Quiz 2, 6/30/16

The value of the Euler ϕ-function at the positive integer n is defined to be the number of positive integers less than or equal to n that are relatively prime to n. [Note: ϕ is the Greek letter phi.]

a. Find $\phi(14)$.

b. Find $\phi(17)$.

Extra Credit. Show that n is prime if and only if $\phi(n) = n - 1$.

Solution. a. The positive integers less than or equal to 14 that are relatively prime to 14 are $1, 3, 5, 9, 11, 13$. Thus $\phi(14) = 6$.

b. The positive integers less than or equal to 17 that are relatively prime to 17 are $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16$. Thus $\phi(17) = 16$.

Proof of the extra credit question:

($\Rightarrow$): Suppose n is prime. We want to show that $\phi(n) = n-1$. If n is prime, every positive integer less than n is relatively prime to n, so $\phi(n) = n - 1$.

($\Leftarrow$): Suppose $\phi(n) = n - 1$. We want to show that n is prime. We may prove the contrapositive of this statement. Suppose n is not prime. We want to show that $\phi(n) \neq n - 1$.

Suppose n is composite. Then there are positive integers a and b less than n that divide n: $n = ab$. Thus $\phi(n) \leq n-1-2$. That is, $\phi(n) \leq n-3$. Therefore $\phi(n) < n - 2$. In particular, $\phi(n) \neq n - 1$.

Therefore, n is prime $\Leftrightarrow \phi(n) = n - 1$. □