Math-113, Homework 9, non textbook problems

A. Let $A = \{x, y\}$ and $B = \{z, w\}$ be two alphabets.

- Consider a group homomorphism
$$\phi : F[A] \to F[B], \qquad x \mapsto z^{-1}w^2, \quad y \mapsto w^{-1}z.$$
Prove that ϕ is an isomorphism. *Hint: You can find an inverse*
- Consider a group homomorphism
$$\psi : F[A] \to F[B], \quad x \mapsto z^{-1}w^2, \quad y \mapsto z.$$
Prove that ψ is not surjective. Is it injective?
Hint: For the last question determine if $F[A]$ is isomorphic to $\psi(F[A])$.

B*.(Optional, not for the grade) Consider two groups given by generators and relations
$$G_1 = \langle r, s \,|\, r^n = s^2 = srsr = 1\rangle, \qquad G_2 = \langle s_1, s_2 \,|\, s_1^2 = s_2^2 = (s_1s_2)^n = 1\rangle.$$

- List all elements of G_2.
- Prove that $G_2 \simeq D_{2n}$. What is the geometric meaning of s_1, s_2?
- Prove that $G_1 \simeq G_2$.

C. Let A be an abelian group. Denote by
$$\mathcal{H} = \{\phi : A \to A \;\big|\; \phi \text{ is a group homomorphism}\}$$
the collection of all homomorphisms from A to itself. Define two binary operations $+, * : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ as follows: For each $f, g \in \mathcal{H}$
$$(f + g) : A \to A, \qquad (f + g)(a) = f(a) + g(a) \tag{1a}$$
$$(f * g) : A \to A, \qquad (f * g)(a) = f(g(a)) \tag{1b}$$

- Prove that both operations are well-defined, namely that $f + g$ and $f * g$ are always group homomorphisms.
- Prove that $(\mathcal{H}, +, *)$ is a ring.
- ~~Is it a division ring? Prove or provide a counterexample.~~ (will be left for next HW)

Remark 1: The above ring is called the *endomorphism* ring of A and denoted by $End(A)$. A well known example of such rings comes from linear algebra, when A is a vector space (which is an abelian group w.r.t. addition).

D. Let A be an abelian group. Now let
$$\mathcal{F} = \{f : A \to A\}$$
be the collection of all maps from A to itself. Define binary operations $+, * : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ as in (1).

- Prove that (R1) is satisfied, i.e. that $(\mathcal{F}, +)$ is an Abelian group. What is an identity of $(\mathcal{F}, +)$?
- Prove that (R2) is satisfied, i.e. that $(\mathcal{F}, *)$ is associative.
- Exactly one of the two distributive laws in (R3) fails in general. Determine which distributive law (left or right) holds for $(\mathcal{F}, +, *)$. Prove one distributive law and provide a counterexample to the other.