Math-113, Homework 10, non-textbook problems

A. (*This is a continuation of Problem A from HW9*) Let A be an abelian group. Denote by

$$\mathcal{H} = \{\phi : A \to A \mid \phi \text{ is a group homomorphism}\}$$

the collection of all homomorphisms from A to itself. Define two binary operations $+, * : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ as follows: For each $f, g \in \mathcal{H}$

$$(f+g) : A \to A, \qquad (f+g)(a) = f(a) + g(a)$$
$$(f*g) : A \to A, \qquad (f*g)(a) = f(g(a))$$

Determine wheter $\mathcal{H}$ is a division ring. Prove or provide counterexample.

B. Consider the set of complex integers

$$\mathbb{Z}[\mathbf{i}] = \{n + m\mathbf{i} \mid n, m \in \mathbb{Z}\} \qquad \subset \quad \mathbb{C}$$

- Prove that $\mathbb{Z}[i]$ is a ring. This ring is called the *Gaussian Integers.*
- Prove that the group of units

  $$(\mathbb{Z}[\mathbf{i}])^{\times} = \{1, -1, \mathbf{i}, -\mathbf{i}\}$$

  has exactly four elements.
  Hint: Recall from HW1 how $|x + \mathbf{i}y| = \sqrt{x^2 + y^2}$ *behaves w.r.t. multiplication of complex numbers.*
- Is the group of units $((\mathbb{Z}[\mathbf{i}])^{\times}, \times)$ abelian? Determine its isomorphism class.
- Prove that $1 + \mathbf{i}$ is a "Gaussian prime", namely that whenever for $a, b \in \mathbb{Z}[\mathbf{i}]$

  $$ab = 1 + \mathbf{i} \qquad \Rightarrow \qquad \text{either } a \text{ is a unit, or } b \text{ is a unit.}$$

C*. (Optional, not for the grade) Recall that the group of units in integers is

$$\mathbb{Z}^{\times} = \{1, -1\} \simeq \mathbb{Z}_2,$$

similarly

$$(\mathbb{Z}[\mathbf{i}])^{\times} = \{1, -1, \mathbf{i}, -\mathbf{i}\} \simeq \ldots$$

What will be the group of units for integer quaternions?