Quiz 3, 7/08/16

Prove that for every positive integer n,

$$\sum_{j=1}^{n} j2^j = (n-1)2^{n+1} + 2.$$

Proof. We proceed by induction.

BASIS STEP: We prove that the statement is true when $n = 1$.

The left-hand side of the equation is

$$1 \cdot 2^1 = 1 \cdot 2 = 2.$$

The right-hand side of the equation is

$$(1-1)2^{1+1} + 2 = 0 \cdot 2^2 + 2 = 0 + 2 = 2.$$

Since the left-hand side and the right-hand side are equal, we have shown the basis step is true.

INDUCTIVE STEP: We want to show that if

$$\sum_{j=1}^{n} j2^j = (n-1)2^{n+1} + 2,$$

then

$$\sum_{j=1}^{n+1} j2^j = n2^{n+2} + 2.$$

Splitting the left-hand side into its first n terms followed by its last term and invoking the inductive hypothesis, we have

$$\begin{aligned}
\sum_{j=1}^{n+1} j2^j &= [1 \cdot 2 + 2 \cdot 2^2 + \cdots + n \cdot 2^n] + (n+1) \cdot 2^{n+1} \\
&= \sum_{j=1}^{n} j2^j + (n+1) \cdot 2^{n+1} \\
&= (n-1)2^{n+1} + 2 + (n+1) \cdot 2^{n+1} \\
&= 2^{n+1} \cdot [(n-1) + (n+1)] + 2 \\
&= 2^{n+1} \cdot 2n + 2 \\
&= n \cdot 2^{n+1+1} + 2 \\
&= n \cdot 2^{n+2} + 2.
\end{aligned}$$

This concludes the inductive step.

Since we showed both the basis step and the inductive step are valid, by mathematical induction, we have shown that the formula is correct. $\square$