4.5 The Superposition Principle and Undetermined Coefficients Revisited

**Theorem 3** (Superposition Principle). If $y_1$ is a solution to the equation

$$ay'' + by' + cy = f_1(t),$$

and $y_2$ is a solution to

$$ay'' + by' + cy = f_2(t),$$

then for any constants $k_1$ and $k_2$, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t).$$

**Proof.**

$$ay'' + by' + cy = a(k_1y_1 + k_2y_2)'' + b(k_1y_1 + k_2y_2)' + c(k_1y_1 + k_2y_2)$$

$$= k_1(ay_1'' + by_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2)$$

$$= k_1f_1(t) + k_2f_2(t).$$

**Example 1.** Given that $y_1(t) = (1/4)\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t + \cos 2t$, find solutions to the following:

a) $y'' + 2y' + 4y = t + \cos 2t$.

b) $y'' + 2y' + 4y = 2t - 3\cos 2t$.

c) $y'' + 2y' + 4y = 11t - 12\cos 2t$.

Thus, by superposition principle, the general solution to a nonhomogeneous equation is the sum of the general solution to the homogeneous equation and one particular solution. That is, if the general solution to $ay'' + by' + cy = 0$ is $c_1y_1(t) + c_2y_2(t)$, and if a particular solution to $ay'' + by' + cy = f(t)$ is $y_p(t)$, then the general solution to $ay'' + by' + cy = f(t)$ is $c_1y_1(t) + c_2y_2(t) + y_p(t)$.

**Theorem 4** (Existence and Uniqueness: Nonhomogeneous Case). For any real numbers $a, b, c, t_0, y_0, y_1$, with $a \neq 0$, suppose $y_p(t)$ is a particular solution to

$$ay'' + by' + cy = f(t)$$

in an interval $I$ containing $t_0$, and that $y_1(t), y_2(t)$ are linearly independent solutions to the associated homogeneous equation

$$ay'' + by' + cy = 0$$

in $I$. Then there exists a unique solution in $I$ to the initial value problem

$$ay'' + by' + cy = f(t), \quad y(t_0) = y_0, y'(t_0) = y_1,$$

and it is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t),$$

for the appropriate choice of the constants $c_1, c_2$.

**Example 2.** Find a general solution to $y'' = 2y' - y + 2e^x$, if a particular solution is $y_p(x) = x^2e^x$. 

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To find a particular solution to the differential equation
\[ ay'' + by' + cy = P_m(t)e^{rt}, \]
where \( P_m(T) \) is a polynomial of degree \( m \), use the form
\[ y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}; \quad (5) \]
if \( r \) is not a root of the associated auxiliary equation, take \( s = 0 \); if \( r \) is a simple root of the associated auxiliary equation, take \( s = 1 \); and if \( r \) is a double root of the associated auxiliary equation, take \( s = 2 \).

To find a particular solution to the differential equation
\[ ay'' + by' + cy = P_m(t)e^{\alpha t} \cos \beta t + Q_n(t)e^{\alpha t} \sin \beta t, \quad \beta \neq 0, \]
where \( P_m(t) \) is a polynomial of degree \( m \) and \( Q_n(t) \) is a polynomial of degree \( n \), use the form
\[ y_p(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t} \cos \beta t + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t} \sin \beta t, \quad (6) \]
where \( k \) is the larger of \( m \) and \( n \). If \( \alpha + i\beta \) is not a root of the associated auxiliary equation, take \( s = 0 \); if \( \alpha + i\beta \) is a root of the associated auxiliary equation, take \( s = 1 \).

Example 3. Decide whether the method of undetermined coefficients together with superposition principle can be applied to find a particular solution of the following equation. Do not solve the equation.
\[ 2y'' - y' + 6y = t^2 e^{-t} \sin t - 8t \cos 3t + 10t. \]

Example 4. Find a general solution to \( y''(x) + 6y'(x) + 10y(x) = 10x^4 + 24x^3 + 2x^2 - 12x + 18 \).

Example 5. Find the solution to the initial value problem:
\[ y'' + y' - 12y = e^t + e^{2t} - 1; \quad y(0) = 1, \quad y'(0) = 3. \]

Example 6. Determine the form of a particular solution for the differential equation. Do not solve.
\[ y'' + 5y' + 6y = \sin t - \cos 2t. \]