Math-113, Homework 1, non-textbook problems

A. Let $\mathbb{C} = \mathbb{R}^2$ denote the set of vectors on a two-dimensional plane. Every element $[x, y] \in \mathbb{C}$ can be described by its cartesian coordinates x and y. Define a binary operation $\star : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ as follows:

$$[a, b] \star [c, d] = [ac - bd, ad + bc]$$

for all $z_1 = [a, b] \in \mathbb{C}$ and $z_2 = [c, d] \in \mathbb{C}$. Prove that the binary operation $\times$ satisfies the following properties:

- $z_1 \star z_2 = z_2 \star z_1$ for all $z_1, z_2 \in \mathbb{C}$. In other words, this binary operation is commutative.
- $(z_1 \star z_2) \star z_3 = z_1 \star (z_2 \star z_3)$ for all $z_1, z_2, z_3 \in \mathbb{C}$. In other words, this binary operation is associative.

Remark 1. Operation $\star$ defined above is called *multiplication of complex numbers* and elements $[x, y] \in \mathbb{C}$ are commonly written as

$$[x, y] = x + \mathbf{i} \cdot y, \tag{1}$$

where $\mathbf{i}$ is the so-called imaginary unit (See more details in Section 1 of the textbook).

B. Let $\mathbb{C}$ be the set of complex numbers as before. Now consider a pair of binary operations: $\star : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined in exercise A, together with the usual addition of vectors in the plane

$$+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}, \qquad (a + \mathbf{i} \cdot b) + (c + \mathbf{i} \cdot d) = (a + c) + \mathbf{i} \cdot (b + d).$$

Prove the following statements by direct computation in coordinates:

- $(0 + \mathbf{i} \cdot 1) \star (0 + \mathbf{i} \cdot 1) = -1 + \mathbf{i} \cdot 0$
- $(z_1 + z_2) \star z_3 = z_1 \star z_3 + z_2 \star z_3 \qquad$ for all $z_1, z_2, z_3 \in \mathbb{C}$

Remark 2. The last property you have just proved is called distributivity. We will return to the example of complex numbers later, once we define the notion of a *ring*.

C. Let $\Phi : \mathbb{C} \to \mathbb{C}$ be a map defined as

$$\Phi(a + \mathbf{i} \cdot b) = a + \mathbf{i} \cdot (-b).$$

Prove that Φ is an isomorphism of a binary structure $(\mathbb{C}, \star)$ with itself.

Remark 3. An isomorphism (of a binary structure) with itself is called *automorphism* (of a binary structure).

D. Let $\Psi : S \to S'$ be an isomorphism between binary structures $(S, \star)$ and $(S', \star')$. Prove that Ψ^{-1} satisfies the homomorphism property.