4.2 Homogeneous Linear Equations: The General Solution

Definition. *A set of functions* $f_1, \ldots, f_n$ *is* linearly independent *if the only way to have a linear combination of these function to be zero is to have all the weights to be zero. That is,*

$$c_1 f_1 + \cdots + c_n f_n = 0 \Rightarrow c_1 = \cdots = c_n = 0.$$

Two functions are linearly independent when no function is a constant multiple of the other. Functions that are not linearly independent are linearly dependent*.*

Example 1. *Determine whether the functions* y_1 *and* y_2 *are linearly independent on the interval* $(0, 1)$*.*

a) $y_1(t) = e^{3t}, y_2(t) = e^{-4t}$.

b) $y_1(t) = 0, y_2(t) = e^{t}$.

A differential equation is an equation that involves a derivative of a function. To solve a differential equation means to find that function. We begin our study of the *linear second-order constant-coefficient* differential equation

$$ay'' + by' + c = f(t), \quad a \neq 0 \tag{1}$$

with the *homogeneous* equation, where $f(t) = 0$:

$$ay'' + by' + c = 0, \quad a \neq 0. \tag{2}$$

Every homogeneous equation has the trivial solution: the zero function. To find linearly independent non-trivial solutions of (2), we observe that a solution y must have the property that its second derivative is a linear combination of its first derivative and the function itself. Thus we may try a solution of the form $y = e^{rt}$, because the derivatives of e^{rt} are just constant multiples of e^{rt}. By substituting $y = e^{rt}$ into (2), we obtain

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \Rightarrow e^{rt}(ar^2 + br + c) = 0.$$

Since $e^{rt} \neq 0$ for all t, we obtain

$$ar^2 + br + c = 0. \tag{3}$$

Equation (3) is called the *auxiliary* equation for (2). The roots of the auxiliary equation are:

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac > 0$, there are two distinct real roots, when $b^2 - 4ac = 0$, there is a double real root, and when $b^2 - 4ac < 0$, there are two complex conjugate roots. In this section we consider the real roots.

Suppose $y_1(t), \ldots, y_n(t)$ are linearly independent solutions of (2). That is, suppose

$$\begin{gathered} ay_1'' + by_1' + cy_1 = 0, \\ \vdots \\ ay_n'' + by_n' + cy_n = 0. \end{gathered}$$

Then $y(t) = c_1 y_1(t) + \cdots + c_n y_n(t)$ is also a solution to (2):

$$\begin{aligned} ay'' + by' + cy &= a(c_1 y_1 + \cdots + c_n y_n)'' + b(c_1 y_1 + \cdots + c_n y_n)' + c(c_1 y_1 + \cdots + c_n y_n) \\ &= a(c_1 y_1'' + \cdots + c_n y_n'') + b(c_1 y_1' + \cdots + c_n y_n') + c(c_1 y_1 + \cdots + c_n y_n) \\ &= c_1(ay_1'' + by_1' + cy_1) + \cdots + c_n(ay_n'' + by_n' + cy_n) \\ &= 0 + \cdots + 0 \\ &= 0. \end{aligned}$$

Example 2. *Find a general solution to* $y'' - 5y' + 6y = 0$.

An *initial value problem* is a differential equation, together with a set of constraints on the function and its derivatives.

Example 3. *Solve the initial value problem* $y'' + y' = 0$; $y(0) = 2, y'(0) = 1$.

Theorem 1 (Existence and Uniqueness: Homogeneous Case)**.** *For every real numbers* a, b, c, t_0, y_0, y_1, *with* $a \neq 0$, *for all* $t \in \mathbb{R}$, *there exists a* unique *solution to the initial value problem*

$$ay'' + by' + cy = 0; \quad y(t_0) = y_0, \quad y'(t_0) = y_1. \tag{4}$$

Theorem 2 (Representation of Solutions to Initial Value Problem)**.** *If* $y_1(t)$ *and* $y_2(t)$ *are any two linearly independent solutions, on* $(-\infty, \infty)$, *to the homogeneous equation (2), then unique constants* c_1 *and* c_2 *can always be found so that* $c_1y_1(t) + c_2y_2(t)$ *satisfies the initial value problem (4) on* $(-\infty, \infty)$.

Lemma 1. *For any real numbers* a, b, c, *with* $a \neq 0$, *if* $y_1(t)$ *and* $y_2(t)$ *are solutions to the homogeneous equation (2), and if the equality*

$$y_1(\tau)y_2'(\tau) - y_1'(\tau)y_2(\tau) = 0 \tag{5}$$

holds at any point τ, *then* y_1 *and* y_2 *are linearly dependent on* $(-\infty, \infty)$. *The expression on the left-hand side of (5) is called the* Wronskian *of* y_1 *and* y_2 *at the point* τ.

If the auxiliary equation (3) has distinct roots r_1 and r_2, then both $y_1(t) = e^{r_1t}$ and $y_2(t) = e^{r_2t}$ are solutions to (2) and a general solution is $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$.

Question: What if we have only a repeated root? How do we find two linearly independent solutions?

Answer: If the auxiliary equation (3) has a repeated root r, then both $y_1(t) = e^{rt}$ and $y_2(t) = te^{rt}$ are solutions to (2) and a general solution is $y(t) = c_1e^{rt} + c_2te^{rt}$.

Example 4. *Solve* $y'' - 4y' + 4y = 0$.