Chapter 1

Propositional Logic, Predicates, and Equivalence

- A statement or a proposition is a sentence that is true (T) or false (F) but not both.
- The symbol $\neg$ denotes *not*, $\wedge$ denotes *and*, and $\vee$ denotes *or*.
- If p is a proposition, the negation of p, $\neg p$, has opposite truth value from p. If p is true, $\neg p$ is false, if p is false, then $\neg p$ is true.
- If p and q are propositions, the conjunction of p and q, $p \wedge q$, is true when both p and q are true, and is false otherwise.
- If p and q are propositions, the disjunction of p and q, $p \vee q$, is false when both p and q are false, and is true otherwise.
- The symbol $\equiv$ or $\Leftrightarrow$ denotes equivalent truth value.
- The symbol $\rightarrow$ denotes implication. The symbol $\leftrightarrow$ indicates if and only if.
- If propositions p and q are equivalent, they are both true or both false, that is, they both have the same truth value.
- A tautology is a statement that is always true. A contradiction is a statement that is always false.
- *DeMorgan's Laws.*

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

- If p and q are propositions, the conditional "if p then q" (or "p only if q" or "q if p), denoted by $p \rightarrow q$, is false when p is true and q is false; otherwise it is true. p is a sufficient condition for q and q is a necessary condition for p.
- The contrapositive of a conditional statement of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- If p and q are propositions, the biconditional "p if and only if q," denoted by $p \leftrightarrow q$, is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated iff.

- A predicate is a sentence that contains one or more variables and becomes a proposition when specific values are substituted for the variables. The domain of a predicate variable consists of all values that may be substituted in place of the variable.
- Let $P(x)$ be a predicate and D the domain of x. A universal proposition is a statement of the form

$$\forall x \text{ in } D, P(x).$$

  It is defined to be true if, and only if, $P(x)$ is true for every x in D. It is defined to be false if, and only if, $P(x)$ is false for at least one x in D. A value of x for which $P(x)$ is false is called a counterexample to the universal proposition.
- Let $P(x)$ be a predicate and D the domain of x. An existential proposition is a statement of the form

$$\exists x \text{ in } D \text{ such that } P(x).$$

  It is defined to be true if, and only if, $P(x)$ is true for at least one x in D. It is defined to be false if, and only if, $P(x)$ is false for all x in D.
- Logic formulas

$$\begin{aligned}
p \to q &\equiv \neg p \vee q \\
p \to q &\equiv \neg q \to \neg p \\
p \to q &\not\equiv q \to p \\
p \to q &\not\equiv \neg p \to \neg q \\
\neg(p \to q) &\equiv p \wedge \neg q \\
\neg(\forall x \text{ in } D, P(x)) &\equiv \exists x \text{ in } D \text{ such that } \neg P(x) \\
\neg(\exists x \text{ in } D \text{ such that } P(x)) &\equiv \forall x \text{ in } D, \neg P(x)
\end{aligned}$$

Chapter 2

Basic Structures: Sets, Functions, Sequences, and Sums

2.1 Sets

Goal: To introduce the basic terminology of set theory.

2.1.1 Definition of set

A set is an unordered collection of objects. The objects in a set are called the elements, or members of the set. A set is said to contain its elements. The notation $a \in A$ denotes that a is an element of the set A. If a is not a member of A, write $a \notin A$.

2.1.2 Describing Sets

Roster Method and Set-Builder Notation.

Remark: When specifying the elements of sets the number of times an element is listed and the order in which the elements are listed do not matter.

2.1.3 Some Important Sets in Mathematics

$$
\begin{aligned}
\mathbb{N} &= \text{natural numbers} & &= \{0, 1, 2, 3, \ldots\} \\
\mathbb{Z} &= \text{integers} & &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\
\mathbb{Z}^+ &= \text{positive integers} & &= \{1, 2, 3, \ldots\} \\
\mathbb{R} &= \text{set of real numbers} \\
\mathbb{R}^+ &= \text{set of positive real numbers} \\
\mathbb{C} &= \text{set of complex numbers} \\
\mathbb{Q} &= \text{set of rational numbers}
\end{aligned}
$$

2.1.4 Empty Set and Universal Set

Remark: Distinguish between $\emptyset$ and $\{\emptyset\}$: the empty set is the set with no elements and that it is a subset of every set.

2.1.5 Subsets and Set Equality

(The set A is a subset of B) $\Leftrightarrow [\forall x(x \in A \to x \in B)]$.
Notation: $A \subseteq B$.

Theorem 1. *For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.*

Proof. Let S be a set. To show that $\emptyset \subseteq S$ we must show that $\forall x(x \in \emptyset \to x \in S)$ is true. Because the empty set contains no elements, if follows that $x \in \emptyset$ is always false. It follows that the conditional statement $x \in \emptyset \to x \in S$ is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is true. Therefore, $\forall x(x \in \emptyset \to x \in S)$ is true. This completes the proof of (i).

To show that $S \subseteq S$, let x be an arbitrary element of S. WTS that x is an element of S. But this is trivial. This completes the proof of (ii). □

Two sets are equal if and only if they have the same elements. That is,

$$(A = B) \Leftrightarrow ((A \subseteq B) \text{ and } (B \subseteq A))$$

or

$$(A = B) \Leftrightarrow (\forall x(x \in A \to x \in B) \wedge \forall x(x \in B \to x \in A))$$

2.1.6 Cardinality of Sets

The cardinality of a finite set S is the number of distinct elements in S, depicted by $|S|$.

2.1.7 Tuples

An n-tuple is an ordered list of objects, such as $(a_1, \ldots, a_n)$.

