4.3 Auxiliary Equations with Complex Roots

Suppose we want to solve

$$ay'' + by' + cy = 0. \tag{1}$$

However, the auxiliary equation does not have real roots. As an example, consider the *simple harmonic* equation $y'' + y = 0$, so called because of its relation to the vibration of a musical tone, which has solutions $y_1(t) = \sin t$ and $y_2(t) = \cos t$. The auxiliary equation of the simple harmonic equation is $r^2 + 1 = 0$, which has imaginary roots $r = \pm i$, where $i^2 = -1$. Could we use the ideas from the previous section and have the solutions e^{it} and e^{-it}? Euler says yes: In the quadratic equation $ar^2 + br + c = 0$, when $b^2 - 4ac < 0$, the roots are complex conjugate numbers

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a} i \quad \text{and} \quad r_2 = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a} i.$$

We need to clarify the meaning of e^{it}. Consider the following:

$$\begin{aligned} e^{it} &= 1 + (it) + \frac{(it)^2}{2!} + \cdots + \frac{(it)^n}{n!} + \cdots \\ &= 1 + ti - \frac{t^2}{2!} - \frac{t^3}{3!} i + \frac{t^4}{4!} + \frac{t^5}{5!} i + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\right) \\ &= \cos t + i \sin t. \end{aligned}$$

Thus we have proved *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{2}$$

Thus if a root of the auxiliary equation is $r_1 = \alpha + i\beta$, then a solution is $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$.

Lemma 2. *Let $z(t) = u(t) + iv(t)$ be a solution to (1), where $a, b, c \in \mathbb{R}$. Then, the real part $u(t)$ and the imaginary part $v(t)$ are real-valued functions of (1).*

Thus, by Lemma 2, we can say that if the complex conjugate roots of the auxiliary equation are $\alpha \pm \beta i$, then two linearly independent solutions to (1) are $e^{\alpha t}\cos\beta t$ and $e^{\alpha t}\sin\beta t$. Therefore a general solution of (1) is

$$y(t) = c_1 e^{\alpha t}\cos\beta t + c_2 e^{\alpha t}\sin\beta t, \tag{3}$$

where c_1, c_2 are arbitrary constants.

Example 1. *Find a general solution.*

a) $4y'' - 4y' + 26y = 0.$

b) $y''' - y'' + 2y = 0.$

Example 2. *Solve the initial value problem $y'' - 2y' + 2y = 0$; $y(\pi) = e^{\pi}, y'(\pi) = 0$.*

Throughout this section we have assumed that the coefficients a, b, c in the differential equation were real numbers. If we allow them to be complex constants, then the roots r_1, r_2 of the auxiliary equation are, in general, also complex and not necessarily conjugates of each other. When $r_1 \neq r_2$, a general solution still has the form $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, and c_1, c_2 are now arbitrary complex-valued constants.

A complex differential equation may also be regarded as a system of two real differential equations because we can always work separately with its real and imaginary parts. We will discuss systems in chapter 9.