Math-113, Homework 12, non-textbook problems

A. Let p be a prime and $f(x) \in \mathbb{Z}_p[x]$ be a polynomial in one variable over $\mathbb{Z}_p$. Prove that for all $t \in \mathbb{Z}_p$, we have

$$(f(t))^p = f(t^p).$$

B. Let p be a prime and suppose that $\mathbb{F}$ is a finite field with p^d elements, where $d \in \mathbb{N}$. Prove that

$$x^{p^d} - x = \prod_{b \in \mathbb{F}} (x - b). \tag{1}$$

where the product is taken over all elements of $\mathbb{F}$. *Hint: Prove that every $b \in \mathbb{F}$ is a root of the l.h.s. of (1)*

Remark. Finite fields with p^d elements do exist for all primes p and all natural numbers $d \in \mathbb{N}$. When $d = 1$ we already familiar with fields $\mathbb{Z}_p$ coming from modular arithmetic. As for the case $d > 1$, the explicit construction of such fields is more complicated and is usually the subject of the second course in Abstract Algebra.