2.1.8 Cartesian Product

The Cartesian product of sets A_i, $i = 1, \ldots, n$, is $\{(a_1, \ldots, a_n) | a_i \in A_i, i = 1, \ldots, n\}$

2.2 Set Operations

Set operations include union $A \cup B$, intersection $A \cap B$, difference $A - B$, and complement $\bar{A}$.

Goal: How set identities are established and to introduce the most important such identities.

Remark: The relationship between set identities and logical equivalences becomes clear when set operations are expressed using set builder notation and logical operators.

Show several different ways to prove a set identity, namely by showing that each side is a subset of the other, by a membership table, by the use of logical equivalences, or by using set identities that have already been established. Explain that the set identities in Table 1 are analogous to the propositional equivalences in Section 1.3. We touch briefly on how to count elements in the union of two sets, foreshadowing the treatment of inclusion-exclusion in Chapter 8.

2.3 Functions

Goal: To introduce the concept of a function, the notion of one-to-one functions, onto functions, and the floor and ceiling functions.

2.3.1 Function

A *function* (mapping or transformation) $f : A \to B$ is a relation from a set A (called domain) to a set B (called codomain), such that to each $x \in A$, exactly one $y \in B$ is assigned. Thus the function f is a subset of $A \times B$.

2.3.2 Graph of a function

The graph of a function $f : A \to B$ is the set of ordered pairs $\{(a, b) \mid a \in A \text{ and } b \in B\}$, where A and B are nonempty sets. $f(S)$ is called the image of the set S under f:

$$f(S) = \{f(s) \mid s \in S\} = \{t \mid \exists s \in S(t = f(s))\}.$$

2.3.3 1-1 Function

A function $f : A \to B$ is one-to-one (1-1 or injective) iff

$$\forall x, z \in A, (f(x) = f(z) \Rightarrow x = z)$$

Alternatively, we may use the contrapositive statement:

$$\forall x, z \in A, (z \neq x \Rightarrow f(z) \neq f(x))$$

2.3.4 Onto Function

A function $f : A \to B$ is onto (surjective) iff

$$\forall y \in B \exists x \in A(f(x) = y$$

2.3.5 Floor and Ceiling Functions

The floor function (or the greatest integer function), depicted by $\lfloor x \rfloor$ (or $[x]$), rounds x down to the nearest integer less than or equal to x. The ceiling function, depicted by $\lceil x \rceil$, rounds x up to the closest integer greater than or equal to x.

2.3.6 Logarithm and Factorial

Notation: $\log x$ in the textbook is used for logarithm base 2.

The factorial function $f : \mathbb{N} \to \mathbb{Z}^+$, denoted by $f(n) = n!$, assigns the product of first n positive integers. We define $0! = 1$ so that we do not have to write exceptions in our formulas.

Stirling's formula:

$$n! \sim \sqrt{2\pi n}(n/e)^n$$

The symbol $\sim$ is read as "is asymptotic to," that is, $\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}(n/e)^n} = 1$.

2.3.7 combining functions

We may add, subtract, multiply, and compose functions to derive new functions: $f + g, f - g, fg, f \circ g$.

2.3.8 Bijection

A function f is a bijection (or a 1-1 correspondence) iff f is both 1-1 and onto. Such a function is bijective.

2.3.9 Identity Function

The identity function: $\iota : A \to A$ is defined by $\iota(x) = x$ for all $x \in A$.

2.3.10 Inverse Function

When f is a bijection, $f : A \to B$, we may define an inverse function, denoted by $f^{-1} : B \to A$, such that $f^{-1}(B) = A$.

2.4 Sequences and Summations

Goals: To introduce terminology used for sequences and summations. To introduce recurrence relations and some methods for solving them. To work with summations and establish several important summation formulae.

2.4.1 Definition

Sequences are ordered lists of elements. A sequence is a function $f : \mathbb{Z} \to S$.

2.4.2 Notation

$a_n = f(n)$ indicates the nth term of the sequence. $\{a_n\}$ indicates a sequence with an individual term a_n.

A finite sequence is called a string.

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

2.4.3 Geometric progression

A geometric progression is a sequence of the form $a, ar, ar^2, \ldots$, where the intial term a and the common ratio r are real numbers.

2.4.4 Arithmetic progression

An arithmetic progression is a sequence of the form $a, a + d, a + 2d, \ldots$, where the initial term a and the commond difference d are real numbers.

2.4.5 The Fibonacci sequence

The Fibonacci sequence, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, 4, \ldots$.

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a closed formula, for the terms of the sequence.

Advice: The first part of this section deals with sequences. Recurrence relations are introduced and the method of iteration for solving them is discussed. Example 11 illustrates how recurrence relations are used to solve a problem involving compound interest. The topic of integer sequences is covered, which requires more critical and creative thinking than the other material. Examples 1215 involve conjecturing a formula or rule for generating the terms of a sequence when only the first few terms are known. Encourage to try the On-Line Encyclopedia of Integer Sequences, mentioned in this section.

The second part of the section introduces summation notation. Make sure you can work with the different forms of this notation and with shifting indices in summations. In particular, this will be helpful later when we prove summation formulae using mathematical induction. You should also understand that sequences and strings are just special types of functions.

2.4.6 Geometric Series

$$\sum_{0}^{n} ar^n = \frac{a - ar^{n+1}}{1 - r}, r \neq 1$$

Proof.

$$S_n = a + ar^2 + \cdots + ar^n$$
$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n+1}$$

Subtract the two equations and solve for S_n:

$$\begin{aligned} S_n - rS_n &= a - ar^{n+1} \\ \therefore S_n(1-r) &= a(1-r^{n+1}) \\ \therefore S_n &= \frac{a(1-r^{n+1})}{1-r} \end{aligned}$$

$\square$