Math-113, Homework 13, non-textbook problems

The following set of problems is a continuation of exercise 26.30. Please complete textbook exercise 26.30 first. Below we denote by Nil(X), the nilradical of a ring X.

A. Let R be a commutative ring. For every ideal $I \subseteq R$ define the **radical of** I, denoted as $\sqrt{I}$, to be the set

$$\sqrt{I} = \{r \,:\, r^k \in I, \text{ for some } k \in \mathbb{N}\}.$$

- Show that $\sqrt{I}$ is an ideal.
- Show that $I \subseteq \sqrt{I}$.
- Show that $\mathrm{Nil}(R) = \sqrt{(0)}$. Here (0) stands for the zero ideal of R.
- Let I_1, I_2 be ideals, is it true that $\sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$? Prove or provide a counterexample.

B. An ideal I is said to be **radical** if $I = \sqrt{I}$. Let $R = \mathbb{C}[x]$ be the ring of polynomials in one variable with complex coefficients.

- Find at least one ideal $I \subseteq R$, which is not radical.
- For the example constructed above, is it true that R/I has zero divisors? Give an example of a zero divisor, of prove that there are none.
- Consider the quotient ring $\sqrt{I}/I$ for the example you have constructed. This quotient ring is, in particular, a vector space over $\mathbb{C}$. Is it finite dimensional? If yes, find a basis and describe multiplication of basis elements.

C*. (Optional, not for the grade) Show that for every ideal $I \subseteq R$ in a commutative ring R we have

$$\sqrt{I}/I \cong \mathrm{Nil}(R/I).$$