Poisson geometry of large quantum groups

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The big quantum group $U_\epsilon(\mathfrak{g})$ is the Hopf algebra with the same generators as $U_q(\mathfrak{g})$ and relations in which q is replaced by ϵ.

Generators $K_i^{\pm 1}, E_i, F_i$, $1 \leq i \leq n$. Relations: e.g. for $U_\epsilon(\mathfrak{sl}(2))$:

$$KEK^{-1} = \epsilon^2 E, \quad KFK^{-1} = \epsilon^{-2} F, \quad EF - FE = \frac{K - K^{-1}}{\epsilon - \epsilon^{-1}}.$$

Big quantum groups II

Punchline:

- The big quantum group $U_\epsilon(\mathfrak{g})$ has the central subalgebra

$$Z_\epsilon(\mathfrak{g}) := \mathbb{C}[E_{\beta_1}^N, \ldots, E_{\beta_\ell}^N, F_{\beta_1}^N, \ldots, F_{\beta_\ell}^N][K_1^{\pm N}, \ldots, K_n^{\pm N}],$$

  where $E_{\beta_1}, \ldots, E_{\beta_\ell}, F_{\beta_1}, \ldots F_{\beta_\ell}$ are the root vectors of $U_\epsilon(\mathfrak{g})$, and
- $U_\epsilon(\mathfrak{g})$ is a finitely generated, free $Z_\epsilon(\mathfrak{g})$-module.

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The small quantum group of Lusztig is

$$u_\zeta(\mathfrak{g}) = U_\zeta(\mathfrak{g})/\mathfrak{m}_0 U_\zeta(\mathfrak{g}),$$

where $\mathfrak{m}_0 := (E^N_{\beta_j}, F^N_{\beta_j}, K^N_i - 1, \forall i, j) \in \mathrm{MaxSpec} Z_\zeta(\mathfrak{g}) \cong \mathbb{C}^{2\ell} \times (\mathbb{C}^*)^n$.

Bundle of algebras over $\mathrm{MaxSpec} Z_\epsilon(\mathfrak{g})$ with fibers $U_\epsilon(\mathfrak{g})/\mathfrak{m} U_\epsilon(\mathfrak{g})$ for maximal ideals $\mathfrak{m}$ of $Z_\epsilon(\mathfrak{g})$. Lusztig's small quantum group is one fiber.

Orders

General picture (Procesi, Artin, Kaplansky, Posner): A is a $\mathbb{C}$-algebra, which is module-finite over a central subalgebra Z.

$\mathrm{Irr}(A)$=set of irreducible (finite dimensional) reps of A.

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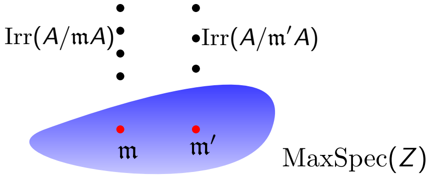
Problem. Describe he fibers of ψ, which are $\psi^{-1}(\mathfrak{m}) = \mathrm{Irr}(A/\mathfrak{m}A)$.



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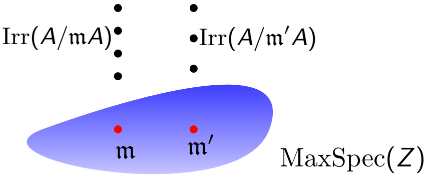

If A is a free Z-module, we have a bundle of algebras over $\mathrm{MaxSpec}(Z)$ with fibers $A/\mathfrak{m}A$. [Not needed ingredient].

Poisson orders

Definition. A $\mathbb{C}$-algebra A, which is module-finite over a central subalgebra Z is called a Poisson order if

1. Z is equipped with the structure of a Poisson algebra and
2. $\exists$ a linear map $\partial : Z \to \mathrm{Der}_{\mathbb{C}}(A)$ such that ∂_z extends the Hamiltonian derivation $\{z, -\}$ of Z for all $z \in Z$.

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Theorem [Brown–Gordon, De Concini–Kac–Procesi]

Let (A, Z) be a Poisson order.
For every $\mathfrak{m}, \mathfrak{m}'$ in the same symplectic leaf of $\mathrm{MaxSpec} Z$,

$$A/\mathfrak{m}A \cong A/\mathfrak{m}'A.$$

In particular, there is a dimension-preserving bijection between the corresponding fibers of ψ:

$$\mathrm{Irr}(A/\mathfrak{m}A) \leftrightarrow \mathrm{Irr}(A/\mathfrak{m}'A).$$

Specialization

An algebra A is obtained by specialization, if there is a $\mathbb{C}$-algebra R and a central element $\hbar \in R$ that is not a zero divisor, such that

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Then $(A, \mathcal{Z}(A))$ has a canonical structure of a Poisson order. In terms of the projection $\pi : R \to R/\hbar R \cong$ and a linear section $\iota : A \to R$:

- The Poisson structure on Z is $\{z_1, z_2\} = \pi\left(\frac{[\iota(z_1), \iota(z_2)]}{\hbar}\right)$
- The linear map $\partial : Z \to \mathrm{Der}_{\mathbb{C}}(A)$ is $\partial_z(a) = \pi\left(\frac{[\iota(z_1), \widetilde{a}]}{\hbar}\right)$, where $\widetilde{a} \in \pi^{-1}(a)$.

