Math-113, Homework 7, non-textbook problems

A. Prove that the direct product $X = G_1 \times G_2$ of two groups satisfy the so-called *universal property*:

$$\begin{array}{l}\text{For every group } W \text{ and pair of group homomorphisms } \phi_1 : W \to G_1 \text{ and } \phi_2 : W \to G_2 \\ \text{there exists a unique group homomorphism } \phi : W \to X \text{ s.t. } \phi_1 = \pi_1 \circ \phi \text{ and } \phi_2 = \pi_2 \circ \phi.\end{array} \tag{1}$$

$$\begin{tikzcd}
W \arrow[dr, dashed, "\phi"] \arrow[drr, bend left, "\phi_2"] \arrow[ddr, bend right, "\phi_1"'] & & \\
& X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & G_2 \\
& G_1 &
\end{tikzcd}$$

Hint: First construct a function ϕ and prove that it is unique. Then show that the function ϕ, which you have constructed satisfy homomorphism property.

B*. (optional, not for the grade) Prove that if there exists another group Y together with a pair of maps $\pi_1' : Y \to G_1$ and $\pi_2' : Y \to G_2$, s.t. the triple (Y, π_1', π_2') satisfy universal property (1), then $Y \simeq X$.

Remark. Note that the statement of the Problem B implies that the group X satisfying universal property (1) is unique up to isomorphism. This allows one to use (1) as an alternative definition of a direct product of groups. Of course, we will not use this approach in the current course, however, it is worth noting that such definition of a "product" can be used in many areas of mathematics well beyond the framework of groups and group homomorphisms.