4.6 Variation of Parameters

4.6.1 Wronskian and Linear Independence

Suppose that we have found two solutions y_1 and y_2 of the differential equation $L(y) = ay'' + by' + cy = 0$ and we are interested in whether they are linearly independent. That is, whether there are scalars c_1 and c_2 (other than $c_1 = c_2 = 0$) such that

$$c_1y_1 + c_2y_2 = 0.$$

If there are, that is, if y_1 and y_2 are dependent, it follows by differentiation that

$$c_1y_1' + c_2y_2' = 0.$$

Evaluating these functions at an arbitrary point $t_0 \in I$, we have

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= 0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= 0 \end{aligned}$$

which may be regarded as a system of equations in the "unknowns" c_1 and c_2. Its determinant of coefficients is the number

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

and we know that the only way the system can have a nontrivial solution $(c_1, c_2) \neq (0,0)$ is for this determinant to be zero. We define the *Wronskian* of y_1 and y_2 to be the function

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

Then we have proved that

$$\text{linear dependence of } y_1 \text{ and } y_2 \Rightarrow W(t) = 0 \text{ for all } t \in I.$$

An equivalent statement is that if there is any point $t \in I$ for which $W(t) \neq 0$, then y_1 and y_2 are linearly independent.

Theorem. *Suppose that y_1 and y_2 are linearly independent solutions of the homogeneous linear equation $L(y) = 0$ in the interval I. Then their Wronskian is nowhere zero in I.*

Proof. Suppose the contrary. Then there is a point $t_0 \in I$ for which $W(t_0) = 0$. This implies that the system

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= 0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= 0 \end{aligned}$$

has a nontrivial solution $(c_1, c_2) \neq (0,0)$. Choose such a solution and define the function $\phi = c_1y_1 + c_2y_2$. Since y_1 and y_2 are solutions of $L(y) = 0$, so is ϕ. Moreover,

$$\begin{aligned} \phi(t_0) &= c_1y_1(t_0) + c_2y_2(t_0) = 0 \\ \phi'(t_0) &= c_1y_1'(t_0) + c_2y_2'(t_0) = 0 \end{aligned}$$

because c_1 and c_2 satisfy the above system. Hence ϕ satisfies the initial conditions $\phi(t_0) = 0$ and $\phi'(t_0) = 0$.

Now comes the deep part! According to the Uniqueness Theorem, there is only one solution of $L(y) = 0$ satisfying given initial conditions. It is obvious that the zero function $\psi(t) = 0$ is a solution satisfying the initial conditions $\psi(t_0) = 0$ and $\psi'(t_0) = 0$. Since ϕ is, too, ϕ must be the zero function. Hence

$$c_1y_1 + c_2y_2 = 0 \quad \text{where } c_1 \text{ and } c_2 \text{ are not both zero.}$$

But that's impossible! For if it were true, y_1 and y_2 would be linearly dependent, whereas by hypothesis they are independent. Our opening supposition (that the theorem is false) is incorrect; the theorem is true. □

4.6.2 Lagrange's Method of Variation of Parameters

Suppose that two independent solutions y_1 and y_2 of the homogeneous linear equation

$$L(y) = a(t)y'' + b(t)y' + c(t)y = 0$$

are known. Then of course the function $\phi = v_1 y_1 + v_2 y_2$ is also a solution if v_1 and v_2 are constants. The idea (due to Lagrange) is to try a solution of the nonhomogeneous equation $L(y) = f$ in this form, but with v_1 and v_2 as unknown functions to be determined. Let

$$y_p(t) = v_1 y_1(t) + v_2 y_2(t)$$

be this trial solution. In order to determine v_1 and v_2, two conditions must be imposed. One is the fact that y_p is supposed to be a solution of the nonhomogeneous equation, that is, $L(y_p) = f$. The other may be imposed more or less arbitrary; Lagrange's idea was to treat the variables v_1 and v_2 as "pseudo-constants."

To see what we mean, note that

$$y' = v_1 y_1' + v_1' y_1 + v_2 y_2' + v_2' y_2.$$

If v_1 and v_2 were true constants, the terms $v_1' y_1$ and $v_2' y_2$ would not appear. The condition we impose is that their *sum* drops out, that is,

$$v_1' y_1 + v_2' y_2 = 0. \tag{1}$$

This reduces the formula for y' to $y' = v_1 y_1' + v_2 y_2'$, from which

$$y'' = v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2'.$$

The beauty of this is that no second derivatives of the unknown functions v_1 and v_2 appear. When we substitute in

$$L(y) = ay'' + by' + cy = f$$

we will have only a first-order problem to solve:

$$\begin{aligned}
&L(y) = f\\
&\therefore a(v_1 y_1'' + v_1' y_1' + v_2 y_2'' + v_2' y_2') + b(v_1 y_1' + v_2 y_2') + c(v_1 y_1 + v_2 y_2) = f\\
&\therefore v_1(a y_1'' + b y_1' + c y_1) + v_2(a y_2'' + b y_2' + c y_2) + a(v_1' y_1' + v_2' y_2') = f.
\end{aligned}$$

Since y_1 and y_2 are solutions of $L(y) = 0$, the first two parentheses are zero, so the condition that our trial solution satistifes $L(y) = f$ reduces to

$$a(v_1' y_1' + v_2' y_2') = f$$

or

$$v_1' y_1' + v_2' y_2' = f/a. \tag{2}$$

Equations (1) and (2) constitute a system in the unknowns v_1' and v_2', namely

$$\begin{aligned}
y_1 v_1' + y_2 v_2' &= 0\\
y_1' v_1' + y_2' v_2' &= f/a
\end{aligned}$$

A remarkable aspect of this system is that its determinant of coefficients is the Wronskian of y_1 and y_2,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Since y_1 and y_2 are linearly indpendent solutions of $L(y) = 0$, the Wronskian is nowhere zero, so we know that the system can be solve by Cramer's Rule. In other words, we can find v_1' and v_2'. Integration will yield v_1 and v_2, so the method works.

Example 1. *Find a general solution to* $y'' + 9y = \sec^2(3t)$.

Example 2. *Find a particular solution to* $2x''(t) - 2x'(t) - 4x(t) = 2e^{2t}$ *by (a) undetermined coefficients and (b) variation of parameters, and compare the two methods.*

Example 3. *Find a general solution to* $y'' - 6y' + 9y = t^{-3}e^{3t}$.

As you can see, Lagrange's method of variation of parameters may be messy. Its virtue is that it is infallible. The method of undetermined coefficients is usually easier when it works, but it applies only in special cases.