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In our applications $\hbar = q - \epsilon$ for some $\epsilon \in \mathbb{C}$ and the algebra R is over $\mathbb{C}[q^{\pm 1}][\prod(q - c_j)^{-1}]$, $c_j \in \mathbb{C}$, $c_j \neq \epsilon$.

However, $\mathcal{Z}(A)$ is usually singular and we need to construct a large smooth Poisson subalgebra $Z \subset \mathcal{Z}(A)$ turning (A, Z) into a Poisson order.

The De Concini–Kac–Procesi work

De Concini–Kac–Procesi proved the following:

1. $U_\epsilon(\mathfrak{g})$ is a specialization, so $(U_\epsilon(\mathfrak{g}), \mathcal{Z}(U_\epsilon(\mathfrak{g})))$ is a Poisson order.
2. $Z_\epsilon(\mathfrak{g})$ is a Poisson subalgebra of $\mathcal{Z}(U_\epsilon(\mathfrak{g}))$.
3. $Z_\epsilon(\mathfrak{g})$ is isomorphic to the coordinate ring of the dual Poisson algebraic group of a simple algebraic group G with the standard Poisson structure $\Rightarrow$ Invariance of $U_\epsilon(\mathfrak{g})/\mathfrak{m}U_\epsilon(\mathfrak{g})$ along conjugacy classes of G.

Goals of the talk

1. In the late 1990s Andruskiewitsch and Schneider initiated a program for classifying pointed Hopf algebras.
   Intuitive introduction to the underlying notions.
2. In that program a large family of (infinite dimensional) Hopf algebras arises, which contains as special cases:
   (a) all multiparameter big quantum groups at roots of unity,
   (b) all multiparameter big quantum supergroups at roots of unity
   (c) exceptional families that can be viewed as quantizations of finite dimensional simple Lie algebras in prime characteristic.
3. We will describe extensions of the DKP results to all of these algebras.
4. DKP rely on concrete calculations of Poisson brackets and reductions to rank 2 cases. This does not work for supergroups: 13 kinds of additional Serre relations on up to 4 generators.
5. We use methods based on perfect pairings of restricted and non-restricted integral forms and avoid any ad hoc calculations of Poisson brackets. Sketch.

Yetter–Drinfeld modules

All pointed Hopf algebras are deformations of smash products

$$\mathfrak{B}(V)\#\mathbb{C}[G]$$

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- G is a group,
- V is a Yetter–Drinfeld module of G,
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- $\mathfrak{B}(V)$ is the Nichols algebra of V.

Best understood is the case of abelian groups G. For an abelian group G, a Yetter–Drinfeld module is a G-graded G-module V:

$$V = \oplus_{g \in G} V_g \quad \text{where} \quad V_g \quad \text{are } G\text{-modules.}$$

V is of diagonal type if G acts by characters on V_g.

Examples

Take $G := \mathbb{Z}^n = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$, so $\mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[K_1^{\pm 1}, \ldots, K_n^{\pm 1}]$. For

$$\mathfrak{q} = (q_{ij}) \in M_n(\mathbb{C}^*)$$

consider the Yetter-Drinfeld module $V = \mathbb{C}E_1 \oplus \cdots \oplus \mathbb{C}E_n$ such that

$$K_i.E_j = q_{ij}E_j \quad \text{and} \quad V_{\alpha_j} = \mathbb{C}E_j.$$

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Example 2. $\mathfrak{sl}(m|n)$ type: q_{ij} symmetric and

$$q_{ii} = \begin{cases} \epsilon^{-2}, & i < m \\ -1, & i = m \\ \epsilon^2, & i > m \end{cases} \qquad q_{i,i-1} = \begin{cases} \epsilon, & i \leq m \\ \epsilon^{-1}, & i > m \end{cases} \qquad \text{other } q_{ij} = 1.$$

Nichols algebras

A Yetter–Drinfeld module V is a braided vector space with braiding

$$\sigma(E_i \otimes E_j) = q_{ij} E_j \otimes E_i.$$

The tensor algebra $T(V)$ is a Hopf algebra in the category of braided vector spaces.

