6 Counting

6.1 The Basics of Counting

Combinatorics, the study of arrangements of objects, is an important part of discrete mathematics. This subject was studied as long ago as the seventeenth century, when combinatorial questions arose in the study of gambling games. Enumeration, the counting of objects with certain properties, is an important part of combinatorics.

Suppose that a password on a computer system consists of six, seven, or eight characters. Each of these characters must be a digit or a letter of the alphabet. Each password must contain at least one digit. How many such passwords are there? The techniques needed to answer this question and a wide variety of other counting problems will be introduced in this section.

Basic Counting Principles

There are two basic rules for counting: product rule and sum rule.

**Theorem** (THE PRODUCT RULE). Suppose that a procedure can be broken down into a sequence of two tasks. If there are $n_1$ ways to do the first task, and for each of these ways of doing the first task, there are $n_2$ ways to do the second task, then there are $n_1n_2$ ways to do the procedure.

**Example 1.** An office building contains 27 floors and has 37 offices on each floor. How many offices are in the building?

**Solution.** By the product rule there are $27 \cdot 37 = 999$ offices.

An extended version of the product rule is often useful. Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_m$ in sequence. If each task $T_i, i = 1, 2, \ldots, n$, can be done in $n_i$ ways, regardless of how the previous tasks were done, then there are $n_1n_2\cdots n_m$ ways to carry out the procedure. This version of the product rule can be proved by mathematical induction from the product rule for two tasks.

**Example 2.** A particular brand of shirt comes in 12 colors, has a male version and a female version, and comes in three sizes for each sex. How many different types of this shirt are made?

**Solution.** By the product rule there are $12 \cdot 2 \cdot 3 = 72$ different types of shirt.

**Example 3** (Counting Functions). How many functions are there from a set with $m$ elements to a set with $n$ elements?

**Solution.** For each of the $m$ elements in the domain, there are $n$ choices from the codomain. Hence, by the product rule there are $n \cdot n \cdots n = n^m$ functions from a set with $m$ elements to one with $n$ elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements.

**Example 4** (The Telephone Numbering Plan). The North American numbering plan (NANP) specifies the format of telephone numbers in the U.S., Canada, and many other parts of North America. A telephone number in this plan consists of 10 digits, which are split into a three-digit area code, a three-digit office code, and a four-digit station code. Because of signaling considerations, there are certain restrictions on some of these digits. To specify the allowable format, let $X$ denote a digit that can take any of the values 0 through 9, let $N$ denote a digit that can take any of the values 2 through 9, and let $Y$ denote a digit that must be a 0 or a 1. Two numbering plans, which will be called the old plan, and the new plan, will be discussed. (The old plan, in use in the 1960s, has been replaced by the new plan, but the recent rapid growth in demand for new numbers for mobile phones and devices will eventually make even this new plan obsolete. In this example, the letters used to represent digits follow the conventions of the North American Numbering Plan.) As will be shown, the new plan allows the use of more numbers. In the old plan, the formats of the area code, office code, and station code are NYX, NNX, and XXXX, respectively, so that telephone numbers had the form
NYX-NNX-XXXX. In the new plan, the formats of these codes are NXX, NXX, and XXXX, respectively, so that telephone numbers have the form NXX-NNX-XXXX. How many different North American telephone numbers are possible under the old plan and under the new plan?

Solution. By the product rule, there are \(8 \cdot 2 \cdot 10 = 160\) area codes with format NYX and \(8 \cdot 10 \cdot 10 = 800\) area codes with format NXX. Similarly, by the product rule, there are \(8 \cdot 8 \cdot 10 = 640\) office codes with format NNX. The product rule also shows that there are \(10 \cdot 10 \cdot 10 \cdot 10 = 10,000\) station codes with format XXXX. Consequently, applying the product rule again, it follows that under the old plan there are

\[
160 \cdot 640 \cdot 10,000 = 1,024,000,000
\]
different numbers available in North America. Under the new plan, there are

\[
800 \cdot 800 \cdot 10,000 = 6,400,000,000
\]
different numbers available.

Definition. Given a set \(S\), the power set of \(S\) is the set of all subsets of the set \(S\). The power set of \(S\) is denoted by \(\mathcal{P}(S)\).

Proposition (Counting Subsets of a Finite Set). \(|\mathcal{P}(S)| = 2^{|S|}\).

Proof. Let \(S\) be a finite set. List the elements of \(S\) in arbitrary order. There is a one-to-one correspondence between subsets of \(S\) and bit strings of length \(|S|\). Namely, a subset of \(S\) is associated with the bit string with a 1 in the \(i\)th position if the \(i\)th element in the list is in the subset, and a 0 in this position otherwise. By the product rule, there are \(2^{|S|}\) bit strings of length \(|S|\). Hence, \(|\mathcal{P}(S)| = 2^{|S|}\).

