1, \quad \text{and} \quad n'_w = n_w, \ \text{for } w \neq v \ .$$



Define generators

$$\hat{e}^{(v)}(z) := \left[\mathrm{Tr}\,(z-\Phi_v)^{-1}, \hat{\mathbf{e}}\right] \ ,$$

$$\hat{f}^{(v)}(z) := -\left[\mathrm{Tr}\,(z-\Phi_v)^{-1}, \hat{\mathbf{f}}\right] \ .$$

and its action on crystal configurations is

$$\hat{e}^{(v)}(z)|\Lambda\rangle = \sum_{\substack{\square\in\Lambda^+ \\ \hat{\square}=v}} \frac{1}{z-\phi_\square} \times \hat{E}(\Lambda\to\Lambda+\square)|\Lambda+\square\rangle \ ,$$

$$\hat{f}^{(v)}(z)|\Lambda\rangle = \sum_{\substack{\square\in\Lambda^- \\ \hat{\square}=v}} \frac{1}{z-\phi_\square} \times \hat{F}(\Lambda\to\Lambda-\square)|\Lambda-\square\rangle \ .$$

$$\hat{\psi}^{(v)}(z)|\Lambda\rangle = \hat{\psi}^{(v)}_\Lambda(z) \times |\Lambda\rangle \ .$$

Need

$$\hat{E}(\Lambda\to\Lambda+\square) := \frac{\langle\Psi_{\Lambda+\square}|\hat{\mathbf{e}}|\Psi_\Lambda\rangle}{\langle\Psi_{\Lambda+\square}|\Psi_{\Lambda+\square}\rangle}$$

$$\hat{F}(\Lambda\to\Lambda-\square) := \frac{\langle\Psi_{\Lambda-\square}|\hat{\mathbf{f}}|\Psi_\Lambda\rangle}{\langle\Psi_{\Lambda-\square}|\Psi_{\Lambda-\square}\rangle}$$

The correct formula:

$$\hat{\mathbf{e}}\,\Psi_\Lambda = \sum_{\square\in\Lambda^+} \frac{\mathrm{Eul}_\Lambda}{\mathrm{Eul}_{\Lambda,\Lambda+\square}} \Psi_{\Lambda+\square}\ .$$

$$\hat{\mathbf{f}}\,\Psi_\Lambda = \sum_{\square\in\Lambda^-} \frac{\mathrm{Eul}_\Lambda}{\mathrm{Eul}_{\Lambda-\square,\Lambda}} \Psi_{\Lambda-\square}\ .$$

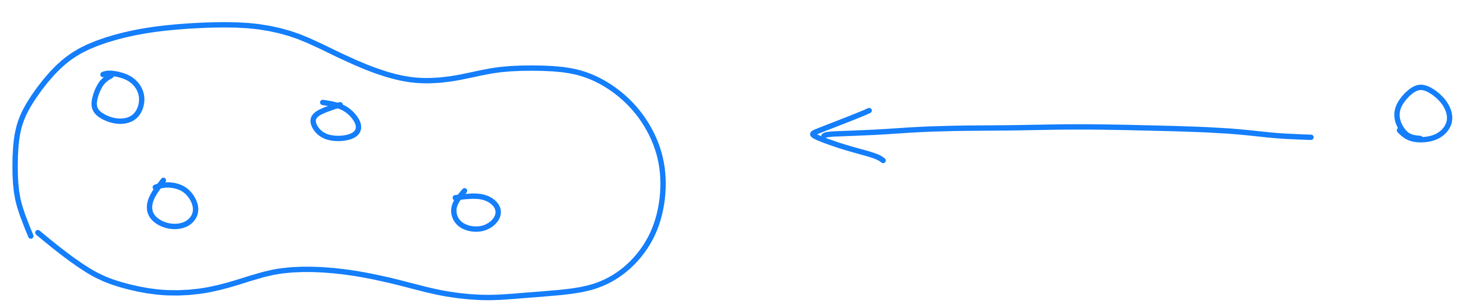
$$\mathcal{M}_\Sigma \times \mathcal{M}_{\Sigma+\square}$$

$$I_1 \supset I_2$$

Mathematically, this is derived by the Fourier-Mukai transform with the incident locus as a kernel [Nakajima,⋯]

