4.4 Non-homogeneous Equations: the Method of Undetermined Coefficients

We saw previously that the homogeneous equation

$$ay'' + by' + cy = 0$$

has infinitely many solutions that are linear combinations of two linearly independent solutions. Now let's consider the non-homogeneous equation

$$ay'' + by' + cy = f(t). \tag{1}$$

Our goal is to find a single *particular* solution to (1).

Example 1. *Find a particular solution to* $y'' + 3y' + 2y = 3t$*.*

We may proceed with guess and check and observe that $y_p(t) = At$ does not work; however, $y_p(t) = At+B$ does work! So in general, if the non-homogeneity $f(t)$ on the right side of (1) is a polynomial of degree n, we should try for a particular solution a polynomial of degree n as well.

If the non-homogeneity $f(t)$ in (1) is e^{kt}, we should try a multiple of that function, and if $f(t)$ is a sine or cosine function, we should try a linear combination of sine and cosine functions.

However, if $f(t)$ is already a solution to the associated homogeneous equation, this method fails. In these situations, we append a t to our guess, as we have done when we needed an extra solution.

To find a particular solution to the differential equation

$$ay'' + by' + cy = Ct^m e^{rt},$$

where m is a nonnegative integer, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}, \tag{2}$$

with

(i) $s = 0$ if r is not a root of the associated auxiliary equation;

(ii) $s = 1$ is r is a simple root of the associated auxiliary equation; and

(iii) $s = 2$ is r is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay'' + by' + cy = \begin{cases} Ct^m e^{\alpha t}\cos\beta t \\ \text{or} \\ Ct^m e^{\alpha t}\sin\beta t \end{cases}$$

for $\beta \neq 0$, use the form

$$y_p(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{\alpha t}\cos\beta t + t^s(B_m t^m + \cdots + B_1 t + B_0)e^{\alpha t}\sin\beta t, \tag{3}$$

with

(iv) $s = 0$ if $\alpha + \beta i$ is not a root of the associated auxiliary equation; and

(v) $s = 1$ if $\alpha + \beta i$ is a root of the associated auxiliary equation.

Example 2. *Decide whether or not the method of undetermined coefficients can be applied to find a particular solution of*

(i) $ty'' - y' + 2y = \sin 3t$*,*

(ii) $2\omega''(x) - 3\omega(x) = 4x\sin^2 x + 4x\cos^2 x$*.*

Example 3. *Find a particular solution to* $y'' - 2y' + y = 8e^t$*.*

Example 4. *Determine the form of a particular solution for the differential equation* $y'' - y' - 12y = 2t^6 e^{-3t}$*.*

Example 5. *Use the method of undetermined coefficients to find a particular solution to* $y^{(4)} - 3y'' - 8y = \sin t$*.*