The product rule is often phrased in terms of sets in this way: If \(A_1, A_2, \ldots, A_m\) are finite sets, then the number of elements in the Cartesian product of these sets is the product of the number of elements in each set. To relate this to the product rule, note that the task of choosing an element in the Cartesian product \(A_1 \times A_2 \times \cdots \times A_m\) is done by choosing an element in \(A_1\), an element in \(A_2\), \ldots, and an element in \(A_m\). By the product rule it follows that

\[
|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdots \cdot |A_m|.
\]

Theorem (THE SUM RULE). If a task can be done either in one of \(n_1\) ways or in one of \(n_2\) ways, where none of the set of \(n_1\) ways is the same as any of the set of \(n_2\) ways, then there are \(n_1 + n_2\) ways to do the task.

We can extend the sum rule to more than two tasks. Suppose that a task can be done in one of \(n_1\) ways, in one of \(n_2\) ways, \ldots, or in one of \(n_m\) ways, where none of the set of \(n_i\) ways of doing the task is the same as any of the set of \(n_j\) ways, for all pairs \(i\) and \(j\) with \(1 \leq i < j \leq m\). Then the number of ways to do the task is \(n_1 + n_2 + \cdots + n_m\). This extended version of the sum rule is often useful in counting problems. This version of the sum rule can be proved using mathematical induction from the sum rule for two sets.

The sum rule can be phrased in terms of sets as: If \(A_1, A_2, \ldots, A_m\) are pairwise disjoint-finite sets, then the number of elements in the union of these sets is the sum of the numbers of elements in the sets. To relate this to our statement of the sum rule, note there are \(|A_i|\) ways to choose an element from \(A_i\) for \(i = 1, 2, \ldots, m\). Because the sets are pairwise disjoint, when we select an element from one of the sets \(A_i\), we do not also select an element from a different set \(A_j\). Consequently, by the sum rule, because we cannot select an element from two of these sets at the same time, the number of ways to choose an element from one of the sets, which is the number of elements in the union, is

\[
|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m| \text{ when } A_i \cap A_j = \emptyset \text{ for all } i, j.
\]

This equality applies only when the sets in question are pairwise disjoint. The situation is much more complicated when these sets have elements in common.
More Complex Counting Problems

Sometimes we need to use both sum and product rules.

**Example 5.** How many bit strings are there of length six or less, not counting the empty string?

*Solution.* We use the sum rule, adding the number of bit strings of each length up to 6. If we include the empty string, then we get $2^6 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = 2^7 - 1 = 127$ (using the formula for the sum of a geometric progression).

**Example 6.** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

*Solution.* Let $P$ be the total number of possible passwords, and let $P_6, P_7, \text{ and } P_8$ denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule, $P = P_6 + P_7 + P_8$. We will now find $P_6, P_7, \text{ and } P_8$. Finding $P_6$ directly is difficult. To find $P_6$ it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is $36^6$, and the number of strings with no digits is $26^6$. Hence, $P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560$.

Similarly, we have

$$P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920$$

and

$$P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880$$

Consequently

$$P = P_6 + P_7 + P_8 = 2,684,483,063,360.$$  

**Theorem** (THE SUBTRACTION RULE). If a task can be done in either $n_1$ ways or $n_2$ ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the principle of inclusion-exclusion, especially when it is used to count the number of elements in the union of two sets. Suppose that $A_1$ and $A_2$ are sets. Then, there are $|A_1|$ ways to select an element from $A_1$ and $|A_2|$ ways to select an element from $A_2$. The number of ways to select an element from $A_1$ or from $A_2$, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from $A_1$ and the number of ways to select an element from $A_2$, minus the number of ways to select an element that is in both $A_1$ and $A_2$. Because there are $|A_1 \cup A_2|$ ways to select an element in either $A_1$ or in $A_2$, and $|A_1 \cap A_2|$ ways to select an element common to both sets, we have

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$ 

**Example 7.** Every student in a discrete mathematics class is either a computer science or a mathematics major or is a joint major in these two subjects. How many students are in the class if there are 38 computer science majors (including joint majors), 23 mathematics majors (including joint majors), and 7 joint majors?

*Solution.* This is an application of the inclusion-exclusion principle: $38 + 23 - 7 = 54$ (we need to subtract the 7 double majors counted twice in the sum).
Tree Diagrams

Counting problems can be solved using tree diagrams. To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them.

Example 8. Use a tree diagram to find the number of bit strings of length four with no three consecutive 0s.

Solution. We draw the tree, with its root at the top. We show a branch for each of the possibilities 0 and 1, for each bit in order, except that we do not allow three consecutive 0’s. Since there are 13 leaves, the answer is 13.