Physically, we need to bring in particles from infinity. Along the process Some low-frequency modes get exchanged with high-frequency modes

BPS bound state



$$\Psi = \Psi_{|\omega|<\Lambda_{\mathrm{cf}}}\left(x_{|\omega|<\Lambda_{\mathrm{cf}}}\right)\Psi_{|\omega|>\Lambda_{\mathrm{cf}}}\left(x_{|\omega|<\Lambda_{\mathrm{cf}}}, x_{|\omega|>\Lambda_{\mathrm{cf}}}\right) + O(\mathbf{s}^{-1})\ .$$

Highly non-trivial cancellations!

For example, for one of the Serre relations of $Y(\widehat{\mathfrak{gl}}_{3|1})$

$$\mathrm{Sym}_{z_1,z_2}\left[e^{(2)}(z_1),\left[e^{(3)}(w_1),\left[e^{(2)}(z_2),e^{(1)}(w_2)\right\}\right\}\right\}$$

$$A_2 := \operatorname*{Res}_{z_1,z_2,w_1,w_2}\langle\Lambda|A_1|\Lambda_0\rangle =$$

$$= [1,2,4,3]+[1,3,4,2]-[2,1,3,4]+[2,1,4,3]-[2,3,1,4]+[2,4,1,3]+$$
$$+[2,4,3,1]-[3,1,2,4]+[3,1,4,2]-[3,2,1,4]+[3,4,1,2]+[3,4,2,1]-$$
$$-[4,1,2,3]-[4,1,3,2]-[4,2,1,3]-[4,3,1,2] = 0\ !$$

$$[2,4,1,3]=-\frac{1}{48}\,,\quad [4,2,1,3]=-\frac{1}{96}\,,\quad [2,1,4,3]=-\frac{1}{48}\,,\quad [1,2,4,3]=\frac{1}{32}\,,$$
$$[4,1,2,3]=\frac{1}{64}\,,\quad [1,4,2,3]=\frac{1}{64}\,,\quad [4,1,3,2]=-\frac{1}{64}\,,\quad [1,4,3,2]=-\frac{1}{64}\,,$$
$$[2,4,3,1]=\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+\hbar_2)}\,,\quad [4,2,3,1]=\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,$$
$$[2,3,4,1]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,4,1]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,2,1]=-\frac{2\hbar_1+\hbar_2}{48\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,2,1]=-\frac{(2\hbar_1+\hbar_2)^2}{24\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,1,3,4]=-\frac{2\hbar_1+\hbar_2}{24\,(4\hbar_1+3\hbar_2)}\,,\quad [1,2,3,4]=\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,$$
$$[2,3,1,4]=\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,\quad [3,2,1,4]=-\frac{(2\hbar_1+\hbar_2)^2}{12\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,2,4]=-\frac{2\hbar_1+\hbar_2}{16\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,2,4]=\frac{(2\hbar_1+\hbar_2)^2}{8\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[4,3,1,2]=\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+\hbar_2)}\,,\quad [3,4,1,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,,$$
$$[1,3,4,2]=-\frac{2\hbar_1+\hbar_2}{32\,(4\hbar_1+3\hbar_2)}\,,\quad [3,1,4,2]=\frac{(2\hbar_1+\hbar_2)^2}{16\,(4\hbar_1+\hbar_2)\,(4\hbar_1+3\hbar_2)}\,.$$

Summary

- BPS/DT/PT counting for toric CY3: solved by crystal melting
- We defined a new algebra, the BPS quiver Yangian, in terms of CY3 data
- We have a well-defined representation of quiver Yangian in terms of crystal melting
- The representation is derived by equivariant localization in supersymmetric quantum mechanics

New Physics and new Mathematics!!