9.7 Nonhomogeneous Linear Systems

We may extend the techniques of undetermined coefficients and variation of parameters, that we used to solve equation $ay'' + by' + cy = f(t)$, to nonhomogeneous linear systems.

9.7.1 Undetermined Coefficients

We may use the method of undetermined coefficients to find a particular solution to the nonhomogeneous linear system

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{f}(t)$$

where A is an $n \times n$ constant matrix and the entries of $\mathbf{f}(t)$ are polynomials, exponential functions, sines, cosines, or finite sums and products of these functions.

Example 1. *Find a general solution to the system* $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$, *where* $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $\mathbf{f}(t) = \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$.

9.7.2 Variation of Parameters

Suppose $X(t)$ be a fundamental matrix for the homogeneous system

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{0}. \tag{1}$$

A general solution to (1) is $X(t)\mathbf{c}$, where $\mathbf{c}$ is a constant vector. We seek a particular solution to the nonhomogeneous system

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t) \tag{2}$$

of the form

$$\mathbf{x}_p(t) = X(t)\mathbf{v}(t), \tag{3}$$

where $\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_n(t) \end{bmatrix}$ is a vector function of t to be determined.

To derive a formula for $\mathbf{v}(t)$, we first differentiate (3), using product rule, to obtain

$$\mathbf{x}_p'(t) = X(t)\mathbf{v}'(t) + X'(t)\mathbf{v}(t).$$

Substituting the expressions for $\mathbf{x}_p$ and $\mathbf{x}_p'$ into (2) yields

$$X\mathbf{v}' + X'\mathbf{v} = AX\mathbf{v} + \mathbf{f}. \tag{4}$$

Since X satisfies the matrix equation $X' = AX$, equation (4) becomes

$$\begin{aligned} X\mathbf{v}' + AX\mathbf{v} &= AX\mathbf{v} + \mathbf{f}, \\ X\mathbf{v}' &= \mathbf{f}. \end{aligned}$$

Multiplying both sides of the last equation by X^{-1} (which exists because the columns of X are linearly independent) gives

$$\mathbf{v}' = X^{-1}\mathbf{f}.$$

Integrating, we obtain

$$\mathbf{v}(t) = \int X^{-1}(t)\mathbf{f}(t)dt.$$

Hence, a particular solution to (2) is

$$\mathbf{x}_p(t) = X(t)\mathbf{v}(t) = X(t)\int X^{-1}(t)\mathbf{f}(t)dt. \tag{5}$$

Combining (5) with the solution $X(t)\mathbf{c}$ to the homogeneous system yields the following general solution to (2):

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t)\int X^{-1}(t)\mathbf{f}(t)dt. \tag{6}$$

Given an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we solve for $\mathbf{c}$ in (6). Expressing $\mathbf{x}(t)$ using a definite integral, we have

$$\mathbf{x}(t) = X(t)\mathbf{c} + X(t)\int_{t_0}^{t} X^{-1}(s)\mathbf{f}(s)ds.$$

From the initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, we find

$$\mathbf{x}_0 = \mathbf{x}(t_0) = X(t_0)\mathbf{c} + X(t_0)\int_{t_0}^{t_0} X^{-1}(s)\mathbf{f}(s)ds = X(t_0)\mathbf{c}.$$

Solving for $\mathbf{c}$, we have $\mathbf{c} = X^{-1}(t_0)\mathbf{x}_0$. Thus, the solution to the initial value problem

$$\mathbf{x}'(t) - A(t)\mathbf{x}(t) = \mathbf{f}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = X(t)X^{-1}(t_0)\mathbf{x}_0 + X(t)\int_{t_0}^{t} X^{-1}(s)\mathbf{f}(s)ds.$$

To apply the variation of parameters formula, we must first determine a fundamental solution matrix $X(t)$ for the homogeneous system. If the entries of A depend on t, the determination of $X(t)$ may be extremely difficult, entailing, perhaps, a matrix power series.

Example 2. *Find a general solution to the system* $\mathbf{x}' - A\mathbf{x} = \mathbf{f}$, *where* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and* $\mathbf{f}(t) = \begin{bmatrix} 8\sin t \\ 0 \end{bmatrix}$.

Example 3. *Find the solution to* $\mathbf{x}'(t) - \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}\mathbf{x}(t) = \begin{bmatrix} 4t \\ -4t-2 \end{bmatrix}$ *that satisfies the initial condition*

a) $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

b) $\mathbf{x}(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$