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Example 1 $\rightsquigarrow u_\epsilon^+(\mathfrak{g})$,
Example 2 $\rightsquigarrow u_\epsilon^+(\mathfrak{sl}(m|n))$.

Root systems etc

Finite dimensional Nichols algebras $\mathfrak{B}(V)$ were classified by Heckenberger following earlier work of Andruskiewitsch–Schneider.

Heckenberger and Angiono: Root systems $\Delta^{\mathfrak{q}}_{+} \supseteq$Cartan roots $\mathfrak{O}^{\mathfrak{q}}_{+}$ (like even/odd roots for Lie superalgebras), e.g. $\mathfrak{sl}(m|n)$-case α_m non-Cartan simple root, the other simple roots are Cartan, a Weyl groupoid, Lusztig isomorphisms on

$$u_{\mathfrak{q}} := D(\mathfrak{B}(V)\#\mathbb{C}[\mathbb{Z}^n]),$$

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Angiono: (1) Explicit relations of $\mathfrak{B}(V)$: (complicated) Serre relations and power relations $E_\alpha^{N_\alpha} = 0$ for $\alpha \in \mathfrak{O}_+^{\mathfrak{q}}$ and $E_i^{N_i} = 0$.

(2) Defined distinguished pre-Nichols algebras

$\widetilde{\mathfrak{B}}(V) := T(V)/$(complicated Serre relations and $E_i^{N_i}$, i non-Cartan vertex).

Dictionary I

Starting with a braiding matrix $\mathfrak{q} = (q_{ij}) \in M_n(\mathbb{C}^*)$, consider:

$$U_\mathfrak{q} := D(\widetilde{\mathfrak{B}}(V)\#\mathbb{C}[\mathbb{Z}^n]) \rightsquigarrow U_\epsilon(\mathfrak{g}) \quad \text{large quantum group},$$

$$U_\mathfrak{q}^{\geq} := \widetilde{\mathfrak{B}}(V)\#\mathbb{C}[\mathbb{Z}^n] \rightsquigarrow U_\epsilon^{\geq}(\mathfrak{g}) \quad \text{large quantum Borel subalg},$$

$$U_\mathfrak{q}^{+} := \widetilde{\mathfrak{B}}(V) \rightsquigarrow U_\epsilon^{+}(\mathfrak{g}) \quad \text{large quantum unipotent subalg},$$

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Example 2 $\rightsquigarrow U_\epsilon^{+}(\mathfrak{sl}(m|n))$ without the parity element σ.
Compared to Benkart–Kang-Kashiwara our operators are

$$E_i = \sigma^{p(i)} e_i, \quad F_i = (\epsilon - \epsilon^{-1}) f_i, \quad K_i^{\pm 1} = \sigma^{p(i)} t_i^{\pm 1}.$$

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Assumption: The braiding matrix $\mathfrak{q}$ belongs to a 1-parameter family, i.e. it is obtained from some one-parameter braiding matrix $\mathbf{q} = (\mathbf{q}_{ij} = b_{ij}\nu^{m_{ij}})$ with the same root system by setting

$$\nu \mapsto \epsilon \in \mathbb{C}^*.$$

Dictionary II

Generators of the large quantum group $U_{\mathfrak{q}}^{\geq}$:

$$E_i, F_i, K_i^{\pm 1}, L_i^{\pm 1}, \quad 1 \leq i \leq n.$$

Relations:

$$K_i E_j = q_{ij} E_j K_i, \qquad L_i E_j = q_{ji}^{-1} E_j L_i,$$
$$K_i F_j = q_{ij}^{-1} F_j K_i, \qquad L_i F_j = q_{ji} F_j L_i,$$
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Root vectors: E_β, F_β, $\beta \in \Delta_+^{\mathfrak{q}}$, PBW bases of $U_{\mathfrak{q}}^{\pm}$. For non-Cartan roots $E_\beta^{N_\beta} = F_\beta^{N_\beta} = 0$.

Construction of Poisson orders

Theorem 1 [Andruskiewitsch–Angiono–Y]

1. Each large quantum group U_{q} is a specialization of a non-restricted integral form of $U_{\mathbf{q}}$. This makes $(U_{\mathrm{q}}, \mathcal{Z}(U_{\mathrm{q}}))$ a Poisson order.
2. The central Hopf subalgebra
$$Z_{\mathrm{q}} := \mathbb{C}[E_\beta^{N_\beta}, F_\beta^{N_\beta}, K_\beta^{\pm N_\beta}, L_\beta^{\pm N_\beta}, \text{Cartan roots } \beta]$$
of U_{q} (defined by Angiono) is invariant under the Lusztig's isomorphisms and U_{q} is module-finite over it.
3. Z_{q} is a Poisson subalgebra of $\mathcal{Z}(U_{\mathrm{q}})$, so $(U_{\mathrm{q}}, Z_{\mathrm{q}})$ a Poisson order.

