Quiz 6, 7/29/16

Use generating functions to solve

$$a_n = 3a_{n-1} + 2^n + 5, \quad a_0 = 1.$$

Proof.

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} a_n x^n \\
&= a_0 + \sum_{n=1}^{\infty} a_n x^n \\
&= 1 + \sum_{n=1}^{\infty} a_n x^n \\
&= 1 + \sum_{n=1}^{\infty} (3a_{n-1} + 2^n + 5) x^n \\
&= 1 + 3\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} 2^n x^n + 5\sum_{n=1}^{\infty} x^n \\
&= 1 + 3x\sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} (2x)^n + 5\sum_{n=1}^{\infty} x^n \\
&= 1 + 3x\sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} (2x)^n + 5\sum_{n=1}^{\infty} x^n \\
&= 1 + 3xG(x) + \frac{2x}{1-2x} + \frac{5x}{1-x}.
\end{aligned}$$

Now we solve for $G(x)$:

$$\begin{aligned}
G(x) - 3xG(x) &= 1 + \frac{2x}{1-2x} + \frac{5x}{1-x} \\
\therefore G(x)(1-3x) &= 1 + \frac{2x}{1-2x} + \frac{5x}{1-x}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
G(x) &= \frac{1}{1-3x} \cdot \frac{(1-2x)(1-x) + 2x(1-x) + 5x(1-2x)}{(1-2x)(1-x)} \\
&= \frac{1 - 3x + 2x^2 + 2x - 2x^2 + 5x - 10x^2}{(1-3x)(1-2x)(1-x)} \\
&= \frac{-10x^2 + 4x + 1}{(1-3x)(1-2x)(1-x)} \\
&= \frac{11/2}{1-3x} + \frac{-2}{1-2x} + \frac{-5/2}{1-x} \\
&= \sum_{n=0}^{\infty} \left[\frac{11}{2}(3x)^n - 2(2x)^n - \frac{5}{2}(x)^n \right] \\
&= \sum_{n=0}^{\infty} \left[\frac{11}{2}(3)^n - 2(2)^n - \frac{5}{2} \right] (x)^n
\end{aligned}$$

Comparing to $G(x) = \sum_{n=0}^{\infty} a_n x^n$, we obtain:

$$a_n = \frac{11}{2}(3)^n - 2(2)^n - \frac{5}{2}. \qquad \square$$