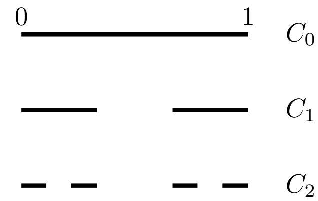
Math-104, Homework 11 (Extra credit)

Problem 1. Define a sequence of sets C_n as follows: Let $C_0 = [0, 1]$ be a closed interval of the real line from 0 to 1. Define C_1 by deleting the open interval of length 1/3 in the middle



$$C_1 = C_0 \backslash \left(\frac{1}{3}, \frac{2}{3}\right) = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

Then define a sequence of sets recursively. On each step, suppose that C_n is a disjoint union of $k = 2^n$ closed intervals:

$$C_n = [a_1, b_1] \sqcup [a_2, b_2] \sqcup \cdots \sqcup [a_k, b_k],$$

Let C_{n+1} be obtained by deleting the middle one third from each of the closed intervals:

$$C_{n+1} = \left[a_1, \frac{2a_1 + b_1}{3}\right] \sqcup \left[\frac{a_1 + 2b_1}{3}, b_1\right] \sqcup \cdots \sqcup \left[a_k, \frac{2a_k + b_k}{3}\right] \sqcup \left[\frac{a_k + 2b_k}{3}, b_k\right].$$

For example $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, e.t.c.

Denote by $\mathcal{C}$ the intersection of all C_n

$$\mathcal{C} = \bigcap_{n=0}^{\infty} C_n,$$

i.e. the set of points in $\mathbb{R}$ which belong simultaneously to all C_n. Let $f : [0, 1] \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 1, & x \in \mathcal{C}, \\ 0, & x \notin \mathcal{C}. \end{cases}$$

- Prove that $3^{-k} \in \mathcal{C}$ for all $k \in \mathbb{N}$. In particular, this implies that $\mathcal{C}$ has infinitely many elements.
- Prove that $f(x)$ is discontinuous at all $x \in \mathcal{C}$.
- Prove that $f(x)$ is integrable on $[0, 1]$ and find the value of

$$\int_0^1 f(x)\mathrm{d}x$$