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Punchline for (3): Z_{q}^+ is the fixed subalgebra of the center $\mathcal{Z}(U_{\mathrm{q}}^+)$ under a set of skew-derivations, which descend from $U_{\mathbf{q}}^+$.

Prove a general fact that all such are Poisson subalgebras of specializations, and extend to Z_{q} via Lusztig's isomorphisms.

Description of Poisson orders

Poisson orders:

- $(U_{\mathfrak{q}}, Z_{\mathfrak{q}})$ (large quantum groups), $(U_{\mathfrak{q}}^{\geq}, Z_{\mathfrak{q}}^{\geq})$ (large quantum Borels).

Problem: Here $Z_{\mathfrak{q}}$ and $Z_{\mathfrak{q}}^{\geq}$ are Poisson–Hopf algebras. What are the (connected) Poisson algebraic groups

$M_{\mathfrak{q}} := \mathrm{MaxSpec} Z_{\mathfrak{q}}$ and $M_{\mathfrak{q}}^{\geq} := \mathrm{MaxSpec} Z_{\mathfrak{q}}^{\geq}$?

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Theorem 2 [Andruskiewitsch–Angiono–Y]

1. For each large quantum group $U_{\mathfrak{q}}$,

$$M_{\mathfrak{q}} \cong B_{\mathfrak{q}}^{+} \times B_{\mathfrak{q}}^{-}, \quad M_{\mathfrak{q}}^{\geq} \cong B_{\mathfrak{q}}^{+},$$

   where $B_{\mathfrak{q}}^{\pm}$ are opposite Borel subgroups of a semisimple algebraic group $G_{\mathfrak{q}}$ of adjoint type with root system built from the Cartan roots of the large quantum group.

2. They are the dual Poisson algebraic groups of $G_{\mathfrak{q}}$ and $B_{\mathfrak{q}}^{-}$ with (nonstandard) Poisson structures with empty Belavin–Drinfeld triples.

Key argument

The second part of the theorem is proved without any direct calculations of Poisson brackets and reductions to low rank cases.

De Concini–Kac–Procesi relied on reductions to rank 2 cases (classical Serre relations on 2 generators). Angiono's list of relations involves 13 additional different kinds of Serre relations on up to 4 generators.

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Punchline: We construct a restricted and a non-restricted integral forms $U^{\mathrm{res}\pm}_{\mathbf{q},\mathcal{A}}$ and $U^{\pm}_{\mathbf{q},\mathcal{A}}$ of $U^{\pm}_{\mathbf{q}}$ and an $\mathcal{A}$-linear perfect pairing

$$U^{\mathrm{res}+}_{\mathbf{q},\mathcal{A}} \times U^{-}_{\mathbf{q},\mathcal{A}} \to \mathcal{A}.$$

1. From the pairing we give a description of the dual tangent Lie bialgebra of $M^{\geq}_{\mathrm{q}}$ in terms of $U_{\mathbf{q}}$ and then prove that it is isomorphic to $\mathrm{Lie}\, B^{-}_{\mathrm{q}}$.
2. Then we use Lie theory to prove that the dual tangent Lie bialgebra of M_{q} is isomorphic to $\mathrm{Lie}\, G_{\mathrm{q}}$.

Symplectic leaves and irreps

$B_{\mathrm{q}}^{+} \times B_{\mathrm{q}}^{-}$ is identified with a subset of

$$G_{\mathrm{q}} \times \text{(a torus, which is not the max torus of } G_{\mathrm{q}}\text{)}.$$

Theorem 2 [Andruskiewitsch–Angiono–Y]

1. (Large quantum groups) The symplectic leaves of $M_{\mathrm{q}} \cong B_{\mathrm{q}}^{+} \times B_{\mathrm{q}}^{-}$ are in bijection with the conjugacy classes of G_{q} times the points of that extra torus. The algebras $U_{\mathrm{q}}/\mathfrak{m}U_{\mathrm{q}}$ are isomorphic to each other across conjugacy classes.
2. (Large quantum Borels) The torus-orbits of symplectic leaves of $M_{\mathrm{q}}^{\geq} \cong B_{\mathrm{q}}^{+}$ are in bijection with the double Bruhat cells $B_{\mathrm{q}}^{+} \cup B_{\mathrm{q}}^{-} w B_{\mathrm{q}}^{-}$ for Weyl group elements w. The algebras $U_{\mathrm{q}}^{\geq}/\mathfrak{m}U_{\mathrm{q}}^{\geq}$ are isomorphic to each other across double Bruhat cells.

Thank you!