4.5 The Superposition Principle and Undetermined Coefficients Revisited

Theorem 3 (Superposition Principle). *If y_1 is a solution to the equation*

$$ay'' + by' + cy = f_1(t),$$

and y_2 is a solution to

$$ay'' + by' + cy = f_2(t),$$

then for any constants k_1 and k_2, the function $k_1y_1 + k_2y_2$ is a solution to the differential equation

$$ay'' + by' + cy = k_1f_1(t) + k_2f_2(t).$$

Proof.

$$\begin{aligned} ay'' + by' + cy &= a(k_1y_1 + k_2y_2)'' + b(k_1y_1 + k_2y_2)' + c(k_1y_1 + k_2y_2) \\ &= k_1(ay_1'' + by_1' + cy_1) + k_2(ay_2'' + by_2' + cy_2) \\ &= k_1f_1(t) + k_2f_2(t). \end{aligned}$$

□

Example 1. *Given that $y_1(t) = (1/4)\sin 2t$ is a solution to $y'' + 2y' + 4y = \cos 2t$ and that $y_2(t) = t/4 - 1/8$ is a solution to $y'' + 2y' + 4y = t$, find solutions to the following:*

a) $y'' + 2y' + 4y = t + \cos 2t$.

b) $y'' + 2y' + 4y = 2t - 3\cos 2t$.

c) $y'' + 2y' + 4y = 11t - 12\cos 2t$.

Thus, by superposition principle, the general solution to a nonhomogeneous equation is the sum of the general solution to the homogeneous equation and one particular solution. That is, if the general solution to $ay'' + by' + cy = 0$ is $c_1y_1(t) + c_2y_2(t)$, and if a particular solution to $ay'' + by' + cy = f(t)$ is $y_p(t)$, then the general solution to $ay'' + by' + cy = f(t)$ is $c_1y_1(t) + c_2y_2(t) + y_p(t)$.

Theorem 4 (Existence and Uniqueness: Nonhomogeneous Case). *For any real numbers a, b, c, t_0, y_0, y_1, with $a \neq 0$, suppose $y_p(t)$ is a particular solution to*

$$ay'' + by' + cy = f(t) \tag{1}$$

in an interval I containing t_0, and that $y_1(t), y_2(t)$ are linearly independent solutions to the associated homogeneous equation

$$ay'' + by' + cy = 0 \tag{2}$$

in I. Then there exists a unique solution in I to the initial value problem

$$ay'' + by' + cy = f(t), \quad y(t_0) = y_0, y'(t_0) = y_1, \tag{3}$$

and it is given by

$$y(t) = y_p(t) + c_1y_1(t) + c_2y_2(t), \tag{4}$$

for the appropriate choice of the constants c_1, c_2.

Example 2. *Find a general solution to $y'' = 2y' - y + 2e^x$, if a particular solution is $y_p(x) = x^2e^x$.*

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{rt},$$

where $P_m(T)$ is a polynomial of degree m, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}; \tag{5}$$

if r is not a root of the associated auxiliary equation, take $s = 0$; if r is a simple root of the associated auxiliary equation, take $s = 1$; and if r is a double root of the associated auxiliary equation, take $s = 2$.

To find a particular solution to the differential equation

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos\beta t + Q_n(t)e^{\alpha t}\sin\beta t, \quad \beta \neq 0,$$

where $P_m(t)$ is a polynomial of degree m and $Q_n(t)$ is a polynomial of degree n, use the form

$$y_p(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_k t^k + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t, \tag{6}$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation, take $s = 0$; if $\alpha + i\beta$ is a root of the associated auxiliary equation, take $s = 1$.

Example 3. *Decide whether the method of undetermined coefficients together with superposition principle can be applied to find a particular solution of the following equation. Do not solve the equation.*

$$2y'' - y' + 6y = t^2e^{-t}\sin t - 8t\cos 3t + 10^t.$$

Example 4. *Find a general solution to* $y''(x) + 6y'(x) + 10y(x) = 10x^4 + 24x^3 + 2x^2 - 12x + 18.$

Example 5. *Find the solution to the initial value problem:*

$$y'' + y' - 12y = e^t + e^{2t} - 1; \quad y(0) = 1, \quad y'(0) = 3.$$

Example 6. *Determine the form of a particular solution for the differential equation. Do not solve.*

$$y'' + 5y' + 6y = \sin t - \cos 2t.$$