9.6 Complex Eigenvalues

Recall that the homogeneous system

$$\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}, \tag{1}$$

where A is a constant $n \times n$ matrix, has a solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{u}$ if and only if λ is an eigenvalue of A and $\mathbf{u}$ is a corresponding eigenvector.

Now suppose $\lambda_1 = \alpha + i\beta$, with α, β real numbers, is en eigenvalue of A with corresponding eigenvector $\mathbf{z} = \mathbf{a} + i\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are real constant vectors. Note that the complex conjugate of $\mathbf{z}$, namely $\bar{\mathbf{z}} = \mathbf{a} - i\mathbf{b}$ is an eigenvector associated with the eigenvalue $\alpha - i\beta$. The reason is as follows: Take the complex conjugate of $(A - \lambda_1 I)\mathbf{z} = \mathbf{0}$ to obtain $(A - \bar{\lambda}_1 I)\bar{\mathbf{z}} = \mathbf{0}$ because A and I have real entries, that is, $\bar{A} = A$ and $\bar{I} = I$. Since $\lambda_2 = \bar{\lambda}_1$, we see that $\bar{\mathbf{z}}$ is an eigenvector associated with λ_2. Therefore, two linearly independent complex vector solutions to (1) are

$$\mathbf{w}_1(t) = e^{\lambda_1 t}\mathbf{z} = e^{(\alpha+i\beta)t}(\mathbf{a} + i\mathbf{b}), \tag{2}$$

$$\mathbf{w}_2(t) = e^{\lambda_2 t}\bar{\mathbf{z}} = e^{(\alpha-i\beta)t}(\mathbf{a} - i\mathbf{b}), \tag{3}$$

Using Euler's formula, we obtain two real vector solutions.

$$\mathbf{w}_1(t) = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{a} + i\mathbf{b}) = e^{\alpha t}\left[(\cos\beta t\mathbf{a} - \sin\beta t\mathbf{b}) + i(\sin\beta t\mathbf{a} + \cos\beta t\mathbf{b})\right].$$

Thus $\mathbf{w}_1(t) = \mathbf{x}_1(t) + i\mathbf{x}_2(t)$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are two real vector functions. Since $\mathbf{w}_1$ is a solution to (1), then $\mathbf{w}_1'(t) = A\mathbf{w}_1(t)$, that is, $\mathbf{x}_1' + i\mathbf{x}_2' = A\mathbf{x}_1 + iA\mathbf{x}_2$. Equating the real and imaginary parts, we obtain

$$\begin{aligned} \mathbf{x}_1'(t) &= A\mathbf{x}_1(t) \\ \mathbf{x}_2'(t) &= A\mathbf{x}_2(t). \end{aligned}$$

Hence $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are real vector solutions to (1) associated with the complex conjugate eigenvalues $\alpha \pm i\beta$. Furthermore, $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent.

If the real matrix A has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then two linearly independent real vector solutions to $\mathbf{x}'(t) - A\mathbf{x}(t) = \mathbf{0}$ are

$$e^{\alpha t}\cos\beta t\mathbf{a} - e^{\alpha t}\sin\beta t\mathbf{b}, \tag{4}$$

$$e^{\alpha t}\sin\beta t\mathbf{a} + e^{\alpha t}\cos\beta t\mathbf{b}. \tag{5}$$

Example 1. *Find a general solution of the system* $\mathbf{x}'(t) = A\mathbf{x}$ *when* $A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$

Example 2. *Find a fundamental matrix for the system* $\mathbf{x}'(t) = A\mathbf{x}(t)$ *when* $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & 4 \end{bmatrix}$

Example 3. *Find the solution to* $\mathbf{x}'(t) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\mathbf{x}(t)$*, given the initial condition*

a) $\mathbf{x}(0) = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$.

b) $\mathbf{x}(-\pi) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.