6.3 Permutations and Combinations

Permutations

Definition 1. *A permutation of a set of (distinct) objects is an ordering of the objects in row.*

Example 1. *In how many different orders can three runners finish a race if no ties are allowed?*

Solution. The different orders for elements a, b, and c are

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

There are six permutations. □

In general, given a set of n objects, how many permutations does the set have? We are free to select the first element, so we have n choices. Once we have chosen the first element, there are $n-1$ objects remaining, so we would have $n-1$ options for the second element. In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step. This process continues until the last element, for which we have only one choice. By the product rule, we have the following.

Theorem 1. *For any integer n, with $n \geq 1$, the number of permutations of a set with n elements is $n!$.*

We may prove this theorem using mathematical induction.

Example 2. *How many permutations of the letters ABCDEFGH contain the string ABC?*

Solution. Think of the letters ABC as glued together. Thus we really have six objects, namely, the super-letter ABC, and the individual letters D, E, F, G, and H. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters ABCDEFGH in which ABC occurs as a block. □

Definition 2. *An ordered arrangement of r elements of a set is called an r-permutation, denoted by $P(n, r)$ or nPr.*

Example 3. *In how many ways can a set of two positive integers less than 100 be chosen?*

Solution. $99 \times 98 = 9702$ ways. □

Theorem 2. *If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are*

$$P(n, r) = nPr = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

r-permutations of a set with n distinct elements.

Proof. The first element of the permutation can be chosen in n ways because there are n elements in the set. There are $n-1$ ways to choose the second element of the permutation, because there are $n-1$ elements left in the set after using the element picked for the first position. Similarly, there are $n-2$ ways to choose the third element, and so on, until there are exactly $n-(r-1) = n-r+1$ ways to choose the rth element. Consequently, by the product rule, there are

$$n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of the set. Furthermore,

$$\frac{n!}{(n-r)!} = \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)!}{(n-r)!} = n(n-1)(n-2)\cdots(n-r+1).$$

□

Remark. *Note that the formula also works when $r = 0$, because by definition, $0! = 1$.*

Example 4. *Prove that for all integers $n \geq 2$,*

$$P(n, 2) + P(n, 1) = n^2.$$

Combinations

Now we want to count *unordered* selection of objects.

Suppose we want to choose r elements from a set with n elements, in no particular order, that is, we want to select a subset with r elements from a set with n elements. In how many ways can we do so?

Definition 3. *Let n and r be integers with $0 \le r \le n$. The symbol*

$$\binom{n}{r}$$

is read "n choose r" and represents the number of subsets of size r that can be chosen from a set with n elements.

Remark. *There are several notations for an r-combination from a set of n distinct elements: $C(n, r)$, nCr (n, choose r), and $\binom{n}{r}$, the binomial coefficient, which is the topic of the next section.*

Theorem 3. *The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals*

$$C(n, r) = nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. We relate r-combinations to r-permutations. The $P(n, r)$ r-permutations of the set can be obtained by forming the $C(n, r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r, r)$ ways. Consequently, by the product rule,

$$P(n, r) = C(n, r) \cdot P(r, r).$$

Therefore

$$C(n, r) = \frac{P(n, r)}{P(r, r)} = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}.$$ □

Example 5. *How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?*

Solution. Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are

$$C(52, 5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!}$$

different hands of five cards that can be dealt. The number of ways are to select 47 cards from a standard deck of 52 cards is the same:

$$C(52, 47) = \frac{52!}{47!(52-47)!} = \frac{52!}{5!47!}.$$

□

Question: What is the relationship between $C(n, r)$ and $C(n, n-r)$?

Corollary. *Let n and r be nonnegative integers with $r \le n$. Then $C(n, r) = C(n, n-r)$.*

Proof.

$$C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n, r).$$ □

Definition 4. *A combinatorial proof of an identity is a proof that uses counting arguments to prove that both sides of the identity count the same objects but in different ways or a proof that is based on showing that there is a bijection between the sets of objects counted by the two sides of the identity. These two types of proofs are called double counting proofs and bijective proofs, respectively.*

Example 6. *Give a combinatorial proof of $C(n,r) = C(n,n-r)$.*

Proof. Observe that any subset of size r can be specified either by saying which r elements lie in the subset or by saying which $n-r$ elements lie outside the subset. Suppose that A is a set with n elements. Any subset B with r elements completely determines a subset, $A-B$, with $n-r$ elements.

Suppose A has k subsets of size $r : B_1, B_2, \ldots, B_k$. Then each B_i can be paired up with exactly one set of size $n-r$, namely its complement $A-B_i$ as shown below.

Subsets of size r		Subsets of size $n-r$
B_1	$\longleftrightarrow$	$A-B_1$
B_2	$\longleftrightarrow$	$A-B_2$
$\vdots$		$\vdots$
B_k	$\longleftrightarrow$	$A-B_k$

All subsets of size r are listed in the left-hand column, and all subsets of size $n-r$ are listed in the right-hand column. The number of subsets of size r equals the number of subsets of size $n-r$, and so $C(n,r) = C(n,n-r)$. □

The type of reasoning used in this example is called *combinatorial*, because it is obtained by counting things that are combined in different ways.

Example 7. *How many bit strings of length* 10 *have*

1. *exactly three* 0*s?*
2. *more* 0*s than* 1*s?*
3. *at least seven* 1*s?*
4. *at least three* 1*s?*

Solution.

1. There are $C(10,3)$ ways to choose the positions for the 0s, and that is the only choice to be made, so the answer is $C(10,3) = 120$.
2. There are more 0s than 1s if there are fewer than five 1s. Using the same reasoning as in the previous part, together with the sum rule, we obtain the answer $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 210 = 386$. Alternatively, by symmetry, half of all cases in which there are not five 0s have more 0s than 1s; therefore the answer is $(2^{10} - C(10,5))/2 = (1024 - 252)/2 = 386$.
3. We want the number of bit strings with $7, 8, 9$, or 10 1s. By the same reasoning as above, there are $C(10,7) + C(10,8) + C(10,9) + C(10,10) = 120 + 45 + 10 + 1 = 176$ such strings.
4. If a string does not have at least three 1s, then it has $0, 1$, or 2 1s. There are $C(10,0) + C(10,1) + C(10,2) = 1 + 10 + 45 = 56$ such strings. There are $2^{10} = 1024$ strings in all. Therefore there are $1024 - 56 = 968$ strings with at least three 1.

□