Math-113, Homework 11, non-textbook problems

A. Recall that $\mathbb{Z}_{mn} \simeq Z_m \times \mathbb{Z}_n$ as groups if and only if m and n are relatively prime. Now, prove that $\mathbb{Z}_{mn} \simeq Z_m \times \mathbb{Z}_n$ as rings if and only if m and n are relatively prime.

B. Let p be a prime and $k \in \mathbb{N}$ be a natural number. Find the order $|\mathbb{Z}_{p^k}^{\times}|$ of the group of units.
Hint: It is actually easier to calculate the number of zero divisors in $\mathbb{Z}_{p^k}$ first.

C. For each of the four rings R below determine the corresponding group of units $R^{\times}$ and classify it according to the Fundamental Theorem of Finitely Generated Abelian Groups. In all cases justify your answer.

- $\mathbb{Z}_{15}$
- $\mathbb{Z}_{16}$
- $\mathbb{Z}_{20}$
- $\mathbb{Z}_{30}$

D*. (Optional, not for the grade) Find an example of a prime p and a natural number $k \in \mathbb{N}$, for which the group of units $\mathbb{Z}_{p^k}^{\times}$ is not cyclic. *(There is actually only one prime number, for which this can happen.)*